

Computation of the Fourier transform for a continuous integrable function via NFFT

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We investigate the approximation of continuous Fourier transforms by trigonometric sampling polynomials and their efficient evaluation by the nonequispaced fast Fourier transform (NFFT). While the NFFT is traditionally used for the evaluation of trigonometric polynomials, we show that it can also serve as an effective computational tool for the approximation of Fourier transform values. Building on ideas of M. Ehler, K. Gröchenig, and A. Klotz [7], we derive explicit ℓ_∞ error bounds between the Fourier transform and suitable sampling polynomials. The resulting estimates quantify the influence of the sampling width and truncation parameter and provide rigorous accuracy guarantees on entire frequency intervals. In contrast to previous analyses focusing mainly on discrete or L_2 -type errors, our results yield uniform approximation bounds that are directly relevant for practical computations.

The derived theory leads to a simple algorithmic framework: first approximate the Fourier transform by a trigonometric sampling polynomial and then evaluate this polynomial efficiently by the NFFT. Numerical experiments confirm the theoretical convergence rates and demonstrate that accurate approximations of continuous Fourier transforms can be obtained with moderate computational effort.

Key words: computation of Fourier transform, fast Fourier transform, nonequispaced fast Fourier transform, NFFT, maximum error estimates, Poisson summation formula, Wiener amalgam space.

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1 Introduction

The nonequispaced fast Fourier transform (NFFT) is a well-established tool for the efficient evaluation of trigonometric polynomials at arbitrary nodes [6, 4, 22, 19, 11, 2]. Traditionally, the NFFT is employed once the Fourier coefficients of a trigonometric polynomial are already known. In many applications, however, the primary quantity of interest is not the trigonometric polynomial itself but the continuous Fourier transform

$$\hat{f}(v) = \int_{\mathbb{R}} f(x) e^{-2\pi i x v} dx, \quad v \in \mathbb{R}, \quad (1.1)$$

This naturally raises the question whether the NFFT can also be used as a numerical method for computing values of the Fourier transform.

The present paper provides a positive answer to this question. Building upon ideas developed by M. Ehler, K. Gröchenig, and A. Klotz [7], we interpret trigonometric sampling polynomials as approximations of the Fourier transform and derive explicit uniform error estimates. These results show that the NFFT can be employed not only for the evaluation of trigonometric polynomials but also for the efficient computation of values of the continuous Fourier transform. Consequently, the NFFT becomes more than a fast evaluation scheme for trigonometric polynomials. Combined with the approximation results developed in this paper, it yields a practical and theoretically justified method for the computation of continuous Fourier transforms.

More precisely, we consider the computation of the Fourier transform of \hat{f} , see (1.1), under certain decay conditions on f , with $\hat{f} \in C(\mathbb{R})$. We restrict f to the spatial domain $[-L/2, L/2]$ and \hat{f} to the frequency domain $[-P/2, P/2]$, where $L \in 2\mathbb{N}$ and $P \in 2\mathbb{N}$ are sufficiently large positive integers. We sample f at the equispaced points j/P , $j \in [LP]$, with spatial step size $1/P$, where

$$[LP] := \left\{ j \in \mathbb{Z} : -\frac{LP}{2} < j \leq \frac{LP}{2} \right\}$$

denotes the centered index set. The P -periodic trigonometric sampling polynomial of f with degree $LP/2$ is defined by

$$s_{P,L}(v) := \frac{1}{P} \sum_{n \in [LP]} f\left(\frac{n}{P}\right) e^{-2\pi i n v / P}, \quad v \in \mathbb{R}, \quad (1.2)$$

which approximates $\hat{f}(v)$ on the compact frequency interval $[-P/2, P/2]$. We determine approximate values of $\hat{f}(\frac{k}{L})$, $k \in [LP]$, by evaluating the trigonometric sampling polynomial $s_{P,L}(v)$ at $v = k/L$, giving

$$s_{P,L}\left(\frac{k}{L}\right) = \frac{1}{P} \sum_{j \in [LP]} f\left(\frac{j}{P}\right) e^{-2\pi i j k / (LP)}, \quad k \in [LP]. \quad (1.3)$$

This standard approach to computing the Fourier transform is well documented in the literature (see, e.g., [5, pp. 17–23], [19, pp. 507–508], or [7]). Note that the values (1.3)

can be computed efficiently via the fast Fourier transform (FFT), provided L and P are products of powers of the small primes, see [9]. More generally, the trigonometric sampling polynomial $s_{P,L}(v)$ can be evaluated at arbitrary points v with $|v| \leq P/2$ by means of the nonequispaced fast Fourier transform (NFFT). Recall that the NFFT is a fast approximate algorithm to evaluate a 1-periodic trigonometric polynomial

$$\sum_{k \in [M]} c_k e^{2\pi i k x}, \quad x \in \mathbb{R}, \quad (1.4)$$

with given coefficients $c_k \in \mathbb{C}$, $k \in [M]$, at given nonequispaced points $x_j \in [0, 1]$, $j = 1, \dots, N$, where $M \in 2\mathbb{N}$. The fundamental idea of the NFFT is to exploit the FFT to efficiently switch between the spatial and frequency domains, thereby enabling a computationally efficient approximation at arbitrary nonuniform points. Throughout this paper, we assume that $L, P \in 2\mathbb{N}$ are sufficiently large.

In [7], M. Ehler, K. Gröchenig, and A. Klotz established error estimates for this computation of the values $\hat{f}(\frac{k}{L})$ for $k \in [LP]$. Specifically, they studied how the scaled Euclidean approximation error

$$E_{P,L}^*(f) := \frac{1}{\sqrt{L}} \left(\sum_{k \in [LP]} \left| \hat{f}\left(\frac{k}{L}\right) - s_{P,L}\left(\frac{k}{L}\right) \right|^2 \right)^{1/2}$$

depends on P and L , given known decay behavior of $f \in C(\mathbb{R})$ and $\hat{f} \in C(\mathbb{R})$. These estimates are formulated in terms of norms of Wiener amalgam spaces [8].

In this paper, we address the analogous question for the scaled maximum approximation error

$$M_{P,L}(f) := \frac{1}{\sqrt{L}} \max_{|v| \leq P/2} \left| \hat{f}(v) - s_{P,L}(v) \right|, \quad (1.5)$$

again under decay conditions on f and \hat{f} . In particular, we study the discretized version

$$M_{P,L}^*(f) := \frac{1}{\sqrt{L}} \max_{k \in [LP]} \left| \hat{f}\left(\frac{k}{L}\right) - s_{P,L}\left(\frac{k}{L}\right) \right|. \quad (1.6)$$

Obviously, it holds $M_{P,L}^*(f) \leq M_{P,L}(f)$. Therefore we are interested in practicable, explicit upper bounds of $M_{P,L}(f)$. We discuss the following questions:

1. Under which decay assumptions on f and \hat{f} can the Fourier transform \hat{f} be uniformly approximated by the trigonometric sampling polynomial $s_{P,L}$ with explicitly controllable error?
2. How can the NFFT be employed to compute values of the continuous Fourier transform $\hat{f}(v)$ efficiently and with rigorous ℓ_∞ error bounds?

We introduce the so-called decay rate $\delta(f, L)$ of $f \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ with respect to step size $L \in 2\mathbb{N}$, which is defined by (2.2). If $f \in C(\mathbb{R})$ has polynomial or exponential decay (see Example 2.2), then the corresponding decay rate $\delta(f, L)$ can be easily

estimated (see Lemma 2.9 and Lemma 2.13). Applying the Poisson summation formula (2.4), we show in Theorem 2.5 that an upper bound of $M_{P,L}(f)$ is a linear combination of the decay rates $\delta(\hat{f}, P)$ and $\delta(f, L)$, if $f, \hat{f} \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ and $P, L \in 2\mathbb{N}$ are given.

Theorem 2.5 shows that the Fourier transform \hat{f} can be approximated uniformly by the trigonometric sampling polynomial $s_{P,L}$ with an error controlled by the decay rates of f and \hat{f} . Since $s_{P,L}$ is a trigonometric polynomial, its values can be computed efficiently by FFT or NFFT techniques. Consequently, the theorem provides a rigorous foundation for the computation of continuous Fourier transforms by means of the NFFT.

This paper is organized as follows. In Section 2, we recall the Wiener amalgam space $W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$, introduce the decay rate $\delta(f, L)$, and derive uniform error estimates for the scaled maximum approximation error $M_{P,L}(f)$ under polynomial, exponential, and mixed decay conditions on f and \hat{f} . Section 3 demonstrates how the theoretical error bounds translate into practical parameter choices for FFT- and NFFT-based approximations of the Fourier transform. In particular, the numerical experiments confirm that the NFFT yields highly accurate evaluations of $\hat{f}(v)$ throughout the frequency interval while preserving the expected computational efficiency.

2 Uniform approximation by trigonometric sampling polynomials

The *Wiener amalgam space* $W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ is the Banach space consisting of all functions $f : \mathbb{R} \rightarrow \mathbb{C}$ which are *locally* in $C(\mathbb{R})$ and have *globally* an $\ell^1(\mathbb{Z})$ behavior at infinity in the sense that the uniform norms of f over the intervals $[k, k+1]$, $k \in \mathbb{Z}$, form a sequence in $\ell^1(\mathbb{Z})$. The norm of $W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ is given by

$$\|f\|_{W(C(\mathbb{R}), \ell^1(\mathbb{Z}))} := \sum_{k \in \mathbb{Z}} \max_{x \in [0, 1]} |f(x+k)|.$$

The Wiener amalgam space was introduced by N. Wiener [24, p. 73] and was used as convenient function space in Fourier analysis, cf. [8], [13, pp. 103–105], and [7]. The condition $f \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ is strong enough to exclude many pathological functions which play a role in Fourier analysis but are of little practical interest. The Wiener amalgam space $W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ contains all continuous, compactly supported functions and is therefore a dense subspace of the Lebesgue space $L^1(\mathbb{R})$. This Wiener amalgam space has the following property:

Lemma 2.1. (see [13, Lemma 6.1.2]) *If $f \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ and $R > 0$ are given, then*

$$\sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |f(x + Rk)| \leq \left(\frac{1}{R} + 1 \right) \|f\|_{W(C(\mathbb{R}), \ell^1(\mathbb{Z}))}.$$

From Lemma 2.1 with $R = \frac{1}{L}$, $L \in 2\mathbb{N}$, it follows immediately that

$$\sum_{k \in \mathbb{Z}} |f(\frac{k}{L})| \leq \left(\frac{1}{R} + 1 \right) \|f\|_{W(C(\mathbb{R}), \ell^1(\mathbb{Z}))} < \infty \quad (2.1)$$

such that $(f(\frac{k}{L}))_{k \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$ for each $L \in 2\mathbb{N}$.

For each $f \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ we introduce the *decay rate of f with respect to the step size $L \in 2\mathbb{N}$* by

$$\delta(f, L) := \max_{|x| \leq L/2} \sum_{k \in \mathbb{Z} \setminus \{0\}} |f(x + kL)|. \quad (2.2)$$

From Lemma 2.1 it follows immediately that $\delta(f, L) < \infty$ for arbitrary $L \in 2\mathbb{N}$.

Example 2.2. *A function f belongs to the Wiener amalgam space $W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ if and only if $f \in C(\mathbb{R})$ can be majorized by the step function s with jumps at integers, given by $s(x) = m_k$ for $x \in [k, k + 1)$, $k \in \mathbb{Z}$, with $m_k := \max_{x \in [k, k+1]} |f(x)|$ and $\sum_{k \in \mathbb{Z}} m_k < \infty$. Hence it holds*

$$W(C(\mathbb{R}), \ell^1(\mathbb{Z})) \subset C(\mathbb{R}) \cap L^1(\mathbb{R}).$$

In particular, each compactly supported function $f \in C(\mathbb{R})$ belongs to $W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$. If $f \in C(\mathbb{R})$ has polynomial decay, i.e.,

$$\sup_{x \in \mathbb{R}} |f(x)| (1 + |x|)^a \leq c < \infty, \quad a > 1,$$

or exponential decay, i.e.,

$$\sup_{x \in \mathbb{R}} |f(x)| e^{r|x|^\alpha} \leq d < \infty, \quad r, \alpha > 0,$$

then f belongs to $W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$. For corresponding proofs see Lemma 2.9 and Lemma 2.13.

2.1 Poisson summation formula

The key tool for estimating the uniform approximation error is the Poisson summation formula. It allows us to express the error between \hat{f} and the trigonometric sampling polynomial $s_{P,L}$ in terms of the tails of f and \hat{f} , which can then be controlled by decay conditions on these functions.

Lemma 2.3. (see [12]) *Let $P \in 2\mathbb{N}$ be fixed. For each $f \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ with $\hat{f} \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$, the Poisson summation formulae*

$$\sum_{k \in \mathbb{Z}} f(x + kP) = \frac{1}{P} \sum_{n \in \mathbb{Z}} \hat{f}\left(\frac{n}{P}\right) e^{2\pi i nx/P}, \quad (2.3)$$

$$\sum_{k \in \mathbb{Z}} \hat{f}(v + kP) = \frac{1}{P} \sum_{n \in \mathbb{Z}} f\left(\frac{n}{P}\right) e^{-2\pi i nv/P} \quad (2.4)$$

hold for all $x, v \in \mathbb{R}$. All series converge absolutely and uniformly on \mathbb{R} .

Proof. From Lemma 2.1 it follows that

$$f_P(x) := \sum_{k \in \mathbb{Z}} f(x + kP), \quad x \in \mathbb{R},$$

is a continuous, P -periodic function. By the assumption $\hat{f} \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$, it follows from (2.1) that

$$\sum_{k \in \mathbb{Z}} \left| \hat{f}\left(\frac{n}{P}\right) \right| < \infty,$$

so that the P -periodic Fourier series $\frac{1}{P} \sum_{n \in \mathbb{Z}} \hat{f}\left(\frac{n}{P}\right) e^{2\pi i n x / P}$ converges absolutely and uniformly on \mathbb{R} . Both functions coincide, since f_P has the P -periodic Fourier coefficients

$$\frac{1}{P} \int_0^P f_P(x) e^{-2\pi i n x / P} dx = \frac{1}{P} \int_{\mathbb{R}} f(x) e^{-2\pi i n x / P} dx = \frac{1}{P} \hat{f}\left(\frac{n}{P}\right), \quad n \in \mathbb{Z}.$$

This completes the proof of (2.3). Replacing f by \hat{f} yields (2.4). ■

Remark 2.4. *By an example, Y. Katznelson [14] has shown that the Poisson summation formula (2.3) does not hold for all $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ with $\hat{f} \in C(\mathbb{R}) \cap L^1(\mathbb{R})$, illustrating the essential difference between the spaces $W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ and $C(\mathbb{R}) \cap L^1(\mathbb{R})$.*

2.2 The trigonometric sampling polynomial and its fast evaluation

The trigonometric sampling polynomial provides a natural approximation of the Fourier transform \hat{f} on a compact frequency interval, and can be evaluated efficiently at arbitrary points using the NFFT. Now we make this precise.

For $f \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ with $\hat{f} \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ and sufficiently large $L, P \in 2\mathbb{N}$, we define the P -periodic trigonometric sampling polynomial of f with sampling support $[-L/2, L/2]$ and degree $LP/2$ by $s_{P,L}$, see (1.2). Under suitable decay conditions on f and \hat{f} , the polynomial $s_{P,L}$ provides a good uniform approximation of \hat{f} on the compact interval $[-P/2, P/2]$, see Theorem 2.5 and the following results.

The values $s_{P,L}(k/L)$, $k \in [LP]$, on the uniform frequency grid can be computed via the FFT in $\mathcal{O}(LP \log LP)$ operations, provided L and P are products of powers of small primes (see [9]). More generally, the trigonometric sampling polynomial $s_{P,L}$ can be evaluated at K arbitrary points $v_1, \dots, v_K \in [-P/2, P/2]$ by means of the nonequispaced fast Fourier transform (NFFT) in only $\mathcal{O}(LP \log LP + K)$ operations (see [19]). Hence, for each v with $|v| \leq P/2$, the value $s_{P,L}(v)$ is a computationally efficient approximation of $\hat{f}(v)$.

2.3 The uniform approximation error

Let $L, P \in 2\mathbb{N}$ be given. By convenient decay conditions on f and \hat{f} , we can show that $s_{P,L}$ is a good uniform approximation of \hat{f} on the compact interval $[-\frac{P}{2}, \frac{P}{2}]$. To this

end, we use the scaled uniform approximation error (1.5). If we form the maximum only on the grid $\{\frac{k}{L} : k \in [LP]\}$, then we obtain the discretized version of the scaled uniform approximation error (1.6). We derive explicit estimates for $M_{P,L}(f)$ and $M_{P,L}^*(f)$ under various decay conditions on f and \hat{f} . The key decomposition is provided by Theorem 2.5 below, which splits the error into two contributions: one governed by the decay of \hat{f} and one by the decay of f . Explicit bounds for each contribution are then given in Lemma 2.9 and Lemma 2.13, covering the cases of polynomial and exponential decay respectively.

Theorem 2.5. *Let $L, P \in 2\mathbb{N}$ be fixed. Let $f \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ with $\hat{f} \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ be given.*

Then \hat{f} can be uniformly approximated on $[-P/2, P/2]$ by the trigonometric sampling polynomial $s_{P,L}$, and it holds the estimate

$$M_{P,L}^*(f) \leq M_{P,L}(f) \leq \frac{1}{\sqrt{L}} \delta(\hat{f}, P) + \sqrt{L} \delta(f, L), \quad (2.5)$$

where $\delta(\hat{f}, P)$ and $\delta(f, L)$ denote the corresponding decay rates (2.2).

Proof. Splitting both series in the second Poisson summation formula (2.4) into the principal terms and the remainders, we obtain for all $v \in \mathbb{R}$

$$\hat{f}(v) + \sigma_1(v) = s_{P,L}(v) + \sigma_2(v),$$

with

$$\begin{aligned} \sigma_1(v) &:= \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{f}(v + kP), \\ \sigma_2(v) &:= \frac{1}{P} \sum_{n \in \mathbb{Z} \setminus [LP]} f\left(\frac{n}{P}\right) e^{-2\pi i n v / P}. \end{aligned}$$

Hence this provides that

$$M_{P,L}^*(f) \leq M_{P,L}(f) \leq \frac{1}{\sqrt{L}} \max_{|v| \leq P/2} |\sigma_1(v)| + \frac{1}{\sqrt{L}} \max_{|v| \leq P/2} |\sigma_2(v)|.$$

Applying the triangle inequality and (2.2), we obtain

$$\max_{|v| \leq P/2} |\sigma_1(v)| \leq \max_{|v| \leq P/2} \sum_{k \in \mathbb{Z} \setminus \{0\}} |\hat{f}(v + kP)| = \delta(\hat{f}, P).$$

In the expression σ_2 , the summation indices $n \in \mathbb{Z} \setminus [LP]$ can be written in the form $n = m + kLP$ with $m \in [LP]$ and $k \in \mathbb{Z} \setminus \{0\}$, such that

$$\sigma_2(v) = \frac{1}{P} \sum_{m \in [LP]} \sum_{k \in \mathbb{Z} \setminus \{0\}} f\left(\frac{m}{P} + kL\right) e^{-2\pi i m v / P} e^{-2\pi i k L v}.$$

Using triangle inequality and (2.2), this yields

$$\max_{|v| \leq P/2} |\sigma_2(v)| \leq \frac{LP}{P} \max_{m \in [LP]} \sum_{k \in \mathbb{Z} \setminus \{0\}} \left| f\left(\frac{m}{P} + kL\right) \right| \leq L \delta(f, L).$$

This completes the proof. ■

Remark 2.6. For given $f, \hat{f} \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$, we can analogously estimate the maximum approximation error

$$\frac{1}{\sqrt{L}} \max_{|x| \leq P/2} \left| f(x) - \frac{1}{P} \sum_{n \in [LP]} \hat{f}\left(\frac{n}{P}\right) e^{2\pi i nx/P} \right| \quad (2.6)$$

by applying the first Poisson summation formula (2.3). The error (2.6) has the upper bound

$$\frac{1}{\sqrt{L}} \delta(f, P) + \sqrt{L} \delta(\hat{f}, L),$$

which can be further estimated under concrete decay conditions on f and \hat{f} by Lemma 2.9 and Lemma 2.13. For the discretized version of (2.6) we obtain

$$\frac{1}{\sqrt{L}} \max_{\ell \in [LP]} \left| f\left(\frac{\ell}{L}\right) - \frac{1}{P} \sum_{n \in [LP]} \hat{f}\left(\frac{n}{P}\right) e^{2\pi i n \ell / (LP)} \right| \leq \frac{1}{\sqrt{L}} \delta(f, P) + \sqrt{L} \delta(\hat{f}, L).$$

Remark 2.7. The reconstruction in Remark 2.6 presupposes the exact Fourier samples $\hat{f}(n/P)$, $n \in [LP]$. In practice, however, only the computed trigonometric sampling polynomial $s_{P,L}$, see (1.2), is available. We therefore sample $s_{P,L}$ on the finer equispaced grid $\{j/M : j \in [MP]\}$ of the frequency interval $[-P/2, P/2)$, where $M \in 2\mathbb{N}$ with $M \geq 2L$ denotes a frequency oversampling factor, and form the fully computable approximation

$$f_{M,P,L}(x) := \frac{1}{M} \sum_{j \in [MP]} s_{P,L}\left(\frac{j}{M}\right) e^{2\pi i j x / M}, \quad x \in [-L/2, L/2). \quad (2.7)$$

This is the M -point rectangle-rule discretization of the truncated inverse Fourier integral $\int_{-P/2}^{P/2} s_{P,L}(v) e^{2\pi i vx} dv$; on the grid $x = \ell/P$, $\ell \in [LP]$, it can be evaluated by a single inverse FFT of length MP .

Introducing the truncated cardinal series

$$g_{P,L}(x) := \int_{-P/2}^{P/2} s_{P,L}(v) e^{2\pi i vx} dv = \sum_{n \in [LP]} f\left(\frac{n}{P}\right) \text{sinc}(Px - n), \quad \text{sinc}(t) := \frac{\sin(\pi t)}{\pi t}, \quad (2.8)$$

the approximation error splits into three contributions,

$$f(x) - f_{M,P,L}(x) = \underbrace{\int_{|v|>P/2} \hat{f}(v) e^{2\pi i vx} dv}_{\text{band truncation}} + \underbrace{\int_{-P/2}^{P/2} (\hat{f} - s_{P,L})(v) e^{2\pi i vx} dv}_{\text{sampling polynomial}} + \underbrace{(g_{P,L}(x) - f_{M,P,L}(x))}_{\text{frequency discretization}}.$$

The first term is bounded by the L^1 -tail $\eta(\hat{f}, P) := \int_{|v|>P/2} |\hat{f}(v)| dv$, the second by $P \max_{|v|\leq P/2} |\hat{f}(v) - s_{P,L}(v)| = P\sqrt{L} M_{P,L}(f)$, see (1.5). For the third term, a direct evaluation of the geometric sum in (2.7) gives the closed form

$$f_{M,P,L}(x) = \frac{1}{MP} \sum_{n \in [LP]} f(n/P) e^{i\pi\alpha_n/M} \frac{\sin(\pi P\alpha_n)}{\sin(\pi\alpha_n/M)}$$

with $\alpha_n := x - n/P$, hence

$$g_{P,L}(x) - f_{M,P,L}(x) = \frac{1}{P} \sum_{n \in [LP]} f\left(\frac{n}{P}\right) \sin(\pi P\alpha_n) \left[\frac{1}{\pi\alpha_n} - \frac{e^{i\pi\alpha_n/M}}{M \sin(\pi\alpha_n/M)} \right].$$

Since $|\sin(\pi P\alpha_n)| \leq 1$ and, by the elementary inequality $1 - \beta \cot \beta \leq \beta^2$ for $|\beta| \leq \pi/2$, the bracket is bounded in modulus by $M^{-1} \sqrt{1 + \pi^2/4}$ whenever $|\alpha_n| \leq M/2$ (which holds for $|x| \leq L/2$ as $M \geq 2L$), we obtain with Lemma 2.1 (for $R = 1/P$ and $|x| \leq L/2$)

$$|g_{P,L}(x) - f_{M,P,L}(x)| \leq \frac{\sqrt{1 + \pi^2/4}}{MP} \sum_{n \in [LP]} \left| f\left(\frac{n}{P}\right) \right| \leq \frac{(P+1)\sqrt{1 + \pi^2/4}}{MP} \|f\|_{W(C(\mathbb{R}), \ell^1(\mathbb{Z}))}.$$

Combining the three bounds with Theorem 2.5 and $(P+1)/P \leq 2$ yields the uniform error estimate

$$\max_{|x| \leq L/2} |f(x) - f_{M,P,L}(x)| \leq \eta(\hat{f}, P) + P(\delta(\hat{f}, P) + L\delta(f, L)) + \frac{2\sqrt{1 + \pi^2/4}}{M} \|f\|_{W(C(\mathbb{R}), \ell^1(\mathbb{Z}))}. \quad (2.9)$$

The three terms reflect, respectively, the truncation of the inverse transform to the band $[-P/2, P/2]$, the approximation of \hat{f} by $s_{P,L}$ on that band, and the rectangle-rule discretization in the frequency variable.

If \hat{f} has polynomial decay $\sup_v |\hat{f}(v)| (1 + |v|)^b \leq c < \infty$ with $b > 1$, then $\eta(\hat{f}, P) \leq \frac{2c}{b-1} (1 + P/2)^{1-b}$, and with Lemma 2.9 the estimate (2.9) becomes

$$\begin{aligned} \max_{|x| \leq L/2} |f(x) - f_{M,P,L}(x)| &\leq \frac{2c}{b-1} \left(1 + \frac{P}{2}\right)^{1-b} + 2c(2^b - 1) \zeta(b) P^{1-b} + PL\delta(f, L) \\ &+ \frac{2\sqrt{1 + \pi^2/4}}{M} \|f\|_{W(C(\mathbb{R}), \ell^1(\mathbb{Z}))}. \end{aligned} \quad (2.10)$$

For compactly supported f , where $\delta(f, L) = 0$ for sufficiently large L , the spatial error therefore decays as $\mathcal{O}(P^{1-b}) + \mathcal{O}(M^{-1})$, and balancing the two contributions suggests the choice $M \gtrsim P^{b-1}$. We remark that the leading M^{-1} -coefficient is in fact governed by $|s_{P,L}(P/2)| \approx |\hat{f}(P/2)|$, which is already small at the band edge, so that in practice the frequency discretization is markedly less critical than the truncation with respect to P .

Remark 2.8. Roughly speaking, smoothness of f implies polynomial decay of \hat{f} , and smoothness of \hat{f} implies polynomial decay of f . If each derivative $f^{(\ell)}$, $\ell = 0, \dots, b$, belongs to $L^1(\mathbb{R})$, then the Fourier transform of $f^{(\ell)}$ equals $(2\pi v i)^\ell \hat{f}(v)$ and is bounded on \mathbb{R} , so that

$$\sup_{v \in \mathbb{R}} |\hat{f}(v)| (1 + |v|)^b \leq c < \infty.$$

Conversely, if each derivative $\hat{f}^{(\ell)}$, $\ell = 0, \dots, b$, belongs to $L^1(\mathbb{R})$, then the inverse Fourier transform of $\hat{f}^{(\ell)}$ equals $(-2\pi i x)^\ell f(x)$ and is bounded on \mathbb{R} , so that

$$\sup_{x \in \mathbb{R}} |f(x)| (1 + |x|)^b \leq c < \infty.$$

Now we estimate the decay rate (2.2) of a function $f \in C(\mathbb{R})$ with polynomial decay.

Lemma 2.9. Let $L \in 2\mathbb{N}$ be given. Assume that $f \in C(\mathbb{R})$ has polynomial decay

$$\sup_{x \in \mathbb{R}} |f(x)| (1 + |x|)^a \leq c < \infty, \quad a > 1.$$

Then the decay rate of f with respect step size L can be estimated by

$$\delta(f, L) \leq 2c(2^a - 1)\zeta(a)L^{-a}, \quad (2.11)$$

where $\zeta(a)$ denotes the Riemann zeta function

$$\zeta(a) := \sum_{n=0}^{\infty} (n+1)^{-a} < 1 + \frac{1}{a-1}, \quad a > 1.$$

Further f belongs to $W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ with the norm

$$\|f\|_{W(C(\mathbb{R}), \ell^1(\mathbb{Z}))} \leq 2c\zeta(a) < \infty.$$

Proof. By the polynomial decay of f , for all $k \in \mathbb{Z} \setminus \{0\}$ and $|x| \leq L/2$ it holds that

$$|f(x + kL)| \leq c(1 + |x + kL|)^{-a} \leq c(1 + (|k| - \frac{1}{2})L)^{-a}.$$

Hence it follows that

$$\delta(f, L) = \max_{|x| \leq L/2} \sum_{k \in \mathbb{Z} \setminus \{0\}} |f(x + kL)|$$

$$\begin{aligned}
&\leq 2c \sum_{k=1}^{\infty} (1 + (k - \frac{1}{2})L)^{-a} = 2c L^{-a} \sum_{k=1}^{\infty} (k - \frac{1}{2} + \frac{1}{L})^{-a} \\
&< 2c L^{-a} \sum_{n=0}^{\infty} (n + \frac{1}{2})^{-a} = 2c \zeta(a, \frac{1}{2}) L^{-a},
\end{aligned}$$

where $\zeta(a, \frac{1}{2}) := \sum_{n=0}^{\infty} (n + \frac{1}{2})^{-a}$ denotes the *Hurwitz zeta function*. By $\zeta(a, \frac{1}{2}) = 2^a \sum_{n=0}^{\infty} (2n + 1)^{-a}$ we obtain

$$\begin{aligned}
\zeta(a) &= \sum_{n=0}^{\infty} (2n + 1)^{-a} + \sum_{n=0}^{\infty} (2n + 2)^{-a} = \sum_{n=0}^{\infty} (2n + 1)^{-a} + 2^{-a} \sum_{n=0}^{\infty} (n + 1)^{-a} \\
&= 2^{-a} \zeta(a, \frac{1}{2}) + 2^{-a} \zeta(a)
\end{aligned}$$

such that $\zeta(a, \frac{1}{2}) = (2^a - 1) \zeta(a)$, which gives (2.11). For $a \in 2\mathbb{N}$, explicit values of $\zeta(a)$ are known, namely $\zeta(a) = \frac{|B_a|}{2a!} (2\pi)^a$ with the *Bernoulli number* B_a (see [1, pp. 266–267]). The integral test for convergence of series provides that for $a > 1$,

$$\zeta(a) = \sum_{n=0}^{\infty} (n + 1)^{-a} \leq 1 + \int_0^{\infty} (x + 1)^{-a} dx = 1 + \frac{1}{a - 1}.$$

Obviously, we have $f \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$, since it holds by assumption that

$$\|f\|_{W(C(\mathbb{R}), \ell^1(\mathbb{Z}))} \leq c \sum_{n=0}^{\infty} (1 + n)^{-a} + c \sum_{n=1}^{\infty} (1 + (n - 1))^{-a} = 2c \zeta(a) < \infty.$$

This completes the proof. ■

Using the corresponding decay rates of f and \hat{f} which have both polynomial decays, we can estimate the scaled uniform maximum error $M_{P,L}(f)$:

Theorem 2.10. *Let $L, P \in 2\mathbb{N}$ be given. If both functions $f, \hat{f} \in C(\mathbb{R})$ have polynomial decays*

$$\begin{aligned}
\sup_{x \in \mathbb{R}} |f(x)| (1 + |x|)^a &\leq c < \infty, \quad a > 1, \\
\sup_{v \in \mathbb{R}} |\hat{f}(v)| (1 + |v|)^b &\leq d < \infty, \quad b > 1,
\end{aligned}$$

then it holds the error estimate

$$M_{P,L}^*(f) \leq M_{P,L}(f) \leq 2d(2^b - 1) \zeta(b) L^{-1/2} P^{-b} + 2c(2^a - 1) \zeta(a) L^{-a+1/2}. \quad (2.12)$$

Proof. Using Theorem 2.5, it follows that

$$M_{P,L}^*(f) \leq M_{P,L}(f) \leq \frac{1}{\sqrt{L}} \delta(\hat{f}, P) + \sqrt{L} \delta(f, L).$$

Then Lemma 2.9 provides that

$$\begin{aligned}\delta(\hat{f}, P) &\leq 2d(2^b - 1)\zeta(b)P^{-b}, \\ \delta(f, L) &\leq 2c(2^a - 1)\zeta(a)L^{-a}.\end{aligned}$$

This completes the proof. ■

Remark 2.11. We compare Theorem 2.10 with the corresponding result of [7, Theorem 2.1]. From [7, Theorem 2.1] it follows that

$$M_{P,L}^*(f) \leq E_{P,L}^*(f) \leq C(L^{-\alpha} + P^{-\beta})$$

with arbitrary $\alpha \in (0, a - \frac{1}{2})$, $\beta \in (0, b - \frac{1}{2})$, and some constant $C > 0$. Our estimate (2.12) is explicit and more practicable, since all constants are described directly by the assumptions of Theorem 2.10. Moreover, using Theorem 2.10, the scaled Euclidean approximation error can be bounded by

$$E_{P,L}^*(f) \leq \sqrt{LP} M_{P,L}^*(f) \leq 2d(2^b - 1)\zeta(b)L^{-1/2}P^{-b+1/2} + 2c(2^a - 1)\zeta(a)L^{1-a}P^{1/2}.$$

As known, a nontrivial function cannot be bandlimited and space-limited (see [19, pp. 103–104]). If f is space-limited or bandlimited, then we obtain simpler error estimates:

Corollary 2.12. Let $L, P \in 2\mathbb{N}$ be given. If $f, \hat{f} \in C(\mathbb{R})$ satisfy

$$\begin{aligned}\sup_{x \in \mathbb{R}} |f(x)|(1 + |x|)^a &\leq c < \infty, \quad a > 1, \\ \text{supp } \hat{f} &\subseteq \left[-\frac{M}{2}, \frac{M}{2}\right], \quad 0 < M \leq P,\end{aligned}$$

then it holds the error estimate

$$M_{P,L}^*(f) \leq M_{P,L}(f) \leq 2c(2^a - 1)\zeta(a)L^{1/2-a}.$$

If $f, \hat{f} \in C(\mathbb{R})$ satisfy

$$\begin{aligned}\text{supp } f &\subseteq \left[-\frac{M}{2}, \frac{M}{2}\right], \quad 0 < M \leq L, \\ \sup_{v \in \mathbb{R}} |\hat{f}(v)|(1 + |v|)^b &\leq d < \infty, \quad b > 1,\end{aligned}$$

then it holds the error estimate

$$M_{P,L}^*(f) \leq M_{P,L}(f) \leq 2d(2^b - 1)\zeta(b)L^{-1/2}P^{-b}.$$

Proof. This result follows immediately from Theorem 2.10. If $\text{supp } \hat{f} \subseteq \left[-\frac{M}{2}, \frac{M}{2}\right]$ and $M \leq P$, then $\hat{f}(v + kP) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$ and $|v| \leq P/2$, so that $\delta(\hat{f}, P) = 0$. If $\text{supp } f \subseteq \left[-\frac{M}{2}, \frac{M}{2}\right]$ and $M \leq L$, then all samples $f(\frac{n}{P})$ vanish for $n \in \mathbb{Z} \setminus [LP]$, so that $\delta(f, L) = 0$. ■

Now we estimate the decay rate (2.2) of a function with exponential decay.

Lemma 2.13. *Let $L \in 2\mathbb{N}$ be given. Let $f \in C(\mathbb{R})$ be a function with exponential decay*

$$\sup_{x \in \mathbb{R}} |f(x)| e^{r|x|^\alpha} \leq c < \infty, \quad r, \alpha > 0.$$

Assume that in the case $0 < \alpha < 1$ it holds $r(L/2)^\alpha > \frac{1}{\alpha} - 1$.

Then the decay rate of f with respect to step size L can be estimated by

$$\delta(f, L) \leq 2c \gamma(r, \alpha, L) e^{-r(L/2)^\alpha} \quad (2.13)$$

with

$$\gamma(r, \alpha, L) := \begin{cases} 1 + \frac{r^{1/\alpha}}{L(\alpha r(L/2)^\alpha - 1 + \alpha)} & 0 < \alpha < 1, \\ 1 + \frac{1}{rL} & \alpha = 1, \\ 1 + \frac{2(r(L/2)^\alpha + b_\alpha)^{1/\alpha} - r^{1/\alpha} L}{4b_\alpha} & \alpha > 1. \end{cases}$$

For $\alpha > 1$, it holds $b_\alpha := \Gamma(\frac{1}{\alpha} + 1)^{\alpha/(1-\alpha)}$. Further f belongs to $W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$ and has the norm

$$\|f\|_{W(C(\mathbb{R}), \ell^1(\mathbb{Z}))} \leq 2c + \frac{2c}{\alpha} r^{-1/\alpha} \Gamma(\frac{1}{\alpha}) < \infty.$$

Proof. By the exponential decay of f , for all $k \in \mathbb{Z} \setminus \{0\}$ and $|x| \leq L/2$ it holds that

$$|f(x + kL)| \leq c e^{-r|x+kL|^\alpha} \leq c e^{-r(L/2)^\alpha (2|k|-1)^\alpha}.$$

Hence it follows that

$$\begin{aligned} \delta(f, L) &= \max_{|x| \leq L/2} \sum_{k \in \mathbb{Z} \setminus \{0\}} |f(x + kL)| \leq 2c \sum_{k=1}^{\infty} e^{-r(L/2)^\alpha (2k-1)^\alpha} \\ &= 2c \sum_{n=0}^{\infty} e^{-r(L/2)^\alpha (2n+1)^\alpha}. \end{aligned}$$

Setting $u := r(L/2)^\alpha$ for shortness, the integral test for convergence of series yields

$$\sum_{n=0}^{\infty} e^{-u(2n+1)^\alpha} \leq e^{-u} + \int_0^{\infty} e^{-u(2x+1)^\alpha} dx = e^{-u} + \frac{1}{2\alpha} u^{-1/\alpha} \Gamma(\frac{1}{\alpha}, u),$$

where

$$\Gamma(\frac{1}{\alpha}, u) := \int_u^{\infty} t^{1/\alpha-1} e^{-t} dt$$

denotes the *upper incomplete gamma function*. A simple check shows that

$$-\frac{1}{\alpha} u^{-1/\alpha} \Gamma(\frac{1}{\alpha}, t^\alpha u)$$

is a primitive of e^{-ut^α} and therefore

$$\int_1^{\infty} e^{-ut^\alpha} dt = \frac{1}{\alpha} u^{-1/\alpha} \Gamma(\frac{1}{\alpha}, u).$$

By [18, Theorem 1.1 and Proposition 2.7], an upper bound of the upper incomplete gamma function $\Gamma(\frac{1}{\alpha}, u)$ is given by

$$\Gamma\left(\frac{1}{\alpha}, u\right) \leq \begin{cases} \frac{u^{1/\alpha}}{u+1-1/\alpha} e^{-u} & 0 < \alpha < 1, \\ \left(1 + \frac{1}{2u}\right) e^{-u} & \alpha = 1, \\ \frac{\alpha(u+b_\alpha)^{1/\alpha} - u^{1/\alpha}}{b_\alpha} e^{-u} & \alpha > 1. \end{cases}$$

This implies the estimate (2.13). Obviously, we have $f \in W(C(\mathbb{R}), \ell^1(\mathbb{Z}))$, since it holds by assumption and [10, Formula 3.478] that

$$\begin{aligned} \|f\|_{W(C(\mathbb{R}), \ell^1(\mathbb{Z}))} &\leq c \sum_{n=0}^{\infty} e^{-r n^\alpha} + c \sum_{n=1}^{\infty} e^{-r(n-1)^\alpha} = 2c \sum_{n=0}^{\infty} e^{-r n^\alpha} \\ &\leq 2c + 2c \int_0^{\infty} e^{-r t^\alpha} dt = 2c + \frac{2c}{\alpha} r^{-1/\alpha} \Gamma\left(\frac{1}{\alpha}\right) < \infty. \end{aligned}$$

This completes the proof. ■

Using the decay rates of f and \hat{f} which have both exponential decays, we can estimate the scaled uniform maximum error $M_{P,L}(f)$:

Theorem 2.14. *Let $L, P \in 2\mathbb{N}$ be given. Let $f, \hat{f} \in C(\mathbb{R})$ be functions with exponential decay*

$$\begin{aligned} \sup_{x \in \mathbb{R}} |f(x)| e^{r|x|^\alpha} &\leq c < \infty, \quad r, \alpha > 0, \\ \sup_{v \in \mathbb{R}} |\hat{f}(v)| e^{s|v|^\beta} &\leq d < \infty, \quad s, \beta > 0. \end{aligned}$$

Assume that $r(L/2)^\alpha > \frac{1}{\alpha} - 1$ in the case $0 < \alpha < 1$ and that $s(P/2)^\beta > \frac{1}{\beta} - 1$ in the case $0 < \beta < 1$.

Then it holds the error estimate

$$M_{P,L}(f) \leq 2d \gamma(s, \beta, P) L^{-1/2} e^{-s(P/2)^\beta} + 2c \gamma(r, \alpha, L) L^{1/2} e^{-r(L/2)^\alpha}, \quad (2.14)$$

where the constants $\gamma(s, \beta, P)$ and $\gamma(r, \alpha, L)$ are defined in Lemma 2.13.

Proof. Using Theorem 2.5, it follows that

$$M_{P,L}(f) \leq \frac{1}{\sqrt{L}} \delta(\hat{f}, P) + \sqrt{L} \delta(f, L).$$

Then Lemma 2.13 provides that

$$\begin{aligned} \delta(\hat{f}, P) &\leq 2d \gamma(s, \beta, P) e^{-s(P/2)^\beta}, \\ \delta(f, L) &\leq 2c \gamma(r, \alpha, L) e^{-r(L/2)^\alpha}. \end{aligned}$$

This completes the proof. ■

Remark 2.15. We compare Theorem 2.14 with the corresponding result in [7, Theorem 2.3]. Then from [7, Theorem 2.3] it follows immediately that

$$M_{P,L}^*(f) \leq E_{P,L}^*(f) \leq C (e^{-r'(L/2)^\alpha} + e^{-s'(P/2)^\beta})$$

with arbitrary $r' \in (0, r)$, $s' \in (0, s)$, and certain constant $C > 0$. Note that in [7, Theorem 2.3] it is considered only the case $0 < \alpha, \beta \leq 1$. Thus we see that our estimate (2.14) is more general, explicit, and more practicable, since all constants in (2.14) are known from the decay conditions of Theorem 2.14.

Using the decay rates of f and \hat{f} which have mixed decays, we can estimate the scaled uniform maximum error $M_{P,L}(f)$:

Theorem 2.16. Let $L, P \in 2\mathbb{N}$ be given. Let $f, \hat{f} \in C(\mathbb{R})$ be functions with mixed decay

$$\begin{aligned} \sup_{x \in \mathbb{R}} |f(x)| (1 + |x|)^a &\leq c < \infty, \quad a > 1, \\ \sup_{v \in \mathbb{R}} |\hat{f}(v)| e^{s|v|^\beta} &\leq d < \infty, \quad s, \beta > 0. \end{aligned}$$

Assume that $s(P/2)^\beta > \frac{1}{\beta} - 1$ in the case $0 < \beta < 1$. Then it holds the error estimate

$$M_{P,L}(f) \leq 2d \gamma(s, \beta, P) L^{-1/2} e^{-s(P/2)^\beta} + 2c(2^a - 1) \zeta(a) L^{-a+1/2}.$$

Alternatively, let $f, \hat{f} \in C(\mathbb{R})$ be functions with mixed decay

$$\begin{aligned} \sup_{x \in \mathbb{R}} |f(x)| e^{r|x|^\alpha} &\leq c < \infty, \quad r, \alpha > 0, \\ \sup_{v \in \mathbb{R}} |\hat{f}(v)| (1 + |v|)^b &\leq d < \infty, \quad b > 1. \end{aligned}$$

Assume that $r(L/2)^\alpha > \frac{1}{\alpha} - 1$ in the case $0 < \alpha < 1$. Then it holds the error estimate

$$M_{P,L}(f) \leq 2d(2^b - 1) \zeta(b) L^{-1/2} P^{-b} + 2c \gamma(r, \alpha, L) L^{1/2} e^{-r(L/2)^\alpha}. \quad (2.15)$$

Proof. In both cases, Theorem 2.5 yields

$$M_{P,L}(f) \leq \frac{1}{\sqrt{L}} \delta(\hat{f}, P) + \sqrt{L} \delta(f, L).$$

Then the assertion follows immediately from Lemma 2.9 and Lemma 2.13. ■

In practice, $L, P \in 2\mathbb{N}$ are sufficiently large, so that the exponential error terms in the estimates of Theorem 2.16 are negligible. Thus $M_{P,L}(f)$ is dominated by the corresponding polynomial error term.

Remark 2.17. We compare the second case of Theorem 2.16 with the corresponding result in [7, Theorem 2.4]. Then from [7, Theorem 2.4 and (4.21)] it follows immediately that

$$M_{P,L}^*(f) \leq E_{P,L}^*(f) \leq C (e^{-r'(L/2)^\alpha} + P^{-\beta})$$

with arbitrary $r' \in (0, r)$, $\beta \in (0, b - \frac{1}{2})$, and certain constant $C > 0$. Note that in [7, Theorem 2.4] it is considered only the case $0 < \alpha \leq 1$. Thus we see that our estimate (2.15) is more general, explicit, and more practicable, since all constants in (2.15) are known from the decay conditions of Theorem 2.16.

3 Numerical examples

In the previous section, we derived explicit error estimates for the uniform approximation error $M_{P,L}(f)$ under polynomial, exponential, and mixed decay conditions on f and \hat{f} . We now illustrate these theoretical bounds by means of concrete examples. For each example, we specify the relevant decay parameters and state the resulting error estimate from Section 2 explicitly before comparing it numerically with the true approximation error $M_{P,L}(f)$.

Example 3.1. Let M_2 be the centered linear B-spline given by

$$M_2(x) := \begin{cases} 1 - |x| & x \in [-1, 1], \\ 0 & x \in \mathbb{R} \setminus [-1, 1]. \end{cases}$$

Then $f = M_2$ has the Fourier transform

$$\hat{f}(v) = \int_{-1}^1 (1 - |x|) \cos(2\pi vx) dx = (\text{sinc}(\pi v))^2, \quad v \in \mathbb{R}.$$

Since $\text{supp } f = [-1, 1]$, the function $f \in C(\mathbb{R})$ is space-limited with $M = 2$, and $\hat{f} \in C(\mathbb{R})$ has polynomial decay with $b = 2$. If we estimate the even function $\hat{f}(v) (1 + |v|)^2$ separately on the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, \infty)$, we see that $d = \frac{9}{4}$ is a possible choice. By Corollary 2.12, for any $L \geq 2$ it holds

$$M_{P,L}(f) \leq 2d(2^2 - 1) \zeta(2) L^{-1/2} P^{-2} = \frac{\pi^2}{\sqrt{L}} d P^{-2}.$$

Hence the error decays like P^{-2} as $P \rightarrow \infty$, which is confirmed by the numerical results in Figure 3.1a (left). More generally, for the B-spline M_{2m} of even order $2m$, the Fourier transform $\widehat{M_{2m}}$ has polynomial decay with $b = 2m$, and Corollary 2.12 yields the estimate

$$M_{P,L}(f) \leq 2d(2^{2m} - 1) \zeta(2m) L^{-1/2} P^{-2m}.$$

This P^{-2m} decay is illustrated in Figure 3.1b (right, $m = 2$) and Figure 3.2a ($m = 3$).

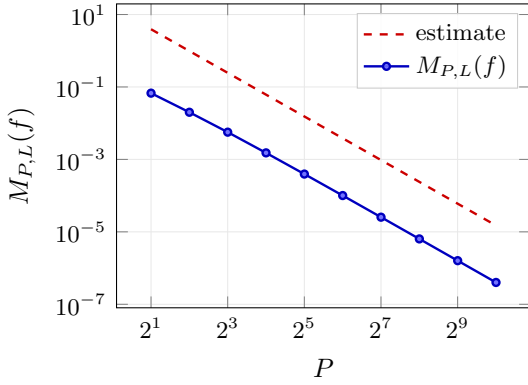
Alternatively, consider the bandlimited function

$$g(x) := \frac{M}{2} \left(\text{sinc} \left(\frac{M}{2} \pi x \right) \right)^2, \quad x \in \mathbb{R},$$

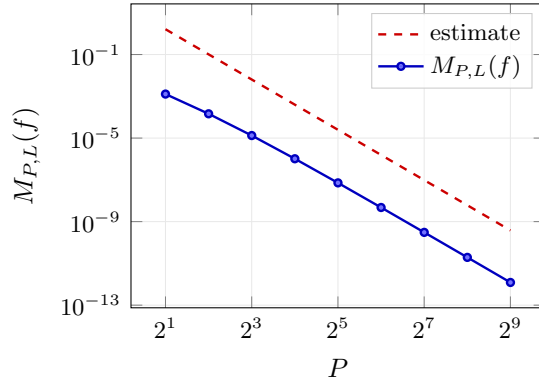
whose Fourier transform is $\hat{g}(v) = M_2(\frac{2}{M}v)$ with $\text{supp } \hat{g} \subseteq [-\frac{M}{2}, \frac{M}{2}]$. Since g has polynomial decay with $a = 2$, Corollary 2.12 gives, for $P \geq M$,

$$M_{P,L}(g) \leq 2c(2^2 - 1) \zeta(2) L^{1/2-2} = \frac{\pi^2}{\sqrt{L}} c L^{-2}.$$

Hence the error decays like $L^{-3/2}$ as $L \rightarrow \infty$.



(a) B-Spline M_2 , order 2 ($L = 2$)



(b) B-Spline M_4 , order 4 ($L = 4$)

Figure 3.1: Estimate (2.12) (Corollary 2.12) and simulated error $M_{P,L}(f)$ for Example 3.1: the B-spline M_2 of order 2 with $L = 2$ (left, decay P^{-2}) and M_4 of order 4 with $L = 4$ (right, decay P^{-4}).

Example 3.2. *The Gaussian function*

$$f(x) = e^{-\pi x^2}, \quad x \in \mathbb{R},$$

has exponential decay with $r = \pi$ and $\alpha = 2$, and its Fourier transform $\hat{f}(v) = e^{-\pi v^2}$ has the same exponential decay with $s = \pi$ and $\beta = 2$. Provided $\pi (L/2)^2 \geq 1$ and $\pi (P/2)^2 \geq 1$, Theorem 2.14 yields

$$M_{P,L}(f) \leq \frac{1}{\sqrt{L}} \left(2 \left(1 + \frac{1}{8} \right) e^{-\pi (P/2)^2} + 2 \left(1 + \frac{1}{8} \right) L e^{-\pi (L/2)^2} \right) \quad (3.1)$$

$$= \frac{9}{4\sqrt{L}} \left(e^{-\pi P^2/4} + L e^{-\pi L^2/4} \right). \quad (3.2)$$

For fixed $L = P$ we obtain

$$M_{L,L}(f) \leq \frac{9(L+1)}{4\sqrt{L}} e^{-\pi L^2/4}.$$

In Figure 3.2b we plot the error $M_{P,P}$ as well as the analytical estimate for $P = 1, \dots, 8$.

Example 3.3. *Let*

$$f(x) := e^{-2\pi|x-\delta|}, \quad x \in \mathbb{R},$$

with shift parameter $\delta \in \mathbb{R}$. Its Fourier transform is

$$\hat{f}(v) = \frac{e^{-2\pi i \delta v}}{\pi(1+v^2)}, \quad v \in \mathbb{R}.$$

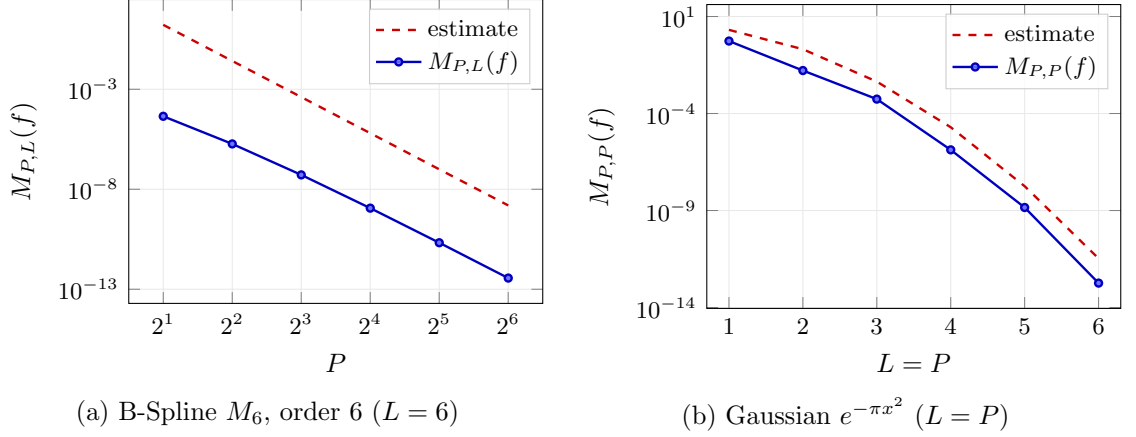


Figure 3.2: Estimate and simulated error $M_{P,L}(f)$: the B-spline M_6 of order 6 with $L = 6$ for Example 3.1 (left, estimate (2.12), decay P^{-6}) and the Gaussian $e^{-\pi x^2}$ with $L = P$ for Example 3.2 (right, estimate $\frac{9(L+1)}{4\sqrt{L}} e^{-\pi L^2/4}$).

Hence f has exponential decay with $r = 2\pi$ and $\alpha = 1$, and \hat{f} has polynomial decay with $b = 2$ and constant $d = 1/\pi$. This is the mixed-decay case of Theorem 2.16. Provided $2\pi(L/2) \geq 1$, i.e., $L \geq 1/\pi$, we obtain

$$M_{P,L}(f) \leq \frac{1}{\sqrt{L}} \left(\frac{2}{\pi} (2^2 - 1) \zeta(2) P^{-2} + 2 \left(1 + \frac{1}{2} \right) L e^{-2\pi(L/2)} \right) = \frac{1}{\sqrt{L}} \left(\frac{2\pi}{P^2} + 3L e^{-\pi L} \right).$$

For sufficiently large L , the exponential term is negligible, and the error is dominated by the polynomial term $\frac{2\pi}{\sqrt{L}P^2}$. A good parameter choice is $L = 8$ and $P = 2^t$, $t \geq 6$; for these values the exponential contribution is below machine precision and the estimate reduces to $M_{P,L}(f) \approx \frac{2\pi}{\sqrt{8}P^2}$, confirming the observed P^{-2} decay in Figure 3.3a.

Example 3.4. For fixed parameter $\beta \in \mathbb{N}$ we consider the function

$$f(x) := \begin{cases} (1 - x^2)^{\beta-1/2} & x \in [-1, 1], \\ 0 & x \in \mathbb{R} \setminus [-1, 1]. \end{cases}$$

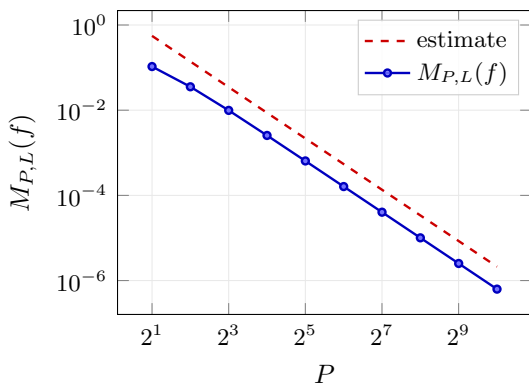
Using [17, p. 8], we determine the corresponding Fourier transform

$$\begin{aligned} \hat{f}(v) &= \int_{-1}^1 (1 - x^2)^{\beta-1/2} e^{-2\pi i v x} dx = 2 \int_0^1 (1 - x^2)^{\beta-1/2} \cos(2\pi v x) dx \\ &= \frac{\pi (2\beta)!}{4^\beta \beta!} \begin{cases} (\pi v)^{-\beta} J_\beta(2\pi v) & v \in \mathbb{R} \setminus \{0\}, \\ \frac{1}{\beta!} & v = 0. \end{cases} \end{aligned} \quad (3.3)$$

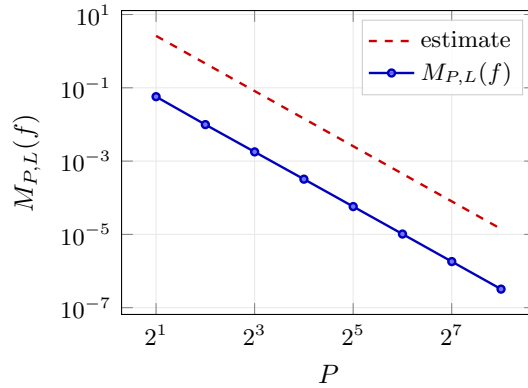
and obtain the estimate [20, Section 5.3]

$$|\hat{f}(v)| \leq \frac{3(2\beta)!}{2^{3/2} 4^\beta \beta! \pi^{\beta-1/2}} |v|^{-\beta-1/2}. \quad (3.4)$$

For $\beta = 2$ we observe by Corollary 2.12 with $M = 1$ and $b = 2$ the decay $P^{-2.5}$, see Figure 3.3b.



(a) mixed decay: $f(x) = e^{-2\pi|x|}$ ($L = 8$)



(b) algebraic decay: $f(x) = (1 - x^2)^{3/2}$ for $x \in [-1, 1]$ ($L = 2$)

Figure 3.3: Estimate and simulated error $M_{P,L}(f)$ for Example 3.3 with $L = 8$ (left, decay P^{-2}) and Example 3.4 with $L = 2$ (right, decay $P^{-2.5}$).

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