Some remarks on regularized Shannon sampling formulas

Melanie Kircheis* Daniel Potts[§] Manfred Tasche[‡]

The fast reconstruction of a bandlimited function from its sample data is an essential problem in signal processing. In this paper, we consider the widely used Gaussian regularized Shannon sampling formula in comparison to regularized Shannon sampling formulas employing alternative window functions, such as the sinh-type window function and the continuous Kaiser–Bessel window function. It is shown that the approximation errors of these regularized Shannon sampling formulas possess an exponential decay with respect to the truncation parameter. The main focus of this work is to address minor gaps in the preceding papers [13, 14] and rigorously prove assumptions that were previously based solely on numerical tests. In doing so, we demonstrate that the sinh-type regularized Shannon sampling formula has the same exponential decay as the continuous Kaiser–Bessel regularized Shannon sampling formula, but both have twice the exponential decay of the Gaussian regularized Shannon sampling formula. Additionally, numerical experiments illustrate the theoretical results.

Key words: Shannon sampling series, regularization, bandlimited function, approximation error, exponential decay, Gaussian regularized Shannon sampling formulas, sinh-type regularized Shannon sampling formulas.

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1 Introduction

In signal processing, the fast reconstruction of a bandlimited function from its sample data is of fundamental importance. A function $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ is called *bandlimited* with *bandwidth* $\delta > 0$, if its Fourier transform

$$(\mathcal{F}f)(\omega) = \hat{f}(\omega) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-\mathrm{i}t\omega}\, \mathrm{d}t\,, \quad \omega \in \mathbb{R}\,, \tag{1.1}$$

*Corresponding author: melanie.kircheis@math.tu-chemnitz.de, Chemnitz University of Technology, Faculty of Mathematics, D–09107 Chemnitz, Germany

[§]potts@mathematik.tu-chemnitz.de, Chemnitz University of Technology, Faculty of Mathematics, D–09107 Chemnitz, Germany

[‡]manfred.tasche@uni-rostock.de, University of Rostock, Institute of Mathematics, D–18051 Rostock, Germany

vanishes for all $|\omega| \ge \delta$. For such a bandlimited function with $\delta \in (0,\, \pi]$ the famous Shannon sampling theorem, see [34, 15, 30], states that

$$f(t) = \sum_{k\in\mathbb{Z}} f(k)\,\mathrm{sinc}(t-k)\,, \quad t \in \mathbb{R}\,, \tag{1.2}$$

where

$$\mathrm{sinc}(t) := \begin{cases} \frac{\sin(\pi t)}{\pi t} & : t \in \mathbb{R}\setminus\{0\}\,, \\ 1 & : t = 0\,, \end{cases} \tag{1.3}$$

denotes the *cardinal sine function.* It is known that the Shannon sampling series (1.2) converges absolutely and uniformly on whole $\mathbb{R}$. However, the practical use of (1.2) is limited, since its evaluation requires infinitely many samples and its truncated version is not a good approximation due to the slow decay of the cardinal sine function, see [12]. In addition to this rather poor convergence, it is known, see [9, 10, 8], that in the presence of noise in the samples $f(k)$, $k \in \mathbb{Z}$, of a bandlimited function $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ the convergence of Shannon sampling series (1.2) may even break down completely. Therefore, it was proposed to consider the regularization of the Shannon sampling series with a suitable window function. Note that many authors such as [7, 18, 28, 20, 31] used window functions in the frequency domain, but the recent study [14] has shown that it is much more beneficial to employ a window function in the spatial domain, cf. [24, 25, 31, 17, 16, 6, 13]. In the following, a *window function* $\varphi:\, \mathbb{R} \to [0,\, 1]$ is an even function in $L^2(\mathbb{R}) \cap C(\mathbb{R})$ which decreases on $[0,\, \infty)$ and fulfills $\varphi(0) = 1$. By $\mathbf{1}_{[-m,\,m]}$ we denote the *characteristic function* of the interval $[-m,\, m]$ with $m \in \mathbb{N}\setminus\{1\}$, i. e., the function

$$\mathbf{1}_{[-m,\,m]}(t) := \begin{cases} 1 & : t \in [-m,\, m]\,, \\ 0 & : t \in \mathbb{R}\setminus[-m,\, m]\,. \end{cases}$$

In this paper, we assume that the bandwidth δ of f fulfills the so-called *oversampling condition* $0 < \delta < \pi$. Then we recover f by the *φ-regularized Shannon sampling formula*

$$\big(R_{\varphi,m}f\big)(t) := \sum_{k\in\mathbb{Z}} f(k)\,\mathrm{sinc}(t-k)\,\varphi(t-k)\,\mathbf{1}_{[-m,\,m]}(t-k)\,, \quad t \in \mathbb{R}\,, \tag{1.4}$$

where $m \in \mathbb{N}\setminus\{1\}$ is the so-called *truncation parameter.* In doing so, we consider the following window functions $\varphi:\, \mathbb{R} \to [0,\, 1]$.

Remark 1.1. The most popular window function, see e. g. [24, 27, 29, 32, 16, 6], is the *Gaussian function*

$$\varphi_{\mathrm{Gauss}}(t) := \mathrm{e}^{-t^2/(2\sigma^2)}\,, \quad t \in \mathbb{R}\,, \tag{1.5}$$

with *variance* $\sigma^2 > 0$. Note that this window function is supported on whole $\mathbb{R}$.

Here we prefer window functions which are compactly supported on the interval $[-m,\, m]$, as studied in [13, 14]. The sinh-*type window function* is defined as

$$\varphi_{\sinh}(t) := \begin{cases} \frac{1}{\sinh\beta}\,\sinh\Big(\beta\,\sqrt{1-\frac{t^2}{m^2}}\Big) & : t \in [-m,\, m]\,, \\ 0 & : t \in \mathbb{R}\setminus[-m,\, m]\,, \end{cases} \tag{1.6}$$

with *shape parameter* $\beta > 0$, see [22]. Then the corresponding expression (1.4) is termed the sinh-*type regularized Shannon sampling formula*. The *continuous Kaiser–Bessel window function* is defined as

$$\varphi_{\mathrm{cKB}}(t) := \begin{cases} \frac{1}{I_0(\beta)-1} \left(I_0(\beta\sqrt{1-t^2/m^2}) - 1\right) & : t \in [-m,\, m]\,, \\ 0 & : t \in \mathbb{R} \setminus [-m,\, m]\,, \end{cases} \tag{1.7}$$

with convenient shape parameter $\beta > 0$, see [22]. Then the corresponding expression (1.4) is called the *continuous Kaiser–Bessel regularized Shannon sampling formula*. We remark that these two window functions (1.6) and (1.7) are well-studied in the context of the nonuniform fast Fourier transform (NFFT), see e. g. [21, Section 6] and [5, 4]. □

Due to the definition of the cardinal sine function (1.3) we have $\mathrm{sinc}(n-k) = \delta_{n,k}$ and therefore the regularized Shannon sampling formula $R_{\varphi,m}f$ in (1.4) has the *interpolation property*

$$\big(R_{\varphi,m}f\big)(n) = f(n)\,, \quad n \in \mathbb{Z}\,. \tag{1.8}$$

Moreover, the use of the characteristic function $\mathbf{1}_{[-m,\,m]}$ in (1.4) leads to *localized sampling* of f, i. e., the computation of $\big(R_{\varphi,m}f\big)(t)$ for any $t \in \mathbb{R} \setminus \mathbb{Z}$ requires only $2m$ samples $f(k)$, where $k \in \mathbb{Z}$ fulfills the condition $|k - t| \le m$. Especially, for $t \in (0,\, 1)$ we obtain the finite sum

$$\big(R_{\varphi,m}f\big)(t) = \sum_{k=1-m}^{m} f(k)\, \mathrm{sinc}(t-k)\, \varphi(t-k)\,.$$

As in many applications, we use *oversampling* of the given bandlimited function f with bandwidth $\delta < \pi$, i. e., the function f is sampled on the integer grid $\mathbb{Z}$.

In this paper, we focus on the φ-regularized Shannon sampling formulas (1.4) for the window functions φ given in Remark 1.1. To compare the corresponding approaches, we present estimates of the uniform approximation error

$$\|f - R_{\varphi,m}f\|_{C_0(\mathbb{R})} := \max_{t\in\mathbb{R}} \big|f(t) - \big(R_{\varphi,m}f\big)(t)\big|\,, \tag{1.9}$$

where $C_0(\mathbb{R})$ denotes the Banach space of continuous functions $g\colon \mathbb{R} \to \mathbb{C}$ vanishing as $|t| \to \infty$ equipped with the norm $\|f\|_{C_0(\mathbb{R})} := \max_{t\in\mathbb{R}} |f(t)|$. Primarily, this work concentrates on addressing minor gaps in the preceding papers [13, 14] and rigorously proving the corresponding assumptions that were previously based solely on numerical experiments.

For this purpose, we initially study the uniform approximation error of general φ-regularized Shannon sampling formulas (1.4) in Section 2. Afterwards, we specify our findings for the window functions φ introduced in Remark 1.1. In particular, Section 3 deals with the Gaussian window function (1.5), while Section 4 is concerned with the sinh-type window function (1.6) and Section 5 with the continuous Kaiser–Bessel window function (1.7).

2 Approximation error of regularized Shannon sampling formulas

Firstly, we estimate the uniform approximation error of the φ-regularized Shannon sampling formula (1.4), analogously to [13, Theorem 3.2] and [14, Theorem 4.1].

Theorem 2.1. *Assume that $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ is bandlimited with bandwidth $\delta \in (0,\, \pi)$. Further let $\varphi : \mathbb{R} \to [0,\, 1]$ be an even function in $L^2(\mathbb{R}) \cap C(\mathbb{R})$ which is decreasing on $[0,\, \infty)$ with $\varphi(0) = 1$, and let $m \in \mathbb{N} \setminus \{1\}$ be given.*
Then the φ-regularized Shannon sampling formula (1.4) satisfies the error estimate

$$\|f - R_{\varphi,m} f\|_{C_0(\mathbb{R})} \le \big(E_1(m) + E_2(m)\big)\, \|f\|_{L^2(\mathbb{R})}\,, \quad m \in \mathbb{N} \setminus \{1\}\,,$$

with the error constants

$$E_1(m) := \max_{\omega \in [-\delta,\, \delta]} \left| 1 - \frac{1}{\sqrt{2\pi}} \int_{\omega-\pi}^{\omega+\pi} \hat{\varphi}(\tau)\, \mathrm{d}\tau \right|, \tag{2.1}$$

$$E_2(m) := \frac{\sqrt{2}}{\pi\, m} \sqrt{\varphi^2(m) + \int_m^\infty \varphi^2(t)\, \mathrm{d}t}\,. \tag{2.2}$$

Proof. (i) Initially, we consider only the case $t \in (0,\, 1)$, where we split the approximation error

$$f(t) - \big(R_{\varphi,m} f\big)(t) = e_1(t) + e_{2,0}(t)\,, \quad t \in (0,\, 1)\,,$$

into the *regularization error*

$$e_1(t) := f(t) - \sum_{k \in \mathbb{Z}} f(k)\, \mathrm{sinc}(t-k)\, \varphi(t-k)\,, \quad t \in \mathbb{R}\,, \tag{2.3}$$

and the *truncation error*

$$\begin{aligned} e_{2,0}(t) &:= \sum_{k \in \mathbb{Z}} f(k)\, \mathrm{sinc}(t-k)\, \varphi(t-k) - \big(R_{\varphi,m} f\big)(t) \\ &= \sum_{k \in \mathbb{Z} \setminus \{1-m,\ldots,m\}} f(k)\, \mathrm{sinc}(t-k)\, \varphi(t-k)\,, \quad t \in (0,\, 1)\,. \end{aligned} \tag{2.4}$$

(ii) To estimate the regularization error (2.3), we start our study by considering the Fourier transform (1.1) of the function $\varphi\,\mathrm{sinc}$, i. e., the term

$$\mathcal{F}(\varphi\, \mathrm{sinc})(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(t)\, \mathrm{sinc}(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t\,.$$

Using the convolution property of $\mathcal{F}$ in $L^2(\mathbb{R})$ (see [21, Theorem 2.26]), we have

$$\mathcal{F}(\varphi\, \mathrm{sinc})(\omega) = \big(\hat{\varphi} \star (\mathcal{F}\mathrm{sinc})\big)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\varphi}(\omega - \tau)\, (\mathcal{F}\mathrm{sinc})(\tau)\, \mathrm{d}\tau\,,$$

and hence by

$$(\mathcal{F}\mathrm{sinc})(\tau) = \frac{1}{\sqrt{2\pi}}\, \mathbf{1}_{[-\pi,\pi]}(\tau)$$

we obtain

$$\mathcal{F}(\varphi\, \mathrm{sinc})(\omega) = \frac{1}{2\pi} \int_{\omega-\pi}^{\omega+\pi} \hat{\varphi}(\tau)\, \mathrm{d}\tau\,.$$

Consequently, using the shifting property of $\mathcal{F}$, the Fourier transform (1.1) of the shifted function $\varphi(t-k)\,\mathrm{sinc}(t-k)$ with $k \in \mathbb{Z}$ reads as

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(t-k)\,\mathrm{sinc}(t-k)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \mathrm{e}^{-\mathrm{i}\omega k}\,\mathcal{F}(\varphi\,\mathrm{sinc})(\omega) = \frac{1}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega k} \int_{\omega-\pi}^{\omega+\pi} \hat{\varphi}(\tau)\,\mathrm{d}\tau\,.$$

Therefore, the Fourier transform of the regularization error e_1 in (2.3) has the form

$$\hat{e}_1(\omega) = \hat{f}(\omega) - \left(\frac{1}{2\pi} \sum_{k\in\mathbb{Z}} f(k)\,\mathrm{e}^{-\mathrm{i}\omega k}\right) \int_{\omega-\pi}^{\omega+\pi} \hat{\varphi}(\tau)\,\mathrm{d}\tau\,. \tag{2.5}$$

Note that since the set of shifted cardinal sine functions $\mathrm{sinc}(\cdot - k)$ with $k \in \mathbb{Z}$ forms an orthonormal system in $L^2(\mathbb{R})$, i.e.,

$$\int_{\mathbb{R}} \mathrm{sinc}(t-k)\,\mathrm{sinc}(t-\ell)\,\mathrm{d}t = \delta_{k,\ell}\,, \quad k,\,\ell \in \mathbb{Z}\,,$$

and the given function f can be represented by the Shannon sampling series (1.2), we obtain that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |f(k)|^2 &= \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} f(k)\,\overline{f(\ell)} \int_{\mathbb{R}} \mathrm{sinc}(t-k)\,\mathrm{sinc}(t-\ell)\,\mathrm{d}t \\ &= \int_{\mathbb{R}} f(t)\,\overline{f(t)}\,\mathrm{d}t = \|f\|^2_{L^2(\mathbb{R})} < \infty\,, \end{aligned} \tag{2.6}$$

and thus the series

$$\sum_{k\in\mathbb{Z}} f(k)\,\mathrm{e}^{-\mathrm{i}\omega k}$$

converges in $L^2([-\pi,\pi])$. Moreover, since f is bandlimited with bandwidth $\delta \in (0,\,\pi)$, we have $\hat{f}(\omega) = 0$ for all $\omega \in \mathbb{R} \setminus [-\delta,\,\delta]$, and thereby the restricted function $\hat{f}\big|_{[-\pi,\,\pi]}$ belongs to $L^2([-\pi,\pi])$. Hence, this restricted function possesses the 2π-periodic Fourier expansion

$$\hat{f}(\omega) = \sum_{k\in\mathbb{Z}} c_k(\hat{f})\,\mathrm{e}^{-\mathrm{i}\omega k}\,, \quad \omega \in [-\pi,\,\pi]\,,$$

with the Fourier coefficients

$$c_k(\hat{f}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\tau)\,\mathrm{e}^{\mathrm{i}k\tau}\,\mathrm{d}\tau = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\tau)\,\mathrm{e}^{\mathrm{i}k\tau}\,\mathrm{d}\tau = \frac{1}{\sqrt{2\pi}}\,f(k)\,, \quad k \in \mathbb{Z}\,,$$

by inverse Fourier transform. In other words, the function $\hat{f}$ can be represented in the form

$$\hat{f}(\omega) = \hat{f}(\omega)\,\mathbf{1}_{[-\delta,\delta]}(\omega) = \frac{1}{\sqrt{2\pi}} \left(\sum_{k\in\mathbb{Z}} f(k)\,\mathrm{e}^{-\mathrm{i}k\omega}\right) \mathbf{1}_{[-\delta,\delta]}(\omega)\,, \quad \omega \in \mathbb{R}\,. \tag{2.7}$$

Introducing the auxiliary function

$$\Delta_{\varphi}(\omega) := \mathbf{1}_{[-\delta,\delta]}(\omega) - \frac{1}{\sqrt{2\pi}} \int_{\omega-\pi}^{\omega+\pi} \hat{\varphi}(\tau)\,\mathrm{d}\tau\,, \quad \omega \in \mathbb{R}\,,$$

we see by inserting (2.7) into (2.5) that

$$\hat{e}_1(\omega) = \hat{f}(\omega)\, \Delta_\varphi(\omega)\,, \quad \omega \in \mathbb{R}\,,$$

and thereby

$$\big|\hat{e}_1(\omega)\big| = \big|\hat{f}(\omega)\big|\, \big|\Delta_\varphi(\omega)\big|\,, \quad \omega \in \mathbb{R}\,.$$

Thus, inverse Fourier transform and the definition (2.1) yields

$$\begin{aligned} |e_1(t)| &\le \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \big|\hat{e}_1(\omega)\big|\, \mathrm{d}\omega = \frac{1}{\sqrt{2\pi}} \int_{-\delta}^{\delta} \big|\hat{f}(\omega)\big|\, \big|\Delta_\varphi(\omega)\big|\, \mathrm{d}\omega \\ &\le \frac{1}{\sqrt{2\pi}} \max_{\omega \in [-\delta, \delta]} \big|\Delta_\varphi(\omega)\big| \int_{-\delta}^{\delta} \big|\hat{f}(\omega)\big|\, \mathrm{d}\omega = \frac{1}{\sqrt{2\pi}}\, E_1(m) \int_{-\delta}^{\delta} \big|\hat{f}(\omega)\big|\, \mathrm{d}\omega\,. \end{aligned}$$

By the Cauchy–Schwarz inequality and the Parseval equality $\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$ we obtain

$$\int_{-\delta}^{\delta} \big|1 \cdot \hat{f}(\omega)\big|\, \mathrm{d}\omega \le \Big(\int_{-\delta}^{\delta} 1^2\, \mathrm{d}\omega\Big)^{1/2} \Big(\int_{-\delta}^{\delta} \big|\hat{f}(\omega)\big|^2\, \mathrm{d}\omega\Big)^{1/2} = \sqrt{2\delta}\, \|\hat{f}\|_{L^2(\mathbb{R})} \le \sqrt{2\pi}\, \|f\|_{L^2(\mathbb{R})}\,.$$

Consequently, we receive the estimate

$$|e_1(t)| \le E_1(m)\, \|f\|_{L^2(\mathbb{R})}\,, \quad t \in \mathbb{R}\,,$$

and hence

$$\max_{t \in \mathbb{R}} |e_1(t)| \le E_1(m)\, \|f\|_{L^2(\mathbb{R})}\,.$$

(iii) Now we estimate the truncation error $e_{2,0}(t)$ for $t \in (0,\, 1)$. By (2.4) and $\varphi(t) \ge 0$, we obtain

$$|e_{2,0}(t)| \le \sum_{k \in \mathbb{Z} \setminus \{1-m, \ldots, m\}} |f(k)|\, |\mathrm{sinc}(t-k)|\, \varphi(t-k)\,, \quad t \in (0,\, 1)\,.$$

For $t \in (0,\, 1)$ and $k \in \mathbb{Z} \setminus \{1-m, \ldots, m\}$, we estimate

$$|\mathrm{sinc}(t-k)| \le \frac{1}{\pi\, |t-k|} \le \frac{1}{\pi m}\,,$$

such that

$$|e_{2,0}(t)| \le \frac{1}{\pi m} \sum_{k \in \mathbb{Z} \setminus \{1-m, \ldots, m\}} |f(k)|\, \varphi(t-k)\,, \quad t \in (0,\, 1)\,.$$

Then the Cauchy–Schwarz inequality implies

$$|e_{2,0}(t)| \le \frac{1}{\pi m} \Big(\sum_{k \in \mathbb{Z} \setminus \{1-m, \ldots, m\}} |f(k)|^2\Big)^{1/2} \Big(\sum_{k \in \mathbb{Z} \setminus \{1-m, \ldots, m\}} \varphi^2(t-k)\Big)^{1/2}, \quad t \in (0,\, 1)\,.$$

From (2.6) it follows that

$$|e_{2,0}(t)| \le \frac{1}{\pi m}\, \|f\|_{L^2(\mathbb{R})} \Bigg(\sum_{k\in\mathbb{Z}\setminus\{1-m,\ldots,m\}} \varphi^2(t-k) \Bigg)^{1/2}, \quad t \in (0,\,1)\,.$$

Since by assumption the window function φ is even and $\varphi\big|_{[0,\infty)}$ decreases, we can estimate the series

$$\begin{aligned}\sum_{k\in\mathbb{Z}\setminus\{1-m,\ldots,m\}} \varphi^2(t-k) &= \sum_{k=-\infty}^{-m} \varphi^2(t-k) + \sum_{k=m+1}^{\infty} \varphi^2(t-k) \\ &= \sum_{k=m}^{\infty} \varphi^2(t+k) + \sum_{k=m+1}^{\infty} \varphi^2(k-t) \\ &\le \sum_{k=m}^{\infty} \varphi^2(k) + \sum_{k=m+1}^{\infty} \varphi^2(k-1) = 2 \sum_{k=m}^{\infty} \varphi^2(k)\,, \quad t\in(0,\,1)\,.\end{aligned}$$

Applying the integral test for convergence of series, we obtain that

$$2\sum_{k=m}^{\infty} \varphi^2(k) = 2\,\varphi^2(m) + 2 \sum_{k=m+1}^{\infty} \varphi^2(k) < 2\,\varphi^2(m) + 2\int_m^\infty \varphi^2(t)\,\mathrm{d}t\,.$$

Thus, for each $t \in (0,\,1)$ we have by definition (2.2) that

$$|e_{2,0}(t)| \le \frac{\sqrt{2}}{\pi m} \left(\varphi^2(m) + \int_m^\infty \varphi^2(t)\,\mathrm{d}t \right)^{1/2} \|f\|_{L^2(\mathbb{R})} = E_2(m)\,\|f\|_{L^2(\mathbb{R})} < \infty\,.$$

Furthermore, by the interpolation property (1.8) of $R_{\varphi,m}f$ we have $e_{2,0}(0) = e_{2,0}(1) = 0$, such that

$$\max_{t\in[0,1]} |e_{2,0}(t)| \le E_2(m)\,\|f\|_{L^2(\mathbb{R})}\,.$$

(iv) By the same technique, the error estimate

$$\max_{t\in[n,n+1]} \big|f(t) - \big(R_{\varphi,m}f\big)(t)\big| \le \big(E_1(m) + E_2(m)\big)\,\|f\|_{L^2(\mathbb{R})}$$

can be shown for the interval $[n,\,n+1]$ with arbitrary $n \in \mathbb{Z}$. On the open interval $(n,\,n+1)$, we decompose the approximation error as

$$f(t+n) - \big(R_{\varphi,m}f\big)(t)(t+n) = e_1(t+n) + e_{2,n}(t)\,, \quad t\in(0,\,1)\,,$$

with

$$\begin{aligned} e_1(t+n) &= f(t+n) - \sum_{k\in\mathbb{Z}} f(k)\,\mathrm{sinc}\big(t-(k-n)\big)\,\varphi\big(t-(k-n)\big) \\ &= f(t+n) - \sum_{\ell\in\mathbb{Z}} f(\ell+n)\,\mathrm{sinc}(t-\ell)\,\varphi(t-\ell)\,, \\ e_{2,n}(t) &:= \sum_{\ell\in\mathbb{Z}\setminus\{1-m,\ldots,m\}} f(\ell+n)\,\mathrm{sinc}(t-\ell)\,\varphi(t-\ell)\,. \end{aligned}$$

As shown in steps (ii) and (iii), we have

$$\|e_1(\cdot + n)\|_{C_0(\mathbb{R})} = \|e_1\|_{C_0(\mathbb{R})}\,,$$
$$|e_{2,n}(t)| \le E_2(m)\,\|f\|_{L^2(\mathbb{R})}\,, \quad t \in (0,\,1)\,.$$

Furthermore, by the interpolation property (1.8) of $R_{\varphi,m}f$, we have $e_{2,n}(0) = e_{2,n}(1) = 0$ for each $n \in \mathbb{Z}$ and thus

$$\max_{t\in[n,n+1]} \big|e_{2,n}(t)\big| \le E_2(m)\,\|f\|_{L^2(\mathbb{R})}\,.$$

Hence, it follows that

$$\begin{aligned}\max_{t\in[n,n+1]} \big|f(t) - \big(R_{\varphi,m}f\big)(t)\big| &\le \|e_1\|_{C_0(\mathbb{R})} + \max_{t\in[n,n+1]} \big|e_{2,n}(t)\big| \\ &\le \big(E_1(m) + E_2(m)\big)\,\|f\|_{L^2(\mathbb{R})}\,,\end{aligned}$$

which completes the proof. ■

3 Regularization with the Gaussian function

In this section we consider the Gaussian function (1.5) with variance $\sigma^2 > 0$, analogous to [13, Theorem 4.1]. In order to achieve fast convergence of the Gaussian regularized Shannon sampling formula, we also study the choice of this variance σ^2.

Theorem 3.1. *Assume that $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ is bandlimited with bandwidth $\delta \in (0,\,\pi)$. Further let φ_{Gauss} be the Gaussian function (1.5) with variance $\sigma^2 = \frac{m}{\pi-\delta}$ and let $m \in \mathbb{N}\setminus\{1\}$ be given.*
Then the Gaussian regularized Shannon sampling formula satisfies the error estimate

$$\big\|f - R_{\mathrm{Gauss},m}f\big\|_{C_0(\mathbb{R})} \le \frac{2\sqrt{2}}{\sqrt{\pi m\,(\pi-\delta)}}\,\mathrm{e}^{-m\,(\pi-\delta)/2}\,\|f\|_{L^2(\mathbb{R})}\,. \tag{3.1}$$

Proof. (i) At first, we estimate the regularization error constant (2.1) for the Gaussian function (1.5). Since the Fourier transform of φ_{Gauss} reads as

$$\hat{\varphi}_{\mathrm{Gauss}}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi_{\mathrm{Gauss}}(t)\,\mathrm{e}^{-\mathrm{i}\,t\omega}\,\mathrm{d}t = \sigma\,\mathrm{e}^{-\omega^2\sigma^2/2}\,, \quad \omega \in \mathbb{R}\,,$$

cf. [21, Example 2.6], we have

$$E_1(m) = \max_{\omega\in[-\delta,\delta]} \left|1 - \frac{\sigma}{\sqrt{2\pi}} \int_{\omega-\pi}^{\omega+\pi} \mathrm{e}^{-\tau^2\sigma^2/2}\,\mathrm{d}\tau\right|.$$

Substituting $s = \tau\sigma/\sqrt{2}$ and using the integral $\int_{\mathbb{R}} \mathrm{e}^{-s^2}\,\mathrm{d}s = \sqrt{\pi}$, we obtain for $\omega \in [-\delta,\,\delta]$ with $\delta \in (0,\,\pi)$ that

$$\Delta_{\mathrm{Gauss}}(\omega) := 1 - \frac{1}{\sqrt{\pi}} \int_{(\omega-\pi)\sigma/\sqrt{2}}^{(\omega+\pi)\sigma/\sqrt{2}} \mathrm{e}^{-s^2}\,\mathrm{d}s$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{\pi}} \left(\int_{\mathbb{R}} \mathrm{e}^{-s^2}\,\mathrm{d}s - \int_{(\omega-\pi)\sigma/\sqrt{2}}^{(\omega+\pi)\sigma/\sqrt{2}} \mathrm{e}^{-s^2}\,\mathrm{d}s \right) \\
&= \frac{1}{\sqrt{\pi}} \left(\int_{-\infty}^{(\omega-\pi)\sigma/\sqrt{2}} \mathrm{e}^{-s^2}\,\mathrm{d}s + \int_{(\omega+\pi)\sigma/\sqrt{2}}^{\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s \right) \\
&= \frac{1}{\sqrt{\pi}} \left(\int_{(\pi-\omega)\sigma/\sqrt{2}}^{\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s + \int_{(\omega+\pi)\sigma/\sqrt{2}}^{\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s \right).
\end{aligned}
$$

Since Δ_{Gauss} is even, we consider only the case $\omega \in [0,\, \delta]$. Applying the inequality

$$
\int_a^{\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s = \int_0^{\infty} \mathrm{e}^{-(t+a)^2}\,\mathrm{d}t \le \mathrm{e}^{-a^2} \int_0^{\infty} \mathrm{e}^{-2at}\,\mathrm{d}t = \frac{1}{2a}\,\mathrm{e}^{-a^2}\,, \quad a > 0\,,
$$

we obtain

$$
0 \le \Delta_{\mathrm{Gauss}}(\omega) \le \frac{1}{\sqrt{2\pi}} \left(\frac{\mathrm{e}^{-(\pi-\omega)^2\sigma^2/2}}{(\pi-\omega)\,\sigma} + \frac{\mathrm{e}^{-(\pi+\omega)^2\sigma^2/2}}{(\pi+\omega)\,\sigma} \right) \le \sqrt{\frac{2}{\pi}}\, \frac{\mathrm{e}^{-(\pi-\omega)^2\sigma^2/2}}{(\pi-\omega)\,\sigma}\,, \quad \omega \in [0,\, \delta]\,.
$$

Consequently, we have for all $\omega \in [-\delta,\, \delta]$ that

$$
0 \le \Delta_{\mathrm{Gauss}}(\omega) \le \sqrt{\frac{2}{\pi}}\, \frac{\mathrm{e}^{-(\pi-|\omega|)^2\sigma^2/2}}{(\pi-|\omega|)\,\sigma}
$$

and hence

$$
E_1(m) \le \sqrt{\frac{2}{\pi}}\, \frac{\mathrm{e}^{-(\pi-\delta)^2\sigma^2/2}}{(\pi-\delta)\,\sigma}\,. \tag{3.2}
$$

(ii) Now we examine the truncation error constant (2.2) for the Gaussian function (1.5). By $\varphi_{\mathrm{Gauss}}^2(m) = \mathrm{e}^{-m^2/\sigma^2}$ and the inequality

$$
\int_m^{\infty} \varphi_{\mathrm{Gauss}}^2(t)\,\mathrm{d}t = \sigma \int_{m/\sigma}^{\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s \le \frac{\sigma^2}{2m}\,\mathrm{e}^{-m^2/\sigma^2}
$$

we obtain

$$
E_2(m) \le \frac{\sqrt{2}}{\pi m} \sqrt{\mathrm{e}^{-m^2/\sigma^2} + \frac{\sigma^2}{2m}\,\mathrm{e}^{-m^2/\sigma^2}} = \frac{\sqrt{2}}{\pi m} \sqrt{1 + \frac{\sigma^2}{2m}}\,\mathrm{e}^{-m^2/(2\sigma^2)}\,. \tag{3.3}
$$

(iii) Finally, we choose the variance σ^2 of the Gaussian function (1.5) such that $E_1(m)$ and $E_2(m)$ possess the same exponential decay with respect to m. From (3.2) and (3.3) it follows that

$$
\sigma^2 := \frac{m}{\pi-\delta}\,. \tag{3.4}
$$

This yields the estimates

$$
\begin{aligned}
E_1(m) &\le \sqrt{\frac{2}{\pi}}\, \frac{1}{\sqrt{m\,(\pi-\delta)}}\,\mathrm{e}^{-m\,(\pi-\delta)/2}\,, \\
E_2(m) &\le \frac{\sqrt{2}}{\pi m} \sqrt{1 + \frac{1}{2\,(\pi-\delta)}}\,\mathrm{e}^{-m\,(\pi-\delta)/2}\,.
\end{aligned}
$$

Note that since $m \in \mathbb{N} \setminus \{1\}$ and $\delta \in (0,\, \pi)$, we have

$$\left(\frac{\sqrt{2}}{\sqrt{\pi m\,(\pi-\delta)}}\right)^{-1} \cdot \frac{\sqrt{2}}{\pi m}\,\sqrt{1+\frac{1}{2\,(\pi-\delta)}} = \sqrt{\frac{2(\pi-\delta)+1}{2\pi m}} \le \sqrt{\frac{2\pi+1}{4\pi}} < 1$$

and therefore

$$E_2(m) \le \frac{\sqrt{2}}{\sqrt{\pi m\,(\pi-\delta)}}\,\mathrm{e}^{-m\,(\pi-\delta)/2}\,.$$

Thus, the Gaussian regularized Shannon sampling formula with the variance (3.4) fulfills the error estimate (3.1). This completes the proof. ■

Note that already in [13, Theorem 4.1] bounds on the approximation error of the Shannon sampling formula (1.4) were shown for the Gaussian function (1.5) with suitably chosen variance σ^2, which is basically the same as the one in Theorem 3.1, only looking slightly different due to the different setting considered in [13].

Remark 3.2. Inspired by [6] one could define a weak form of optimality of the Gaussian regularized Shannon sampling formula by saying that the variance σ^2 of the Gaussian function (1.5) is *optimal*, if $E_1(m)$ and $E_2(m)$ possess the same exponential decay with respect to m. Hence, Theorem 3.1 shows that the choice (3.4) is optimal for the Shannon sampling formula (1.4) with the Gaussian function (1.5) in this weak sense. We remark that in [6] a slightly different optimal variance $\sigma^2 = \frac{m-1}{\pi-\delta}$ is presented for the Gaussian regularizer (1.5), while also considering a slightly different truncation than in (1.4). Nevertheless, both results, Theorem 3.1 and [6, Theorem 1.1], possess the same asymptotic behavior.

Additionally, it should be noted that in [6] the approximation error is estimated only up to an unknown constant, while our error estimate of the Gaussian regularized Shannon sampling formula contains relatively small explicit constants, which is more favorable for practical applications. Moreover, we estimate the approximation error differently by splitting it into the regularization error (2.3) and the truncation error (2.4), which seems more intuitive than the rather artificial analysis presented in [6, Theorem 1.1]. □

This definition of weak optimality has led to the following open question of optimality of the variance (3.4), which could so far only be observed numerically.

Conjecture 3.3. The parameter (3.4) is the optimal variance for the Shannon sampling formula (1.4) with the Gaussian function (1.5) not only in the weak sense of Remark 3.2, but also guarantees the maximum decay rate of the uniform approximation error (1.9).

Example 3.4. In order to present numerical evidence for the optimality of the variance (3.4) of the Gaussian regularized Shannon sampling formula stated in Conjecture 3.3 we consider the regularized Shannon sampling formula (1.4) with the Gaussian function φ_{Gauss} in (1.5) for a given bandlimited function $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ with bandwidth $\delta \in (0,\, \pi]$ and estimate the corresponding approximation error

$$\max_{t\in[-1,\,1]} \big|f(t) - \big(R_{\varphi,m}f\big)(t)\big|\,, \tag{3.5}$$

cf. (1.9), numerically. The error (3.5) shall here be approximated by evaluating a given function f and its approximation $R_{\varphi,m}f$ at equidistant points $t_s \in [-1,\, 1]$, $s = 1, \dots, S$, with $S = 10^5$. Note that by the definition of the regularized Shannon sampling formula (1.4) we have

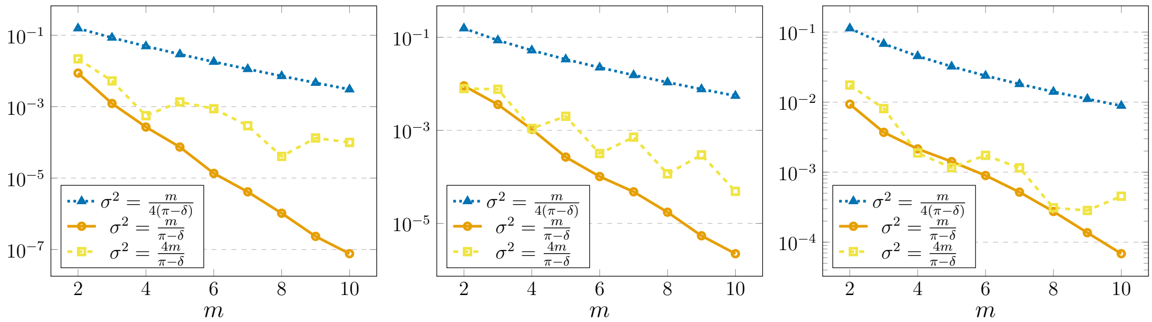
$$\big(R_{\varphi,m}f\big)(t) = \sum_{k=-m-1}^{m+1} f(k)\,\mathrm{sinc}(t-k)\,\varphi(t-k)\,, \quad t \in [-1,\,1]\,.$$

Analogous to [19, Section IV, C] we study the bandlimited function

$$f(t) = \frac{2\delta}{\sqrt{5\pi\delta + 4\pi\sin\delta}}\left[\mathrm{sinc}\left(\frac{\delta t}{\pi}\right) + \frac{1}{2}\,\mathrm{sinc}\left(\frac{\delta(t-1)}{\pi}\right)\right]\,, \quad t \in \mathbb{R}\,, \tag{3.6}$$

with $\|f\|_{L^2(\mathbb{R})} = 1$, for several bandwidth parameters $\delta \in \left\{\frac{\pi}{4},\, \frac{\pi}{2},\, \frac{3\pi}{4}\right\}$, i. e., several oversampling rates $\frac{\pi}{\delta} > 1$. To compare with the variance σ^2 stated in (3.4), we choose the parameter of the Gaussian function (1.5) as $\sigma = \alpha \cdot \frac{m}{\pi-\delta}$ with $\alpha \in \left\{\frac{1}{2}, 1, 2\right\}$.

The corresponding results for different truncation parameters $m \in \{2, 3, \ldots, 10\}$ are displayed in Figure 3.1. It can clearly be seen that both, an increase and a decrease of the variance in (3.4), cause worsened error decay rates with respect to m. Thus, the numerical results give reason to believe that the variance (3.4) of Theorem 3.1 is indeed optimal in terms of the uniform approximation error (1.9), already for very small truncation parameters $m \in \mathbb{N} \setminus \{1\}$. □



(a) $\delta = \frac{\pi}{4}$ (b) $\delta = \frac{\pi}{2}$ (c) $\delta = \frac{3\pi}{4}$

Figure 3.1: Maximum approximation error (3.5) using the Gaussian function φ_{Gauss} in (1.5) with different variances $\sigma^2 \in \left\{\frac{m}{4(\pi-\delta)}, \frac{m}{\pi-\delta}, \frac{4m}{\pi-\delta}\right\}$, for the bandlimited function (3.6) with bandwidths $\delta \in \left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$ and truncation parameters $m \in \{2, 3, \ldots, 10\}$.

Remark 3.5. Note that [27, 26] suggested the *modified Gaussian function*

$$\varphi_{\mathrm{modGauss}}(t) \coloneqq \mathrm{e}^{-t^2/(2\sigma^2)}\,\cos(\lambda t)\,, \quad t \in \mathbb{R}\,, \tag{3.7}$$

with the parameters $\sigma^2 > 0$ and $\lambda \geq 0$ as an improvement to the Gaussian function (1.5). By the same techniques as in Theorem 3.1, however, one can determine that the optimal variance of (3.7) in the weak sense of Remark 3.2 is given by $\sigma^2 = \frac{m}{\pi-\lambda-\delta}$, $0 \leq \lambda < \pi - \delta$, with the corresponding error estimate

$$\big\|f - R_{\mathrm{modGauss},m}f\big\|_{C_0(\mathbb{R})} \leq \frac{2\sqrt{2}}{\sqrt{\pi m\,(\pi-\lambda-\delta)}}\,\mathrm{e}^{-m\,(\pi-\lambda-\delta)/2}\,\|f\|_{L^2(\mathbb{R})}\,.$$

This shows that the approximation error of the regularized Shannon sampling formula with the modified Gaussian function (3.7) has the best exponential decay in the case $\lambda = 0$, therefore proving that the Gaussian function φ_{Gauss} in (1.5) is much more favorable than the modified Gaussian function $\varphi_{\mathrm{modGauss}}$ in (3.7). □

4 Regularization with the sinh-type window function

In this section, we consider the sinh-type window function (1.6) with shape parameter $\beta > 0$, analogous to [13, Theorem 6.1] and [14, Theorem 4.2]. We especially focus on addressing minor gaps in [13, 14] by rigorously proving assumptions up to now based solely on numerical tests. Moreover, we demonstrate that the exponential decay with respect to the truncation parameter $m \in \mathbb{N} \setminus \{1\}$ is twice as fast for the uniform approximation error $\big\|f - R_{\mathrm{sinh},m} f\big\|_{C_0(\mathbb{R})}$ as for the approximation error $\big\|f - R_{\mathrm{Gauss},m} f\big\|_{C_0(\mathbb{R})}$ in Theorem 3.1. To this end, we firstly formulate the following lemma.

Lemma 4.1. *For all $W > 1$ and $\beta > 0$ we have*

$$\Big| \int_1^W \frac{J_1(\beta\,\sqrt{\nu^2-1})}{\sqrt{\nu^2-1}}\,\mathrm{d}\nu \Big| \le \frac{1-\mathrm{e}^{-\beta}}{\beta} + \frac{\sqrt{2}\,\pi}{\sqrt{\beta}}\,, \tag{4.1}$$

where J_1 denotes the Bessel function of first order.

Proof. Substituting $\nu = \cosh t$ in (4.1), we obtain

$$\int_1^W \frac{J_1(\beta\,\sqrt{\nu^2-1})}{\sqrt{\nu^2-1}}\,\mathrm{d}\nu = \int_0^w J_1(\beta\,\sinh t)\,\mathrm{d}t$$

with $w = \mathrm{arcosh} W > 0$. Note that it is known by [11, 6.645–1] that

$$\int_0^\infty J_1(\beta\,\sinh t)\,\mathrm{d}t = I_{1/2}\Big(\frac{\beta}{2}\Big)\,K_{1/2}\Big(\frac{\beta}{2}\Big) = \frac{1-\mathrm{e}^{-\beta}}{\beta} > 0\,,$$

where $I_{1/2}$ and $K_{1/2}$ denote the modified Bessel functions of half order (see [1, 10.2.13, 10.2.14, and 10.2.17]. The additional substitution $s = \beta\,\sinh t$ yields

$$\begin{aligned} \int_0^\infty J_1(\beta\,\sinh t)\,\mathrm{d}t &= \int_0^\infty \frac{J_1(s)}{\sqrt{s^2+\beta^2}}\,\mathrm{d}s = \frac{1-\mathrm{e}^{-\beta}}{\beta}\,, \\ \int_0^w J_1(\beta\,\sinh t)\,\mathrm{d}t &= \int_0^u \frac{J_1(s)}{\sqrt{s^2+\beta^2}}\,\mathrm{d}s\,, \end{aligned} \tag{4.2}$$

where $u = \beta\,\sinh w > 0$.

Let j_k, $k \in \mathbb{N}$, denote the positive zeros of J_1. Note that j_k, $k = 1, \ldots, 40$, are tabulated in [33, p. 748] and that by $(-1)^k\,J_1'(j_k) > 0$, see [1, 9.1.27 and 9.5.2], these zeros are simple. Using these zeros the integral (4.2) can be represented as

$$\frac{1-\mathrm{e}^{-\beta}}{\beta} = \int_0^\infty \frac{J_1(s)}{\sqrt{s^2+\beta^2}}\,\mathrm{d}s = \Big(\int_0^{j_1} + \int_{j_1}^{j_2} + \int_{j_2}^{j_3} + \ldots\Big)\frac{J_1(s)}{\sqrt{s^2+\beta^2}}\,\mathrm{d}s\,. \tag{4.3}$$

The integrand $J_1(s)\,(s^2+\beta^2)^{-1/2}$, $s \in [0,\infty)$, is an oscillating function which tends to zero for $s \to \infty$, since we have

$$\frac{|J_1(s)|}{\sqrt{s^2+\beta^2}} \le \frac{1}{\sqrt{s}\,\sqrt{s^2+\beta^2}}\,, \quad s \in [j_1,\,\infty)\,, \tag{4.4}$$

which can be shown by the same technique as in [23, Lemma 6]. Thus, the right-hand side of (4.3) is a convergent alternating series.

Now we consider the function

$$\mathcal{I}(u) \coloneqq \int_0^u \frac{J_1(s)}{\sqrt{s^2+\beta^2}}\,\mathrm{d}s\,, \quad u \in [0,\,\infty)\,. \tag{4.5}$$

We immediately recognize that $\mathcal{I}(u)$ has relative maxima at j_{2n+1}, $n \in \mathbb{N}$, and relative minima at j_{2n}, $n \in \mathbb{N}$, since we have

$$\mathcal{I}'(j_k) = \frac{J_1(j_k)}{\sqrt{j_k^2+\beta^2}} = 0\,, \quad (-1)^k\,\mathcal{I}''(j_k) = \frac{(-1)^k\,J_1'(j_k)}{\sqrt{j_k^2+\beta^2}} > 0\,, \quad k \in \mathbb{N}\,.$$

In other words, by the oscillatory behavior of the Bessel function J_1, the function $\mathcal{I}(u)$ increases from $\mathcal{I}(0)=0$ to $\mathcal{I}(j_1)$, then decreases from $\mathcal{I}(j_1)$ to $\mathcal{I}(j_2)$, increases again from $\mathcal{I}(j_2)$ to $\mathcal{I}(j_3)$, and so on. By (4.2) and (4.5) it is also easy to see that

$$\frac{1-\mathrm{e}^{-\beta}}{\beta} - \mathcal{I}(j_k) = \left(\int_0^\infty - \int_0^{j_k}\right)\frac{J_1(s)}{\sqrt{s^2+\beta^2}}\,\mathrm{d}s = \int_{j_k}^\infty \frac{J_1(s)}{\sqrt{s^2+\beta^2}}\,\mathrm{d}s\,, \quad k \in \mathbb{N}\,.$$

Thus, combined with the estimate (4.4) this yields

$$\left|\frac{1-\mathrm{e}^{-\beta}}{\beta} - \mathcal{I}(j_k)\right| \le \int_{j_k}^\infty \frac{|J_1(s)|}{\sqrt{s^2+\beta^2}}\,\mathrm{d}s \le \int_{j_k}^\infty \frac{1}{\sqrt{s}\,\sqrt{s^2+\beta^2}}\,\mathrm{d}s\,, \quad k \in \mathbb{N}\,.$$

In addition, from the equivalence relation $\|\boldsymbol{v}\|_1 \le \sqrt{n}\,\|\boldsymbol{v}\|_2$, $\boldsymbol{v} \in \mathbb{R}^n$, of the vector norms it follows that $s+\beta \le \sqrt{2}\,\sqrt{s^2+\beta^2}$, $s,\beta>0$, and therefore

$$\frac{1}{\sqrt{s^2+\beta^2}} \le \frac{\sqrt{2}}{s+\beta}\,, \quad s,\beta > 0\,.$$

Hence, we obtain

$$\left|\frac{1-\mathrm{e}^{-\beta}}{\beta} - \mathcal{I}(j_k)\right| \le \int_{j_k}^\infty \frac{\sqrt{2}}{\sqrt{s}\,(s+\beta)}\,\mathrm{d}s\,, \quad k \in \mathbb{N}\,.$$

Since the antiderivative of $s^{-1/2}\,(s+\beta)^{-1}$ reads as $\frac{2}{\sqrt{\beta}}\,\arctan\sqrt{\frac{s}{\beta}}$ and $\arctan y \le \frac{\pi}{2}$, $y \in \mathbb{R}$, we obtain the estimate

$$\left|\frac{1-\mathrm{e}^{-\beta}}{\beta} - \mathcal{I}(j_k)\right| \le \frac{2\sqrt{2}}{\sqrt{\beta}}\left(\frac{\pi}{2} - \arctan\sqrt{\frac{j_k}{\beta}}\right) = \frac{2\sqrt{2}}{\sqrt{\beta}}\,\arctan\sqrt{\frac{\beta}{j_k}} \le \frac{\sqrt{2}\,\pi}{\sqrt{\beta}}\,.$$

Consequently, as this estimate is valid for all relative extreme values $\mathcal{I}(j_k)$, $k \in \mathbb{N}$, we also obtain

$$\left|\frac{1-\mathrm{e}^{-\beta}}{\beta} - \mathcal{I}(u)\right| \le \frac{\sqrt{2}\,\pi}{\sqrt{\beta}}$$

for all $u > 0$, which immediately implies

$$\frac{1-\mathrm{e}^{-\beta}}{\beta} - \frac{\sqrt{2}\,\pi}{\sqrt{\beta}} \le \mathcal{I}(u) \le \frac{1-\mathrm{e}^{-\beta}}{\beta} + \frac{\sqrt{2}\,\pi}{\sqrt{\beta}}\,.$$

Since $\frac{1-\mathrm{e}^{-\beta}}{\beta} > 0$ and $\frac{\sqrt{2}\,\pi}{\sqrt{\beta}} > 0$ for $\beta > 0$ this yields

$$-\frac{1-\mathrm{e}^{-\beta}}{\beta} - \frac{\sqrt{2}\,\pi}{\sqrt{\beta}} \le \frac{1-\mathrm{e}^{-\beta}}{\beta} - \frac{\sqrt{2}\,\pi}{\sqrt{\beta}} \le \mathcal{I}(u) \le \frac{1-\mathrm{e}^{-\beta}}{\beta} + \frac{\sqrt{2}\,\pi}{\sqrt{\beta}}\,,$$

and thereby the assertion (4.1). This completes the proof. ■

Theorem 4.2. *Let $m \in \mathbb{N} \setminus \{1\}$ be given. Assume that $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ is bandlimited with bandwidth $\delta \in \big(0,\, \frac{m-1}{m}\,\pi\big]$. Further let φ_{sinh} be the* sinh*-type window function* (1.6) *with shape parameter $\beta = m\,(\pi - \delta)$.*
Then the sinh*-type regularized Shannon sampling formula satisfies the error estimate*

$$\big\|f - R_{\mathrm{sinh},m} f\big\|_{C_0(\mathbb{R})} \le \big(4 + 9\,\sqrt{m\,(\pi-\delta)}\,\big)\,\mathrm{e}^{-m\,(\pi-\delta)}\,\|f\|_{L^2(\mathbb{R})}\,. \tag{4.6}$$

Proof. (i) Since φ_{sinh} in (1.6) is compactly supported on $[-m,\, m]$ and $\varphi_{\mathrm{sinh}}(m) = 0$, we have $E_2(m) = 0$. Thus, according to Theorem 2.1, the approximation error can be estimated by

$$\big\|f - R_{\mathrm{sinh},m} f\big\|_{C_0(\mathbb{R})} \le \|f\|_{L^2(\mathbb{R})} \max_{\omega\in[-\delta,\delta]} \big|\Delta_{\mathrm{sinh}}(\omega)\big|\,,$$

where

$$\Delta_{\mathrm{sinh}}(\omega) := 1 - \frac{1}{\sqrt{2\pi}} \int_{\omega-\pi}^{\omega+\pi} \hat{\varphi}_{\mathrm{sinh}}(\tau)\,\mathrm{d}\tau\,, \quad \omega \in [-\delta,\, \delta]\,. \tag{4.7}$$

Following [19, p. 38, 7.58], the Fourier transform of (1.6) has the form

$$\hat{\varphi}_{\mathrm{sinh}}(\tau) = \frac{m\,\sqrt{\pi}}{\sqrt{2}\,\sinh\beta} \cdot \begin{cases} (1-\nu^2)^{-1/2}\, I_1\big(\beta\,\sqrt{1-\nu^2}\big) & : |\nu| < 1\,, \\ (\nu^2-1)^{-1/2}\, J_1\big(\beta\,\sqrt{\nu^2-1}\big) & : |\nu| > 1\,, \end{cases} \tag{4.8}$$

with the scaled frequency $\nu = \frac{m}{\beta}\,\tau$, where J_1 denotes the Bessel function and I_1 the modified Bessel function of first order. Substituting $\tau = \frac{\beta}{m}\,\nu$ in the integral in (4.7), the function Δ_{sinh} reads as

$$\Delta_{\mathrm{sinh}}(\omega) := 1 - \frac{\beta}{\sqrt{2\pi}\,m} \int_{-\nu_1(-\omega)}^{\nu_1(\omega)} \hat{\varphi}_{\mathrm{sinh}}\big(\tfrac{\beta}{m}\,\nu\big)\,\mathrm{d}\nu\,, \quad \omega \in [-\delta,\, \delta]\,, \tag{4.9}$$

with the increasing linear function

$$\nu_1(\omega) := \frac{m}{\beta}\,(\omega + \pi)\,, \quad \omega \in [-\delta,\, \delta]\,. \tag{4.10}$$

(ii) Now we choose the shape parameter of (1.6) in the special form $\beta = m\,(\pi - \delta)$. Thus, we have

$$1 = \nu_1(-\delta) \le \nu_1(\omega) = \frac{\omega + \pi}{\pi - \delta} \le \nu_1(\delta) = \frac{\pi + \delta}{\pi - \delta}\,, \quad \omega \in [-\delta,\, \delta]\,.$$

In view of (4.8) we split (4.9) in the form $\Delta_{\sinh}(\omega) = \Delta_{\sinh,1} - \Delta_{\sinh,2}(\omega)$ with

$$\Delta_{\sinh,1} := 1 - \frac{\beta}{\sinh\beta} \int_0^1 \frac{I_1\big(\beta\,\sqrt{1-\nu^2}\big)}{\sqrt{1-\nu^2}}\,\mathrm{d}\nu\,,$$

$$\Delta_{\sinh,2}(\omega) := \frac{\beta}{2\,\sinh\beta} \left(\int_1^{\nu_1(-\omega)} + \int_1^{\nu_1(\omega)} \right) \frac{J_1\big(\beta\,\sqrt{\nu^2-1}\big)}{\sqrt{\nu^2-1}}\,\mathrm{d}\nu\,.$$

Using [11, 6.681–11] and [1, 10.2.13], we get

$$\int_0^1 \frac{I_1\big(\beta\,\sqrt{1-\nu^2}\big)}{\sqrt{1-\nu^2}}\,\mathrm{d}\nu = \int_0^{\pi/2} I_1(\beta\,\cos\sigma)\,\mathrm{d}\sigma = \frac{\pi}{2}\left(I_{1/2}\Big(\frac{\beta}{2}\Big)\right)^2 = \frac{2}{\beta}\left(\sinh\frac{\beta}{2}\right)^2$$

and hence

$$0 < \Delta_{\sinh,1} = 1 - \frac{2\,\big(\sinh\frac{\beta}{2}\big)^2}{\sinh\beta} = \frac{2\,\mathrm{e}^{-\beta}}{1+\mathrm{e}^{-\beta}} < 2\mathrm{e}^{-\beta}\,. \tag{4.11}$$

By Lemma 4.1 we have

$$\big|\Delta_{\sinh,2}(\omega)\big| \le \frac{\beta}{\sinh\beta}\left(\frac{1-\mathrm{e}^{-\beta}}{\beta} + \frac{\sqrt{2}\,\pi}{\sqrt{\beta}}\right) < \left(2 + \sqrt{\beta}\,\frac{2\,\sqrt{2}\,\pi}{1-\mathrm{e}^{-2\pi}}\right)\mathrm{e}^{-\beta}\,, \quad \omega \in [-\delta,\, \delta]\,, \tag{4.12}$$

since by assumption the parameters $m \in \mathbb{N} \setminus \{1\}$ and $\delta \in \big(0,\, \frac{m-1}{m}\,\pi\big]$ are chosen such that $\beta = m\,(\pi - \delta) \ge \pi$. Additionally, note that

$$\frac{2\,\sqrt{2}\,\pi}{1-\mathrm{e}^{-2\pi}} = 8.902390\ldots < 9$$

holds. Thereby, the terms (4.11) and (4.12) have the same exponential decay $m\,(\pi - \delta)$ and (4.9) can be estimated by

$$\big|\Delta_{\sinh}(\omega)\big| = \Delta_{\sinh,1} + \big|\Delta_{\sinh,2}(\omega)\big| \le \big(4 + 9\,\sqrt{\beta}\,\big)\,\mathrm{e}^{-\beta}\,, \quad \omega \in [-\delta,\, \delta]\,.$$

Thus, the sinh-type regularized Shannon sampling formula with the chosen shape parameter $\beta = m\,(\pi - \delta)$ fulfills the error estimate (4.6). This completes the proof. ■

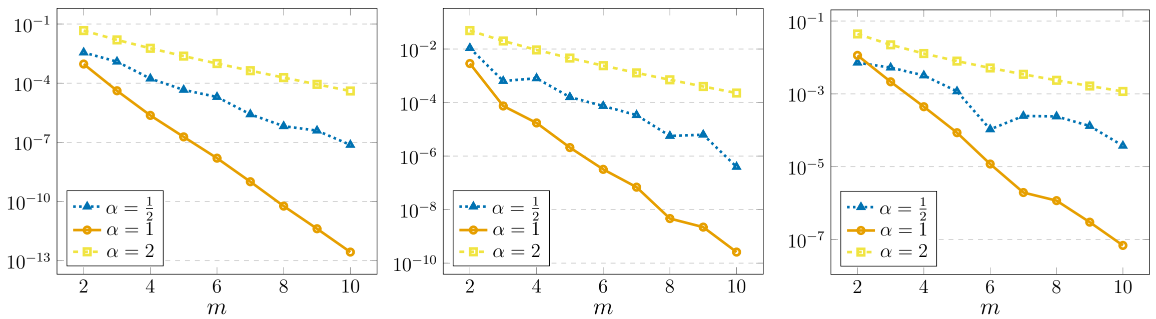
Note that already in [13, Theorem 6.1] and [14, Theorem 4.2] bounds on the approximation error of the Shannon sampling formula (1.4) were shown for the sinh-type window function (1.6) with suitably chosen shape parameter β. Although the respective parameters β look different than the one in Theorem 4.2, they are basically the same, only adapted to the slightly different settings considered in [13, 14]. However, it has to be pointed out that the error constant in Theorem 4.2 is somewhat worsened in comparison to our previous findings in [13, 14] due to the fact that Lemma 4.1 comprises a weaker version of the numerical assumption in [13, p. 25], but nevertheless closes the gap in this previous proof.

In addition, similar to Section 3, the optimality of the shape parameter $\beta = m\,(\pi - \delta)$ for the sinh-type window function (1.6) is still an open problem, which could so far only be observed numerically.

Conjecture 4.3. The parameter $\beta = m\,(\pi - \delta)$ is optimal for the Shannon sampling formula (1.4) with the sinh-type window function (1.6), as it guarantees the maximum decay rate of the uniform approximation error (1.9).

Example 4.4. Analogously as in Example 3.4 we now show numerical evidence for the optimality of the shape parameter $\beta = m\,(\pi - \delta)$ of the sinh-type regularized Shannon sampling formula stated in Conjecture 4.3. More precisely, for the bandlimited function (3.6) with several bandwidth parameters $\delta \in \left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$, i. e., several oversampling rates $\frac{\pi}{\delta} > 1$, we consider the regularized Shannon sampling formula (1.4) with the sinh-type window function φ_{sinh} in (1.6). The corresponding approximation error (3.5) shall again be approximated by evaluating the given function f and its approximation $R_{\varphi,m}f$ at equidistant points $t_s \in [-1,\,1]$, $s = 1, \ldots, S$, with $S = 10^5$. To compare with the parameter in Theorem 4.2, we choose the shape parameter of the sinh-type window function (1.6) as $\beta = \alpha\, m\,(\pi - \delta)$ with $\alpha \in \left\{\frac{1}{2}, 1, 2\right\}$.

The outcomes for different truncation parameters $m \in \{2, 3, \ldots, 10\}$ are depicted in Figure 4.1. As supposed it can clearly be seen that the choice of $\alpha \neq 1$ causes worsened error decay rates with respect to m. Thus, these numerical results bolster the assertion that the shape parameter $\beta = m\,(\pi - \delta)$ of Theorem 4.2 is indeed optimal in terms of the uniform approximation error (1.9), already for very small truncation parameters $m \in \mathbb{N} \setminus \{1\}$. □



(a) $\delta = \frac{\pi}{4}$ (b) $\delta = \frac{\pi}{2}$ (c) $\delta = \frac{3\pi}{4}$

Figure 4.1: Maximum approximation error (3.5) using the sinh-type window function φ_{sinh} in (1.6) with different shape parameters $\beta = \alpha\, m(\pi - \delta)$, $\alpha \in \left\{\frac{1}{2}, 1, 2\right\}$, for the bandlimited function (3.6) with bandwidths $\delta \in \left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$ and truncation parameters $m \in \{2, 3, \ldots, 10\}$.

5 Regularization with the continuous Kaiser–Bessel window function

In this section, we consider the continuous Kaiser–Bessel window function (1.7) with shape parameter $\beta > 0$, analogous to [14, Theorem 4.3]. Once more, we put special emphasis on addressing minor gaps in [14] by rigorously proving assumptions that were previously based solely on numerical tests. Furthermore, we show that the exponential decay with respect to the truncation parameter $m \in \mathbb{N} \setminus \{1\}$ for the uniform approximation error $\|f - R_{\mathrm{cKB},m} f\|_{C_0(\mathbb{R})}$ is the same as for the approximation error $\|f - R_{\sinh,m} f\|_{C_0(\mathbb{R})}$ in Theorem 4.2, only with a slightly worse error constant. To do so, we firstly establish the following lemmas.

Lemma 5.1. *For all $\beta \ge \pi$ we have*

$$\left| I_0(\beta) - \boldsymbol{L}_0(\beta) - 1 + \frac{2}{\pi}\,\mathrm{Si}(\beta) \right| \le \frac{1}{2}\,, \tag{5.1}$$

where $\boldsymbol{L}_0$ denotes the modified Struve function

$$\boldsymbol{L}_0(x) := \sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{\big(\Gamma\big(k+\frac{3}{2}\big)\big)^2} = \frac{2x}{\pi} \sum_{k=0}^{\infty} \frac{x^{2k}}{\big((2k+1)!!\big)^2}\,, \quad x \in \mathbb{R}\,, \tag{5.2}$$

see [1, 12.2.1], and Si *is the* sine integral function

$$\mathrm{Si}(x) := \int_0^x \frac{\sin v}{v}\,\mathrm{d}v\,, \quad x \in \mathbb{R}\,. \tag{5.3}$$

Proof. By [3, Theorem 1] the function $h(x) := I_0(x) - \boldsymbol{L}_0(x)$ is completely monotonic on $[0, \infty)$, i. e., it satisfies $h(x) \ge 0$ and $h'(x) \le 0$ for all $x \in [0, \infty)$. Thereby, we have

$$0 \le h(\beta) \le h(\pi)\,, \quad \beta \ge \pi\,. \tag{5.4}$$

Due to the fact that

$$\mathrm{Si}'(x) = \mathrm{sinc}\big(\tfrac{x}{\pi}\big) \quad \text{and} \quad \mathrm{Si}''(x) = \begin{cases} \frac{x\cos x - \sin x}{x^2} & : x \ne 0\,, \\ 0 & : x = 0\,, \end{cases}$$

the sine integral function (5.3) has its local maxima at $(2k+1)\pi$, $k \in \mathbb{Z} \setminus \{0\}$, and its local minima at $2k\pi$, $k \in \mathbb{Z} \setminus \{0\}$. Moreover, by the definition (5.3) the extremal points of the sine integral function become smaller in magnitude for $x \ge \pi$ when $x \to \infty$, and $\lim_{x\to\infty} \mathrm{Si}(x) = \frac{\pi}{2}$. Thus, for all $\beta \ge \pi$ we have

$$\mathrm{Si}(2\pi) \le \mathrm{Si}(\beta) \le \mathrm{Si}(\pi)\,,$$

and consequently

$$-1 + \frac{2}{\pi}\,\mathrm{Si}(2\pi) \le -1 + \frac{2}{\pi}\,\mathrm{Si}(\beta) \le -1 + \frac{2}{\pi}\,\mathrm{Si}(\pi)\,, \quad \beta \ge \pi\,. \tag{5.5}$$

Combining (5.4) and (5.5), we obtain the inequality

$$\begin{aligned} -0.097176... = -1 + \frac{2}{\pi}\,\mathrm{Si}(2\pi) &\le h(\beta) - 1 + \frac{2}{\pi}\,\mathrm{Si}(\beta) \\ &\le h(\pi) - 1 + \frac{2}{\pi}\,\mathrm{Si}(\pi) = 0.400229\ldots \end{aligned}$$

for all $\beta \ge \pi$, which finally gives (5.1). This completes the proof. ■

Lemma 5.2. *The function*

$$g(x) \coloneqq \frac{\mathrm{e}^x}{x\,(I_0(x) - 1)} \tag{5.6}$$

is monotonously decreasing for $x \ge 1$.

Proof. Using $I_0'(x) = I_1(x)$, see [1, 9.6.27], the derivative of the function (5.6) is given by

$$g'(x) = \frac{\mathrm{e}^x\,(-x - x\,I_1(x) + (x-1)\,I_0(x) + 1)}{x^2\,(I_0(x) - 1)^2}\,.$$

To prove that (5.6) is monotonously decreasing for $x \ge 1$, we need to show that $g'(x) < 0$, $x \ge 1$, or rather

$$\tilde{g}(x) \coloneqq -x - x\,I_1(x) + (x-1)\,I_0(x) + 1 < 0, \quad x \ge 1\,.$$

Note that

$$\tilde{g}(1) = -1 - I_1(1) + 0 + 1 = -I_1(1) = -0.565159\ldots < 0\,.$$

Thus, by showing $\tilde{g}'(x) \le 0$, $x \ge 1$, we see that $\tilde{g}$ is negative for all $x \ge 1$, and thereby g is monotonously decreasing.

In order to do so, we use the formula $(x\,I_1)'(x) = x\,I_0(x)$, see [1, 9.6.28], to compute the derivative

$$\begin{aligned}\tilde{g}'(x) &= -1 - x\,I_0(x) + I_0(x) + (x-1)\,I_1(x)\\ &= (x-1)[I_1(x) - I_0(x)] - 1\,.\end{aligned}$$

Since $I_1(x) \le I_0(x)$ for all $x \ge 0$ by [2, (2.3)] we have $\tilde{g}'(x) < 0$ for all $x \ge 1$, which completes the proof. ■

Theorem 5.3. *Assume that* $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ *is bandlimited with bandwidth* $\delta \in \big(0, \frac{m-1}{m}\pi\big]$. *Further let* φ_{cKB} *be the continuous Kaiser–Bessel window function* (1.7) *with shape parameter* $\beta = m\,(\pi - \delta)$ *and let* $m \in \mathbb{N} \setminus \{1\}$ *be given.*
Then the continuous Kaiser–Bessel regularized Shannon sampling formula satisfies the error estimate

$$\big\|f - R_{\mathrm{cKB},m} f\big\|_{C_0(\mathbb{R})} \le \left(\frac{7}{8}\,m\,(\pi - \delta) + \frac{7}{\pi}\,m^2(\pi - \delta)^2\right)\mathrm{e}^{-m\,(\pi-\delta)}\,\|f\|_{L^2(\mathbb{R})}\,. \tag{5.7}$$

Proof. (i) Since φ_{cKB} in (1.7) is compactly supported on $[-m, m]$ and $\varphi_{\mathrm{cKB}}(m) = 0$, we have $E_2(m) = 0$. Thus, according to Theorem 2.1, the approximation error can be estimated by

$$\big\|f - R_{\mathrm{cKB},m} f\big\|_{C_0(\mathbb{R})} \le \|f\|_{L^2(\mathbb{R})} \max_{\omega \in [-\delta,\delta]} \big|\Delta_{\mathrm{cKB}}(\omega)\big|$$

where

$$\Delta_{\mathrm{cKB}}(\omega) \coloneqq 1 - \frac{1}{\sqrt{2\pi}} \int_{\omega-\pi}^{\omega+\pi} \hat{\varphi}_{\mathrm{cKB}}(\tau)\,\mathrm{d}\tau\,, \quad \omega \in [-\delta, \delta]\,. \tag{5.8}$$

Following [19, p. 3, 1.1, and p. 95, 18.31], the Fourier transform of (1.7) has the form

$$\hat{\varphi}_{\mathrm{cKB}}(\tau) = \frac{m\,\sqrt{2}}{(I_0(\beta)-1)\,\sqrt{\pi}} \cdot \begin{cases} \left(\frac{\sinh\left(\beta\,\sqrt{1-\nu^2}\right)}{\beta\,\sqrt{1-\nu^2}} - \frac{\sin(\beta\nu)}{\beta\nu}\right) & : |\nu| < 1\,, \\ \left(\frac{\sin\left(\beta\,\sqrt{\nu^2-1}\right)}{\beta\,\sqrt{\nu^2-1}} - \frac{\sin(\beta\nu)}{\beta\nu}\right) & : |\nu| > 1\,, \end{cases} \tag{5.9}$$

with the scaled frequency $\nu = \frac{m}{\beta}\,\tau$. Substituting $\tau = \frac{\beta}{m}\,\nu$ in the integral in (5.8), the function Δ_{cKB} reads as

$$\Delta_{\mathrm{cKB}}(\omega) = 1 - \frac{\beta}{m\,\sqrt{2\pi}} \int_{-\nu_1(-\omega)}^{\nu_1(\omega)} \hat{\varphi}_{\mathrm{cKB}}\big(\tfrac{\beta}{m}\,\nu\big)\,\mathrm{d}\nu\,, \quad \omega \in [-\delta,\,\delta]\,, \tag{5.10}$$

with the increasing linear function (4.10).

(ii) Now we choose the shape parameter of (1.7) in the special form $\beta = m\,(\pi - \delta)$. Thus, we have

$$1 = \nu_1(-\delta) \le \nu_1(\omega) = \frac{\omega + \pi}{\pi - \delta} \le \nu_1(\delta) = \frac{\pi + \delta}{\pi - \delta}\,, \quad \omega \in [-\delta,\,\delta]\,.$$

In view of (5.9) we split (5.10) in the form $\Delta_{\mathrm{cKB}}(\omega) = \Delta_{\mathrm{cKB},1} - \Delta_{\mathrm{cKB},2}(\omega)$ with

$$\begin{aligned} \Delta_{\mathrm{cKB},1} &= 1 - \frac{2\beta}{\pi\,(I_0(\beta)-1)} \int_0^1 \left(\frac{\sinh\left(\beta\,\sqrt{1-\nu^2}\right)}{\beta\,\sqrt{1-\nu^2}} - \frac{\sin(\beta\nu)}{\beta\nu}\right) \mathrm{d}\nu\,, \\ \Delta_{\mathrm{cKB},2}(\omega) &= \frac{\beta}{\pi\,(I_0(\beta)-1)} \left(\int_1^{\nu_1(-\omega)} + \int_1^{\nu_1(\omega)}\right) \left(\frac{\sin\left(\beta\,\sqrt{\nu^2-1}\right)}{\beta\,\sqrt{\nu^2-1}} - \frac{\sin(\beta\nu)}{\beta\nu}\right) \mathrm{d}\nu\,. \end{aligned} \tag{5.11}$$

Using [11, 3.997–1] we have

$$\int_0^1 \frac{\sinh\left(\beta\,\sqrt{1-\nu^2}\right)}{\beta\,\sqrt{1-\nu^2}}\,\mathrm{d}\nu = \frac{1}{\beta} \int_0^{\pi/2} \sinh(\beta\,\cos s)\,\mathrm{d}s = \frac{\pi}{2\beta}\,\mathbf{L}_0(\beta)$$

with the modified Struve function (5.2). Additionally, by the definition of the sine integral function (5.3) we have

$$\int_0^1 \frac{\sin(\beta v)}{\beta v}\,\mathrm{d}v = \frac{1}{\beta}\,\mathrm{Si}(\beta)\,,$$

such that we obtain

$$\begin{aligned} \Delta_{\mathrm{cKB},1} &= 1 - \frac{2\beta}{\pi\,\big(I_0(\beta)-1\big)} \left(\frac{\pi}{2\beta}\,\mathbf{L}_0(\beta) - \frac{1}{\beta}\,\mathrm{Si}(\beta)\right) \\ &= \frac{1}{I_0(\beta)-1} \left(I_0(\beta) - \mathbf{L}_0(\beta) - 1 + \frac{2}{\pi}\,\mathrm{Si}(\beta)\right). \end{aligned}$$

Since for $\delta \in \big(0,\,\frac{m-1}{m}\pi\big]$ we have $\beta = m\,(\pi - \delta) \ge \pi$ and therefore $I_0(\beta) > 1$, Lemma 5.1 yields

$$|\Delta_{\mathrm{cKB},1}| \le \frac{1}{2\,\big(I_0(\beta)-1\big)}\,.$$

Now we estimate $\Delta_{\mathrm{cKB},2}(\omega)$ in (5.11) for $\omega \in [-\delta,\, \delta]$ by the triangle inequality as

$$\big|\Delta_{\mathrm{cKB},2}(\omega)\big| \le \frac{\beta}{\big(I_0(\beta)-1\big)} \left(\int_1^{\nu_1(-\omega)} + \int_1^{\nu_1(\omega)}\right)\left|\frac{\sin\big(\beta\sqrt{\nu^2-1}\big)}{\beta\sqrt{\nu^2-1}} - \frac{\sin(\beta\nu)}{\beta\nu}\right| \mathrm{d}\nu\,.$$

By [22, Lemma 4.1] we have

$$\left|\frac{\sin\big(\beta\sqrt{\nu^2-1}\big)}{\beta\sqrt{\nu^2-1}} - \frac{\sin(\beta\nu)}{\beta\nu}\right| \le \frac{2}{\nu^2}\,, \quad \nu \ge 1\,,$$

and therefore

$$|\Delta_{\mathrm{cKB},2}(\omega)| \le \frac{4\beta}{\pi\,\big(I_0(\beta)-1\big)} \int_1^{\infty} \frac{1}{\nu^2}\,\mathrm{d}\nu = \frac{4\beta}{\pi\,\big(I_0(\beta)-1\big)}\,.$$

Thereby, we conclude that

$$|\Delta_{\mathrm{cKB}}(\omega)| \le |\Delta_{\mathrm{cKB},1}| + |\Delta_{\mathrm{cKB2}}(\omega)| \le \frac{1}{I_0(\beta)-1}\left(\frac{1}{2} + \frac{4\beta}{\pi}\right), \quad \omega \in [-\delta,\, \delta]\,.$$

Since by the assumption $0 < \delta \le \frac{m-1}{m}\,\pi$ we have $\beta = m\,(\pi-\delta) \ge \pi$ for $m \in \mathbb{N}\setminus\{1\}$ and by Lemma 5.2 the function $\frac{\mathrm{e}^x}{x\,(I_0(x)-1)}$ is monotonously decreasing for $x \ge 1$, it follows that

$$\frac{\mathrm{e}^{\beta}}{\beta\,(I_0(\beta)-1)} \le \frac{\mathrm{e}^{\pi}}{\pi\,(I_0(\pi)-1)} = 1.644967\ldots < \frac{7}{4}\,.$$

Hence, this yields

$$\frac{1}{I_0(\beta)-1}\left(\frac{1}{2} + \frac{4\beta}{\pi}\right) < \frac{7\beta}{4}\left(\frac{1}{2} + \frac{4\beta}{\pi}\right)\mathrm{e}^{-\beta} = \left(\frac{7}{8}\,\beta + \frac{7}{\pi}\,\beta^2\right)\mathrm{e}^{-\beta}\,.$$

Thus, the continuous Kaiser–Bessel regularized Shannon sampling formula with the chosen shape parameter $\beta = m\,(\pi-\delta)$ fulfills the error estimate (5.7). This completes the proof. ■

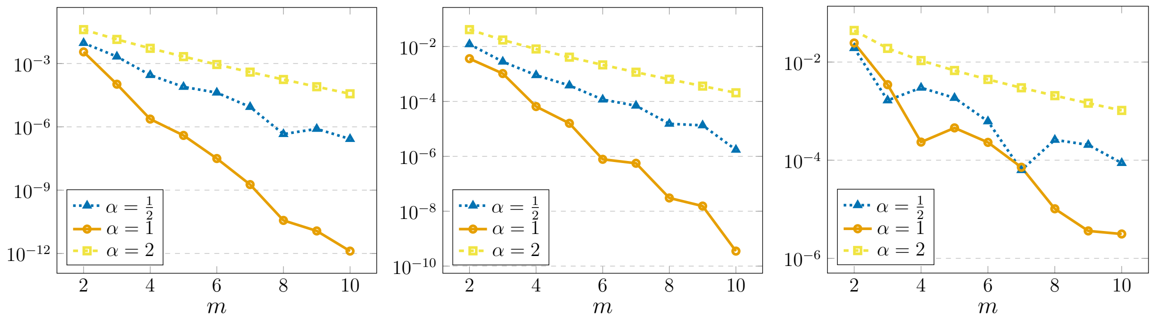
Note that already in [14, Theorem 4.3] bounds on the approximation error of the Shannon sampling formula (1.4) were shown for the continuous Kaiser–Bessel window function (1.7) with suitably chosen shape parameter β. Although the respective parameter β looks different than the one in Theorem 5.3, it is basically the same, only adapted to the slightly different setting considered in [14]. We additionally remark that Theorem 5.3 finally closes the gap in our previous proof in [14] since Lemma 5.1 is a weaker version of the numerical assumption in [13, Figure 4.2], while Lemma 5.2 proves the numerical assumption in [13, p. 23].

Nevertheless, similar to the previous Sections 3 and 4, the optimality of the shape parameter $\beta = m\,(\pi-\delta)$ for the continuous Kaiser–Bessel window function (1.7) is still an open problem, which could so far only be observed numerically.

Conjecture 5.4. The parameter $\beta = m\,(\pi-\delta)$ is optimal for the Shannon sampling formula (1.4) with the continuous Kaiser–Bessel window function (1.7), as it guarantees the maximum decay rate of the uniform approximation error (1.9).

Example 5.5. Analogously as in Example 4.4 we now give numerical evidence for the optimality of the shape parameter $\beta = m\,(\pi - \delta)$ of the continuous Kaiser–Bessel regularized Shannon sampling formula stated in Conjecture 5.4. More precisely, for the bandlimited function (3.6) with several bandwidth parameters $\delta \in \left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$, i. e., several oversampling rates $\frac{\pi}{\delta} > 1$, we consider the regularized Shannon sampling formula (1.4) with the continuous Kaiser–Bessel window function φ_{cKB} in (1.7). The corresponding approximation error (3.5) shall again be approximated by evaluating the given function f and its approximation $R_{\varphi,m}f$ at equidistant points $t_s \in [-1,\,1]$, $s = 1, \ldots, S$, with $S = 10^5$. To compare with the parameter in Theorem 5.3, we choose the shape parameter of the continuous Kaiser–Bessel window function (1.7) as $\beta = \alpha\, m\,(\pi - \delta)$ with $\alpha \in \left\{\frac{1}{2}, 1, 2\right\}$.

The outcomes for different truncation parameters $m \in \{2, 3, \ldots, 10\}$ are depicted in Figure 5.1. As expected it can clearly be seen that the choice of $\alpha \neq 1$ causes worsened error decay rates with respect to m. Thus, these numerical results support the proposition that the shape parameter $\beta = m\,(\pi - \delta)$ of Theorem 5.3 is indeed optimal in terms of the uniform approximation error (1.9), already for very small truncation parameters $m \in \mathbb{N} \setminus \{1\}$. □



(a) $\delta = \frac{\pi}{4}$ (b) $\delta = \frac{\pi}{2}$ (c) $\delta = \frac{3\pi}{4}$

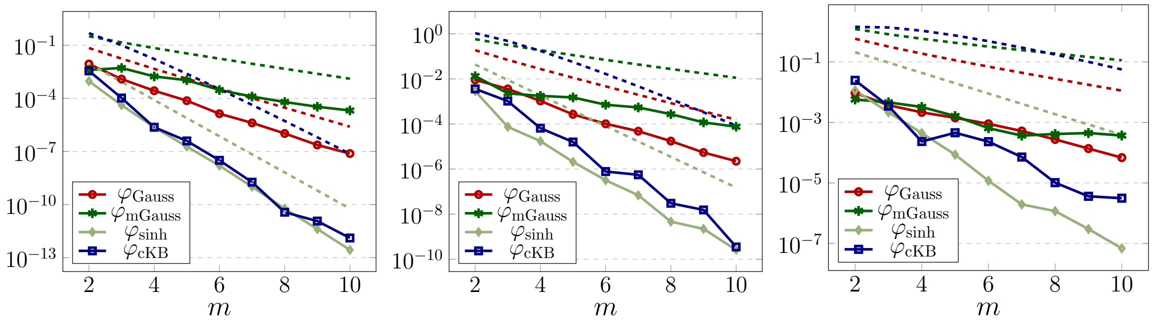
Figure 5.1: Maximum approximation error (3.5) using the continuous Kaiser–Bessel window function φ_{cKB} in (1.7) with different shape parameters $\beta = \alpha\, m(\pi - \delta)$, $\alpha \in \left\{\frac{1}{2}, 1, 2\right\}$, for the bandlimited function (3.6) with bandwidths $\delta \in \left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$ and truncation parameters $m \in \{2, 3, \ldots, 10\}$.

Remark 5.6. Note that the code files for this and all the other experiments are available on https://github.com/melaniekircheis/Optimal-parameter-choice-for-regularized-Shannon-sampling-formulas. □

6 Conclusion

In this paper, we have studied the regularized Shannon sampling formula (1.4) for the widely used Gaussian function (1.5), the sinh-type window function (1.6), and the continuous Kaiser–Bessel window function (1.7). More precisely, for an arbitrary bandlimited function $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ with bandwidth $\delta \in (0,\,\pi)$ we have shown that the uniform approximation error (1.9) of the regularized Shannon sampling formulas of f possess an exponential decay with respect to the truncation parameter m. In doing so, we have demonstrated that the decay rate $m\,(\pi - \delta)$ of the sinh-type regularized Shannon sampling formula, see Theorem 4.2, and the continuous Kaiser–Bessel regularized Shannon sampling formula, see Theorem 5.3, is twice as fast as the decay rate $m\,(\pi - \delta)/2$ of the Gaussian regularized Shannon sampling formula, see Theorem 3.1. Note that the sinh-type regularized Shannon sampling formula is even slightly better than the continuous Kaiser–Bessel regularized Shannon sampling formula due to the constant factors in (4.6) and (5.7), see also Figure 6.1.

The main focus of this work was to elaborate on previous results in [13, 14]. First and foremost, the goal was to rigorously prove all the necessary ingredients for the proofs, thereby improving upon the results in [13, 14] that lacked rigor due to the use of numerical assumptions. In addition, we found that the exponential decay of the approximation error of the regularized Shannon sampling formula (1.4) depends highly on the shape parameter of the corresponding window function. Although the optimality of the variance σ^2 of the Gaussian function and of the shape parameter β of the sinh-type window function and the continuous Kaiser–Bessel function is still an open problem, we have strong reason to believe that our choice is best, even though this is currently only based on numerical observations. We remark that this further emphasizes the superiority of the sinh-type regularized Shannon sampling formula of f, since the approximation errors of the regularized Shannon sampling formulas were compared for the presumably optimal shape parameters each.



(a) $\delta = \frac{\pi}{4}$ (b) $\delta = \frac{\pi}{2}$ (c) $\delta = \frac{3\pi}{4}$

Figure 6.1: Maximum approximation error (3.5) (solid) and error constants (dashed) using $\varphi \in \{\varphi_{\mathrm{Gauss}}, \varphi_{\mathrm{mGauss}}, \varphi_{\mathrm{sinh}}, \varphi_{\mathrm{cKB}}\}$, see (1.5), (3.7), (1.6), and (1.7), for the bandlimited function (3.6) with $\delta \in \left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$ and $m \in \{2, 3, \ldots, 10\}$.

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