How well do fast discrete trigonometric transforms work for parametric numerical integration?

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The fast Fourier transform (FFT) and its real counterparts, the fast algorithms of the discrete cosine transform (DCT) and the discrete sine transform (DST), are frequently employed for the numerical evaluation of the Fourier cosine and Fourier sine transform, thereby acting as parametric quadrature rules based on finitely many samples of a given continuous function. But how accurate are the obtained results? In this paper we study the error occurring if the DCT of type I (DCT-I) and the DST of type I (DST-I) are applied to compute $\int_0^\infty f(x)\cos(2\pi x v)\,\mathrm{d}x$ and $\int_0^\infty f(x)\sin(2\pi x v)\,\mathrm{d}x$ for $f \in L^1(\mathbb{R}_+) \cap C(\mathbb{R}_+)$ and $v \in \mathbb{R}_+$, as well as the error occurring if the Chebyshev coefficients $\frac{2}{\pi}\int_0^\pi h(\cos\theta)\,\cos(n\theta)\,\mathrm{d}\theta$, $n \in \mathbb{N}_0$, of a function $h \in C(I)$ on $I := [-1, 1]$ are computed by the DCT-I.

We present practicable, explicit error estimates under polynomial, exponential, and mixed decay conditions. For the Fourier cosine and sine transform, rational functions lead to a geometric decay in the frequency parameter P combined with an algebraic plateau of order $L^{-3/2}$ in the sampling parameter L, while for meromorphic functions such as $\operatorname{sech}(\pi\,\cdot)$ the critical geometric rate is attained in both parameters. For Chebyshev approximation of a rational function with simple poles in $\mathbb{C} \setminus I$, the quadrature error has the critical geometric rate ρ_0^{-P} with an explicit constant, where $\rho_0 > 1$ is the parameter of the largest Bernstein ellipse of analyticity. Several numerical examples illustrate the theory.

Key words: Fourier cosine transform, Fourier sine transform, Chebyshev coefficients, discrete cosine transform, discrete sine transform, maximum error estimates, Poisson summation formulae, aliasing formula for Chebyshev coefficients, optimal choice of parameters.

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1 Introduction

The FFT and its real counterparts are the most important transforms in modern engineering and science. They are frequently used to compute the Fourier transform and related integral transforms and therefore act as quadrature rules, using a finite number of function samples.

In this paper, we study the occurring errors when discrete trigonometric transforms are employed to compute three closely related integral transforms: the *Fourier cosine transform* and the *Fourier sine transform* of a complex-valued function $f \in L^1(\mathbb{R}_+) \cap C(\mathbb{R}_+)$ defined on the half-line $\mathbb{R}_+ = [0, \infty)$ (see [22]), and the Chebyshev coefficients of a *Chebyshev expansion* of a sufficiently smooth function $h \in C(I)$ on $I := [-1, 1]$. Let

$$\hat{f}_{\cos}(v) := \int\limits_0^\infty f(x) \cos(2\pi\, xv)\, \mathrm{d}x, \quad \hat{f}_{\sin}(v) := \int\limits_0^\infty f(x) \sin(2\pi\, xv)\, \mathrm{d}x\,, \quad v \in \mathbb{R}_+\,, \tag{1.1}$$

and let their discrete numerical approximations be given by

$$c_{P,L}(v) := \tfrac{f(0)}{2P} + \tfrac{1}{P} \sum_{n=1}^{LP/2} f(\tfrac{n}{P}) \cos(\tfrac{2\pi n v}{P}), \quad s_{P,L}(v) := \tfrac{1}{P} \sum_{n=1}^{LP/2} f(\tfrac{n}{P}) \sin(\tfrac{2\pi n v}{P}),\ v \in \mathbb{R}_+, \tag{1.2}$$

which can be efficiently evaluated simultaneously at the $\frac{LP}{2} + 1$ equidistant points $\frac{j}{L}$, $j = 0, \ldots, \frac{LP}{2}$ using fast algorithms of discrete cosine transform of type I (DCT-I) and of discrete sine transform of type I (DST-I), respectively. The trigonometric polynomials $c_{P,L}$ and $s_{P,L}$ are called *sampling polynomials* of f which depend on the samples $f(\frac{n}{P})$, $n = 0, \ldots, \frac{LP}{2}$, on the equidistant grid of $[0, \frac{L}{2}]$, where $L, P \in 2\mathbb{N}$ are sufficiently large. Thus, these sampling polynomials act as *parametric* quadrature rules that simultaneously approximate the integrals in (1.1) for all v in a certain range of $\mathbb{R}_+$ at the same time!

But, how well can the integral transforms $\hat{f}_{\cos}$ and $\hat{f}_{\sin}$ be approximated by their sampling polynomials? How large are the errors

$$E_{P,L}^{\cos}(f) := \tfrac{1}{\sqrt{L}} \max_{v \in [0, \frac{P}{2}]} |\hat{f}_{\cos}(v) - c_{P,L}(v)|, \quad E_{P,L}^{\sin}(f) := \tfrac{1}{\sqrt{L}} \max_{v \in [0, \frac{P}{2}]} |\hat{f}_{\sin}(v) - s_{P,L}(v)|, \tag{1.3}$$

and how does the error decay depend on the properties of f and the choice of L and P?

Furthermore we study a related problem, namely how well the Chebyshev coefficients

$$a_n[h] := \tfrac{2}{\pi} \int\limits_{-1}^{1} \tfrac{1}{\sqrt{1-x^2}}\, h(x)\, T_n(x)\ \mathrm{d}x = \tfrac{2}{\pi} \int\limits_0^\pi h(\cos\theta) \cos(n\theta)\, \mathrm{d}\theta\,, \quad n \in \mathbb{N}_0\,, \tag{1.4}$$

of a function $h \in C(I)$, $I := [-1, 1]$, can be approximated by the DCT-I. Here T_n denotes the n-th Chebyshev polynomial. Denoting by $a_n^{(P)}[h]$ the discrete Chebyshev coefficients computed from the $P+1$ samples $h\big(\cos\frac{\pi j}{P}\big)$ by a DCT-I$(P+1)$, and by $t_{P,L}$ the corresponding truncated Chebyshev series of degree $L \le P$, we study the quadrature error $E_P^{\mathrm{Ch}}(h)$ and the reconstruction error $R_{P,L}^{\mathrm{Ch}}(h) = \|h - t_{P,L}\|_{C(I)}$; all these quantities are defined in Section 3.

Our investigations are of great practical interest, since the considered transforms are regularly approximated using fast trigonometric transforms in practice.

Related literature. General parametric quadrature rules have rarely been investigated (see, e.g. [11]). This paper is inspired by [5], which studies how accurately the Fourier transform is approximated by the centered DFT at discrete points, and by [29], where we investigated the maximum-norm error between the Fourier transform and a complex trigonometric sampling polynomial evaluated by FFT. For earlier error estimates on computing Fourier transforms or

Fourier coefficients via the DFT we refer to [2, 3, 9], and for modifications of the Shannon sampling theorem to [18, 19]. The only other study using Fourier cosine and sine transforms that we are aware of is [4], which however uses different quadrature rules not linked to fast trigonometric transforms and provides error estimates only for analytic functions via contour integrals. Our parametric quadrature formulas can also be interpreted as trapezoidal rules for highly oscillatory integrals; the rich literature on such integrals typically considers non-parametric integrals over finite intervals without connections to fast trigonometric transforms, see e.g. [15, 16, 17].

Regarding the computation of Chebyshev coefficients, early foundational work by D. Elliott [6, 7, 8] investigated error bounds of truncated Chebyshev series via contour integrals, an approach taken up in [35]. The decay of the Chebyshev coefficients is related to the smoothness of h in [21], and [20, 27] study projections onto trigonometric polynomials resp. alternative quadrature rules. All of these works rely on different approximation polynomials and distinct error-estimation frameworks than the ones presented here.

Organization of this paper. In Section 2 we study the parametric quadrature of the Fourier cosine and sine transform in (1.1) using the polynomials (1.2). Subsection 2.1 provides the two main tools, suitable modifications of the Poisson summation formula and the decay rate of f outside $[0, \frac{L}{2}]$. In Subsection 2.2 we provide the main theorem, bounding the quadrature error by the decay rates of f and of $\hat{f}_{\cos}$ resp. $\hat{f}_{\sin}$. Our approach can also be employed in Subsection 2.3 to derive a reconstruction of f from the given samples. In Subsection 2.4 we show for rational and meromorphic functions how L and P influence the new estimates. In Section 3 we derive an aliasing formula and a Chebyshev decay rate for Chebyshev coefficients of a smooth function on $[-1, 1]$, which lead to corresponding new quadrature and reconstruction error estimates. These estimates are shown to be sharp for rational functions. Section 4 provides numerical examples.

2 Parametric quadrature of the Fourier cosine and sine transform

We want to answer the question, how well the parametric integrals in (1.1) can be approximated by a parametric quadrature formula based on DCT or DST and involving only the equidistant function values $f(\frac{n}{P})$, $n = 0, \ldots, \frac{LP}{2}$, for sufficiently large L, $P \in 2\mathbb{N}$.

2.1 Modified Poisson summation formula and decay rates

In this subsection we will provide the two essential ingredients needed to derive the wanted error estimates for the parametric numerical integration based on trigonometric polynomials, which can be accurately evaluated by the fast algorithms of DCT-I and DST-I, respectively.

The natural function space for our analysis is the *Wiener amalgam space* on $\mathbb{R}_+$ given by

$$\mathcal{W}_+ := \Big\{ f \in C(\mathbb{R}_+) : \sum_{j=0}^{\infty} \max_{x \in [0,\,1]} |f(x+j)| < \infty \Big\} \text{ with } \|f\|_{\mathcal{W}_+} := \sum_{j=0}^{\infty} \max_{x \in [0,\,1]} |f(x+j)|. \tag{2.1}$$

Thus $\mathcal{W}_+$ is the Banach space of all $f : \mathbb{R}_+ \to \mathbb{C}$ which are locally in $C(\mathbb{R}_+)$ and have globally an $\ell^1(\mathbb{N}_0)$ behavior at infinity. Wiener amalgam spaces on $\mathbb{R}$ were introduced by N. Wiener [36, p. 73] and are convenient function spaces in Fourier analysis, cf. [10], [14, pp. 103–105], [5]. Membership in $\mathcal{W}_+$ excludes many pathological functions of little practical interest. We conclude from (2.1) for $L \in 2\mathbb{N}$ and $f \in \mathcal{W}_+$ that

$$\sup_{x \in \mathbb{R}_+} \sum_{k=0}^{\infty} |f(x+Lk)| \le \|f\|_{\mathcal{W}_+} < \infty, \tag{2.2}$$

as well as $\sum\limits_{k=0}^{\infty} \big|f\big(\frac{k}{L}\big)\big| \le L\,\|f\|_{\mathcal{W}_+} < \infty$, since $L \ge 2$ and there are at most L function values $f\big(\frac{k}{L}\big)$ in one interval $[j, j+1)$ for $j \in \mathbb{N}_0$, such that $\big(f\big(\frac{k}{L}\big)\big)_{k\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0)$ for each $L \in 2\mathbb{N}$. Obviously we have

$$\mathcal{W}_+ \subset C(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)\,,$$

and each compactly supported $f \in C(\mathbb{R}_+)$ belongs to $\mathcal{W}_+$.

The common analytical tool for our error estimates in Subsection 2.2 will be the Poisson summation formula, which we adapt to the Fourier cosine and Fourier sine transform.

Lemma 2.1. *Let $L, P \in 2\mathbb{N}$. For $f, g \in \mathcal{W}_+$ with $\hat{f}_{\cos}, \hat{g}_{\sin} \in \mathcal{W}_+$ and $g(0) = 0$ we have for $x \in \big[0,\, \frac{L}{2}\big]$*

$$f(x) + \sum_{k=1}^{\infty} \big[f(x+kL) + f(kL-x)\big] = \tfrac{2}{L}\,\hat{f}_{\cos}(0) + \tfrac{4}{L}\sum_{n=1}^{\infty} \hat{f}_{\cos}\big(\tfrac{n}{L}\big)\,\cos\tfrac{2\pi n x}{L}\,, \tag{2.3}$$

$$g(x) + \sum_{k=1}^{\infty} \big[g(x+kL) - g(kL-x)\big] = \tfrac{4}{L}\sum_{n=1}^{\infty} \hat{g}_{\sin}\big(\tfrac{n}{L}\big)\,\sin\tfrac{2\pi n x}{L}\,, \tag{2.4}$$

and for $v \in \big[0,\, \frac{P}{2}\big]$

$$\hat{f}_{\cos}(v) + \sum_{k=1}^{\infty} \big[\hat{f}_{\cos}(v+kP) + \hat{f}_{\cos}(kP-v)\big] = \tfrac{1}{2P}\,f(0) + \tfrac{1}{P}\sum_{n=1}^{\infty} f\big(\tfrac{n}{P}\big)\,\cos\tfrac{2\pi n v}{P}\,, \tag{2.5}$$

$$\hat{g}_{\sin}(v) + \sum_{k=1}^{\infty} \big[\hat{g}_{\sin}(v+kP) - \hat{g}_{\sin}(kP-v)\big] = \tfrac{1}{P}\sum_{n=1}^{\infty} g\big(\tfrac{n}{P}\big)\,\sin\tfrac{2\pi n v}{P}\,, \tag{2.6}$$

and all series converge absolutely and uniformly.

Proof. Let h be defined on $\mathbb{R}$ by $h(x) := \frac{1}{2}(f(x) + g(x))$ and $h(-x) := \frac{1}{2}(f(x) - g(x))$ for $x \in \mathbb{R}_+$, so that f is the even and g the odd part of h on $\mathbb{R}_+$. We observe that

$$\hat{h}(v) = \textstyle\int_{\mathbb{R}} h(x)\,\mathrm{e}^{-2\pi \mathrm{i} x v}\,\mathrm{d}x = \hat{f}_{\cos}(v) - \mathrm{i}\,\hat{g}_{\sin}(v), \quad \hat{h}(-v) = \hat{f}_{\cos}(v) + \mathrm{i}\,\hat{g}_{\sin}(v), \quad v \in \mathbb{R}_+.$$

In particular, h satisfies $\sum\limits_{j\in\mathbb{Z}} \max\limits_{x\in[0,\,1]} |h(x+j)| < \infty$ as well as $\sum\limits_{j\in\mathbb{Z}} \max\limits_{v\in[0,\,1]} |\hat{h}(v+j)| < \infty$, and the formulas of Lemma 2.1 follow from the Poisson summation formulas

$$\sum_{k\in\mathbb{Z}} h(x+kL) = \tfrac{1}{L}\sum_{n\in\mathbb{Z}} \hat{h}\big(\tfrac{n}{L}\big)\,\mathrm{e}^{2\pi \mathrm{i}\, n x/L}\,, \qquad \sum_{k\in\mathbb{Z}} \hat{h}(v+kP) = \tfrac{1}{P}\sum_{n\in\mathbb{Z}} h\big(\tfrac{n}{P}\big)\,\mathrm{e}^{-2\pi \mathrm{i}\, n v/P}$$

for $x, v \in \mathbb{R}$, which can be found e.g. in [13, pp. 15–16] or [26, Theorem 2.28]. ■

We introduce the *(one-sided) decay rate of $f \in \mathcal{W}_+$ with respect to the step size $L \in 2\mathbb{N}$* by

$$\delta_+(f, L) := \max_{|x|\le L/2} \sum_{k=1}^{\infty} |f(x+kL)| \tag{2.7}$$

to measure the decay of f outside the interval $[0, \frac{L}{2})$. By (2.2) we have $\delta_+(f, L) < \|f\|_{\mathcal{W}_+} < \infty$ for all $L \in 2\mathbb{N}$. The two decay classes being most relevant for us are covered by the following two lemmas.

Lemma 2.2. *Let $L \in 2\mathbb{N}$ be given. Assume that $f \in C(\mathbb{R}_+)$ has* polynomial decay

$$\sup_{x\in\mathbb{R}_+} |f(x)|\,(1+x)^a \le c < \infty\,, \quad a > 1\,.$$

Then $f \in \mathcal{W}_+$ and

$$\delta_+(f, L) \le c\,(2^a - 1)\zeta(a)\,L^{-a}\,, \tag{2.8}$$

where ζ denotes the Riemann zeta function

$$\zeta(a) := \sum_{n=0}^{\infty} (n+1)^{-a} \le 1 + \tfrac{1}{a-1}\,, \quad a > 1\,.$$

Proof. The assumption on the decay of f implies for all $k \in \mathbb{N}$ and $|x| < \frac{L}{2}$

$$\big|f(x+kL)\big| \le c\,\big(1+x+kL\big)^{-a} \le c\,\big(1+(k-\tfrac12)L\big)^{-a}\,,$$

and hence

$$\begin{aligned} \delta_+(f,L) &\le c\sum_{k=1}^{\infty}\big(1+(k-\tfrac12)L\big)^{-a} = c\,L^{-a}\sum_{k=1}^{\infty}\big(k-\tfrac12+\tfrac1L\big)^{-a} \\ &< c\,L^{-a}\sum_{n=0}^{\infty}\big(n+\tfrac12\big)^{-a} = c\,\zeta\big(a,\tfrac12\big)\,L^{-a} \end{aligned}$$

with the *Hurwitz zeta function* $\zeta\big(a, \frac12\big) := \sum_{n=0}^{\infty}\big(n+\frac12\big)^{-a} = (2^a-1)\,\zeta(a)$, see e.g. [23, Formula 25.11.11]. In particular,

$$\|f\|_{\mathcal{W}_+} = \sum_{k=0}^{\infty}\max_{x\in[0,\,1]}|f(x+k)| \le c\sum_{k=0}^{\infty}(k+1)^{-a} = c\,\zeta(a) < \infty\,.$$

finishes the proof. ∎

Lemma 2.3. *Let $L \in 2\mathbb{N}$ be given. Let $f \in C(\mathbb{R}_+)$ have* exponential decay

$$\sup_{x\in\mathbb{R}_+} |f(x)|\,\mathrm{e}^{r\,x^\alpha} \le c < \infty\,, \quad r,\,\alpha > 0\,.$$

Then $f \in \mathcal{W}_+$ and, with the abbreviation $u := r\left(\frac{L}{2}\right)^\alpha$ and the upper incomplete gamma function $\Gamma\big(\frac{1}{\alpha}, u\big) := \int_u^\infty t^{\frac{1}{\alpha}-1}\,\mathrm{e}^{-t}\,\mathrm{d}t$ satisfies

$$\delta_+(f,L) \le c\,\mathrm{e}^{-u} + \tfrac{c}{\alpha\,L\,r^{1/\alpha}}\,\Gamma\big(\tfrac{1}{\alpha},\,u\big). \tag{2.9}$$

In particular, if α, r and L satisfy $2\alpha\big(u+1-\frac{1}{\alpha}\big) \ge 1$, we have

$$\delta_+(f,L) \le 2c\,\mathrm{e}^{-r\left(\frac{L}{2}\right)^\alpha}. \tag{2.10}$$

Proof. The assumed exponential decay of f implies for all $k \in \mathbb{N}$ and $|x| \le \frac{L}{2}$ that

$$\big|f(x+kL)\big| \le c\,\mathrm{e}^{-r\,((k-\frac12)L)^\alpha}\,.$$

Since $t \mapsto \mathrm{e}^{-r((t-\frac12)L)^\alpha}$ is decreasing on $[1,\infty)$, the integral test yields

$$\delta_+(f,L) \le c\sum_{k=1}^{\infty}\mathrm{e}^{-r\,((k-\frac12)L)^\alpha} \;\le\; c\,\mathrm{e}^{-r\left(\frac{L}{2}\right)^\alpha} + c\int_1^\infty \mathrm{e}^{-r\,((t-\frac12)L)^\alpha}\,\mathrm{d}t$$

$$= c\,\mathrm{e}^{-u} + \tfrac{c}{L} \int\limits_{L/2}^{\infty} \mathrm{e}^{-r\,y^{\alpha}}\,\mathrm{d}y = c\,\mathrm{e}^{-u} + \tfrac{c}{\alpha\,L\,r^{1/\alpha}}\,\Gamma\big(\tfrac{1}{\alpha},\,u\big)\,.$$

This proves (2.9). If $u > \frac{1}{\alpha} - 1$, then [24, Proposition 2.7] gives $\Gamma\big(\frac{1}{\alpha}, u\big) \le \frac{u^{1/\alpha}}{u+1-\frac{1}{\alpha}}\,\mathrm{e}^{-u}$. Inserting $u^{1/\alpha} = r^{1/\alpha}\frac{L}{2}$
yields (2.10). In particular,

$$\|f\|_{\mathcal{W}_+} \le c \sum_{k=0}^{\infty} \mathrm{e}^{-rk^{\alpha}} \le c + c \int\limits_{0}^{\infty} \mathrm{e}^{-rt^{\alpha}}\,\mathrm{d}t = c + \tfrac{c}{\alpha}\,r^{-1/\alpha}\,\Gamma\big(\tfrac{1}{\alpha}\big) < \infty\,,$$

implies that $f \in \mathcal{W}_+$. Finally, the condition $2\alpha\big(u + 1 - \frac{1}{\alpha}\big) \ge 1$ is satisfied for sufficiently large L and arbitrary $r,\,\alpha > 0$. ■

2.2 Scaled parametric quadrature error

For $f \in \mathcal{W}_+$ with $\hat{f}_{\cos} \in \mathcal{W}_+$ and $L,\,P \in 2\mathbb{N}$, we choose the parametric quadrature rule for $\hat{f}_{\cos}$ and $\hat{f}_{\sin}$ in (1.1) as $c_{P,L}$ and $s_{P,L}$ in (1.2) respectively. We call $c_{P,L}$ and $s_{P,L}$ the *P-periodic sampling cosine polynomial and sine polynomial of f of degree $LP/2$*. Observe that $c_{P,L}$ and $s_{P,L}$ can also be seen as the truncations of the right-hand side of the Poisson summation formula (2.5) and (2.6), respectively. We want to show that under suitable decay conditions on f and $\hat{f}_{\cos}$ resp. $\hat{f}_{\sin}$, the polynomials $c_{P,L}$ and $s_{P,L}$ provide very good uniform approximations of $\hat{f}_{\cos}$ resp. $\hat{f}_{\sin}$ on the compact interval $[0,\,\frac{P}{2}]$ and consider the *scaled parametric quadrature errors* $E^{\cos}_{P,L}(f)$ and $E^{\sin}_{P,L}(f)$ in (1.3).

The next theorem shows that the quadrature error can be directly bounded by the decay rates of f on the one hand and $\hat{f}_{\cos}$ and $\hat{f}_{\sin}$, respectively, on the other hand, where the scaling by $L^{-1/2}$ balances the two error contributions.

Theorem 2.4. *Let $L,\,P \in 2\mathbb{N}$ be fixed and let $f \in \mathcal{W}_+$ with $\hat{f}_{\cos} \in \mathcal{W}_+$ be given. Then $\hat{f}_{\cos}$ can be uniformly approximated on $[0,\,P/2]$ by the P-periodic sampling cosine polynomial $c_{P,L}$ in (1.2) and the error estimate*

$$E^{\cos}_{P,L}(f) = \tfrac{1}{\sqrt{L}} \max_{v\in[0,\frac{P}{2}]} |\hat{f}_{\cos}(v) - c_{P,L}(v)| \le \tfrac{2}{\sqrt{L}}\,\delta_+\big(\hat{f}_{\cos}, P\big) + \sqrt{L}\,\delta_+(f, L) \tag{2.11}$$

holds with the (one-sided) decay rates defined in (2.7) for f and $\hat{f}_{\cos}$.

Furthermore, for $f \in \mathcal{W}_+$ with $\hat{f}_{\sin} \in \mathcal{W}_+$ and $f(0) = 0$ we similarly have

$$E^{\sin}_{P,L}(f) = \tfrac{1}{\sqrt{L}} \max_{v\in[0,\frac{P}{2}]} |\hat{f}_{\sin}(v) - s_{P,L}(v)| \le \tfrac{2}{\sqrt{L}}\,\delta_+\big(\hat{f}_{\sin}, P\big) + \sqrt{L}\,\delta_+(f, L). \tag{2.12}$$

The additional assumption $f(0) = 0$ in the sine case is needed for the continuity of the odd extension of f on $\mathbb{R}$.

Proof. Splitting the two series in the Poisson summation formula (2.5) into the principal terms and the remainders, we obtain $\hat{f}_{\cos}(v) + \sigma_1(v) = c_{P,L}(v) + \sigma_2(v)$ for all $v \in [0,\,\frac{P}{2}]$ with

$$\sigma_1(v) := \sum_{k=1}^{\infty} \big[\hat{f}_{\cos}(v + kP) + \hat{f}_{\cos}(kP - v)\big]\,, \qquad \sigma_2(v) := \tfrac{1}{P} \sum_{n=1+LP/2}^{\infty} f\big(\tfrac{n}{P}\big)\,\cos\tfrac{2\pi n v}{P}\,,$$

and hence $\sqrt{L}\,E^{\cos}_{P,L}(f) \le \max\limits_{v\in[0,P/2]} |\sigma_1(v)| + \max\limits_{v\in[0,P/2]} |\sigma_2(v)|$. The definition of $\delta_+\big(\hat{f}_{\cos}, P\big)$ implies

$$\max_{v\in[0,\,P/2]} |\sigma_1(v)| \le \max_{v\in[0,\,P/2]} \sum_{k=1}^{\infty} \big|\hat{f}_{\cos}(v + kP)\big| + \max_{v\in[0,\,P/2]} \sum_{k=1}^{\infty} \big|\hat{f}_{\cos}(kP - v)\big| \le 2\,\delta_+\big(\hat{f}_{\cos}, P\big)\,.$$

Substituting $n = m + kLP$ with $m = 1 - \frac{LP}{2}, \ldots, \frac{LP}{2}$ and $k \in \mathbb{N}$ in σ_2, we obtain

$$\sigma_2(v) = \frac{1}{P} \sum_{m=1-LP/2}^{LP/2} \sum_{k=1}^{\infty} f\big(\tfrac{m}{P} + kL\big) \cos \frac{2\pi\,(m+kLP)\,v}{P}$$

and therefore

$$\max_{v\in[0,\,P/2]} |\sigma_2(v)| \le \frac{LP}{P} \max_{m=1-LP/2,\ldots,LP/2} \sum_{k=1}^{\infty} \big|f\big(\tfrac{m}{P} + kL\big)\big| \le L\,\delta_+(f,L)\,.$$

Thus (2.11) follows. The estimate (2.12) can be shown similarly. ■

Inserting the decay estimates of Lemmas 2.2 and 2.3 into (2.11) or (2.12), we obtain explicit, practicable bounds. All decay combinations are summarized in Table 2.1. In particular, if $\operatorname{supp} \hat f_{\cos} \subseteq \big[0,\, \frac{K}{2}\big]$ with $K \le P$, then $\hat f_{\cos}(v + kP) = 0$ for all $k \in \mathbb{N}$ and $|v| \le P/2$, so that $\delta_+\big(\hat f_{\cos}, P\big) = 0$. Similarly, $\operatorname{supp} f \subseteq \big[0,\, \frac{K}{2}\big]$ with $K \le L$, implies $\delta_+(f,L) = 0$.

decay of $\hat f_{\cos}$ resp. $\hat f_{\sin}$	bound for $\delta_+(\hat f_{\cos},P)$ resp. $\delta_+(\hat f_{\sin},P)$
$\big\lvert\hat f_{\cos,\sin}(v)\big\rvert\,(1+v)^b \le d,\quad b>1$	$d\,(2^b-1)\zeta(b)\,P^{-b}$
$\big\lvert\hat f_{\cos,\sin}(v)\big\rvert\,\mathrm{e}^{sv^\beta} \le d,\quad s,\beta>0$	$2d\,\mathrm{e}^{-s\,(P/2)^\beta}$
$\operatorname{supp} \hat f_{\cos,\sin} \subseteq \big[0,\,\frac{K}{2}\big],\quad 0<K\le P$	0
decay of f	**bound for $\delta_+(f,L)$**
$\lvert f(x)\rvert\,(1+x)^a \le c,\quad a>1$	$c\,(2^a-1)\zeta(a)\,L^{-a}$
$\lvert f(x)\rvert\,\mathrm{e}^{rx^\alpha} \le c,\quad r,\alpha>0$	$2c\,\mathrm{e}^{-r\,(L/2)^\alpha}$
$\operatorname{supp} f \subseteq \big[0,\,\frac{K}{2}\big],\quad 0<K\le L$	0

Table 2.1: Upper bound for $\delta_+(\hat f_{\cos},P)$ resp. $\delta_+(\hat f_{\sin},P)$ and $\delta_+(f,L)$ for the error estimates (2.11), (2.12), which depend on decay properties of $\hat f_{\cos}$ resp. $\hat f_{\sin}$ and f. Here $\hat f_{\cos,\sin}$ stands for $\hat f_{\cos}$ and $\hat f_{\sin}$, respectively. For the exponential decay estimates we assume, in accordance with (2.10), that $2\beta\big(s\big(\frac{P}{2}\big)^\beta + 1 - \frac{1}{\beta}\big) \ge 1$ and $2\alpha\big(r\big(\frac{L}{2}\big)^\alpha + 1 - \frac{1}{\alpha}\big) \ge 1$, respectively; if these conditions fail, the sharper bound (2.9) has to be used instead.

Remark 2.5. *Note that $\hat f_{\sin}(0) = s_{P,L}(0) = s_{P,L}\big(\frac{P}{2}\big) = 0$, whereas $\hat f_{\sin}\big(\frac{P}{2}\big) \ne 0$ in general, so that $E^{\sin}_{P,L}(f) \ge \frac{1}{\sqrt{L}}\big|\hat f_{\sin}\big(\frac{P}{2}\big)\big|$. No additional hypothesis is needed to cover this endpoint: choosing $x = -\frac{P}{2}$ and $k = 1$ in (2.7) with $L = P$ gives $\delta_+(\hat f_{\sin},P) \ge \big|\hat f_{\sin}\big(\frac{P}{2}\big)\big|$, so that the term $\frac{2}{\sqrt{L}}\,\delta_+(\hat f_{\sin},P)$ in (2.12) already dominates it. In fact this endpoint carries the* entire *sine quadrature error in the sharp example of Remark 2.13, so that requiring $\hat f_{\sin}\big(\frac{P}{2}\big) = 0$ would exclude precisely those functions for which (2.12) is sharp.*

Remark 2.6. ***Fast computation of $c_{P,L}$ and $s_{P,L}$.*** *On the equispaced grid $\big\{\frac{j}{L} : j = 0,\ldots,N\big\}$ with $N := \frac{LP}{2}$, the values*

$$c_{P,L}\big(\tfrac{j}{L}\big) = \frac{1}{2P}\,f(0) + \frac{1}{P}\sum_{k=1}^{N} f\big(\tfrac{k}{P}\big)\cos\frac{\pi\,jk}{N}\,,\quad j = 0,\ldots,N\,,$$

$$s_{P,L}\big(\tfrac{j}{L}\big) = \frac{1}{P}\sum_{k=1}^{N-1} f\big(\tfrac{k}{P}\big)\,\sin\frac{\pi\,jk}{N}\,,\quad j = 1,\ldots,N-1\,,$$

can, up to the endpoint weighting, be simultaneously obtained by fast and stable algorithms for the DCT-I$(N+1)$ *and for the* DST-I$(N-1)$, *respectively with* $\mathcal{O}(N\log N)$ *operations, see [25] and [26, pp. 168–169, 367–381]. A shifted grid* $\left\{\frac{j}{L}+h : j=0,\dots,N\right\}$ *with* $h\in(0,\frac{1}{L})$ *can be handled analogously, by splitting* $\cos\left(\frac{2\pi jk}{LP}+\frac{2\pi hk}{P}\right)$ *and applying one DCT-I*$(N+1)$ *and one DST-I*$(N-1)$.

We also derive a lower estimate of $E^{\cos}_{P,L}$.

Corollary 2.7. *Let* $L,\,P\in 2\mathbb{N}$ *be fixed and let* $f\in\mathcal{W}_+$ *with* $\hat f_{\cos}\in\mathcal{W}_+$ *be given.*
If f *satisfies* $\operatorname{sign}(\hat f_{\cos}(\frac{P}{2}))=\operatorname{sign}\left(\sum_{k=1}^{\infty}\hat f_{\cos}\left(\frac{P}{2}+kP\right)\right)$, *then*

$$\tfrac{1}{\sqrt{L}}\left|\hat f_{\cos}(\tfrac{P}{2})\right|-\sqrt{L}\,\delta_+(f,L)\le E^{\cos}_{P,L}(f). \tag{2.13}$$

If f *is non-negative and monotonically decreasing then*

$$\tfrac{1}{\sqrt{L}}\int\limits_{\frac{L}{2}+\frac{1}{P}}^{\infty} f(x)\,\mathrm{d}x-\tfrac{2}{\sqrt{L}}\,\delta_+\big(\hat f_{\cos},P\big)\le E^{\cos}_{P,L}(f). \tag{2.14}$$

Proof. With the notations in the proof of Theorem 2.4 we have

$$\begin{aligned}\sqrt{L}E^{\cos}_{P,L}(f)&=\max_{v\in[0,\,P/2]}|\hat f_{\cos}(v)-c_{P,L}(v)|=\max_{v\in[0,\,P/2]}|\sigma_2(v)-\sigma_1(v)|\\&\ge\max\{\left|\sigma_2\left(\tfrac{P}{2}\right)\right|-\left|\sigma_1\left(\tfrac{P}{2}\right)\right|,\,\left|\sigma_1\left(\tfrac{P}{2}\right)\right|-\left|\sigma_2\left(\tfrac{P}{2}\right)\right|\}.\end{aligned}$$

1. The assumption $\operatorname{sign}(\hat f_{\cos}(\frac{P}{2}))=\operatorname{sign}\left(\sum_{k=1}^{\infty}\hat f_{\cos}\left(\frac{P}{2}+kP\right)\right)$ implies

$$\left|\sigma_1\left(\tfrac{P}{2}\right)\right|=\left|\hat f_{\cos}\left(\tfrac{P}{2}\right)\right|+2\Big|\sum_{k=1}^{\infty}\hat f_{\cos}\left(\tfrac{P}{2}+kP\right)]\Big|\ge\left|\hat f_{\cos}\left(\tfrac{P}{2}\right)\right|.$$

Assertion (2.13) follows now with $\sigma_2\left(\frac{P}{2}\right)\le L\delta_+(f,L)$.
2. If f is non-negative and monotonically decreasing on $\mathbb{R}_+$, then each rectangle of area $\frac{1}{P}f\left(\frac{n}{P}\right)$ dominates $\int_{n/P}^{(n+1)/P}f(x)\,\mathrm{d}x$, whence

$$\left|\sigma_2(0)\right|=\tfrac{1}{P}\Big|\sum_{n=1+LP/2}^{\infty}f\left(\tfrac{n}{P}\right)\,\cos(0)\Big|\ge\int\limits_{\frac{1+LP/2}{P}}^{\infty}f(x)\,\mathrm{d}x=\int\limits_{\frac{L}{2}+\frac{1}{P}}^{\infty}f(x)\,\mathrm{d}x.$$

Thus $|\hat f_{\cos}(0)-c_{P,L}(0)|\ge|\sigma_2(0)|-|\sigma_1(0)|$ with $\sigma_1(0)\le 2\delta_+(\hat f_{\cos},P)$ yields (2.14). ■

2.3 Reconstruction of f from finite equidistant function values

Our results from Subsection 2.2 also yield an approximate reconstruction formula for f from the given values $f(\frac{n}{P})$, $n=0,\dots,\frac{LP}{2}$. Throughout this subsection we assume $L\le P$, since the samples $\hat f_{\cos}\left(\frac{n}{P}\right)-c_{P,L}\left(\frac{n}{P}\right)$ are taken at arguments $\frac{n}{P}\in\left[0,\frac{L}{2}\right]$, whereas (2.11) controls this difference only on $\left[0,\frac{P}{2}\right]$. Applying (2.3) with L replaced by P, we derive similarly as in (2.11) that

$$\tfrac{1}{\sqrt{L}}\max_{x\in[0,\,P/2]}\Big|f(x)-\tfrac{4}{P}\sideset{}{'}\sum_{n=0}^{LP/2}\hat f_{\cos}\left(\tfrac{n}{P}\right)\,\cos\tfrac{2\pi nx}{P}\Big|\le\tfrac{2}{\sqrt{L}}\,\delta_+(f,P)+4\sqrt{L}\,\delta_+(\hat f_{\cos},L)\,, \tag{2.15}$$

where the prime indicates that the term for $n=0$ is halved. The factor $\frac{4}{P}$ is dictated by the right-hand side of (2.3). Replacing the samples $\hat f_{\cos}(\frac{n}{P})$ by $c_{P,L}(\frac{n}{P})$, we set

$$f_{P,L}(x):=\tfrac{4}{P}\sideset{}{'}\sum_{n=0}^{LP/2}c_{P,L}\left(\tfrac{n}{P}\right)\cos\tfrac{2\pi nx}{P}.$$

For the reconstruction error we obtain from (2.11) and (2.15) that

$$
\begin{aligned}
&\max_{x\in[0,\frac{P}{2}]} |f(x) - f_{P,L}(x)| \\
&\le \max_{x\in[0,\frac{P}{2}]} \Big| f(x) - \tfrac{4}{P} \sum_{n=0}^{LP/2}{}' \hat{f}_{\cos}\big(\tfrac{n}{P}\big) \cos\tfrac{2\pi n x}{P} \Big| + \max_{x\in[0,\frac{P}{2}]} \Big| \tfrac{4}{P} \sum_{n=0}^{LP/2}{}' \big(\hat{f}_{\cos}\big(\tfrac{n}{P}\big) - c_{P,L}\big(\tfrac{n}{P}\big)\big) \cos\tfrac{2\pi n x}{P} \Big| \\
&\le 2\,\delta_+(f,P) + 4L\,\delta_+(\hat{f}_{\cos},L) + \big(2L + \tfrac{4}{P}\big)\big(2\delta_+(\hat{f}_{\cos},P) + L\,\delta_+(f,L)\big)\,, \qquad (2.16)
\end{aligned}
$$

where we use that the second primed sum has $\frac{LP}{2}+1$ terms. For the balanced choice $L = P$ this simplifies to

$$\max_{x\in[0,\frac{P}{2}]} |f(x) - f_{P,L}(x)| \le 8\big(P + \tfrac{1}{P}\big)\,\delta_+(\hat{f}_{\cos},P) + \big(2P^2+6\big)\,\delta_+(f,P)\,,$$

and sufficient decay properties of f then lead by Table 2.1 to corresponding error estimates.

2.4 Parametric quadrature error for rational and meromorphic functions

We apply Theorem 2.4 to a class of rational functions and show how L and P should be chosen. Further, we will give an example of a meromorphic function for which the estimates are almost tight and the error decays geometrically. We focus on $\hat{f}_{\cos}$, similar observations hold for $\hat{f}_{\sin}$.

We start with considering a rational function of the form

$$f(x) = \sum_{j=1}^{J}\sum_{k=1}^{k_j} c_{k,j}(x^2 - z_j^2)^{-k}, \quad x \in \mathbb{R}_+, \qquad (2.17)$$

with $c_{k,j} \in \mathbb{C}$ and pairwise distinct poles $z_1^2,\ldots,z_J^2$ of orders $k_1,\ldots,k_J$, where $z_1,\ldots,z_J$ are assumed to be in the open lower half plane. We set

$$s_0 := \min_{j=1,\ldots,J} |\Im(z_j)| > 0\,, \qquad k_{\max} := \max_{j=1,\ldots,J} k_j\,. \qquad (2.18)$$

Then it can simply be observed that $f \in \mathcal{W}_+$, since $f(x) = \mathcal{O}(x^{-2})$ for $x \to \infty$. For its Fourier cosine transform we observe

Theorem 2.8. *For f of the form* (2.17) *its Fourier cosine transform is given by*

$$\hat{f}_{\cos}(v) = \sum_{j=1}^{J}\sum_{k=1}^{k_j} c_{k,j}\Big(\frac{\pi v}{\mathrm{i}z_j}\Big)^{k-\frac{1}{2}} \frac{\sqrt{\pi}}{(k-1)!} K_{k-\frac{1}{2}}(2\pi\mathrm{i}vz_j), \qquad v \in \mathbb{R}_+,$$

where K_n with $n > 0$ denotes the modified Bessel function of second kind. In particular,

$$|\hat{f}_{\cos}(v)| \le d_0\,(1+v)^{k_{\max}-1}\,\mathrm{e}^{-2\pi s_0 v},$$

for $v \in \mathbb{R}_+$, where the constant $d_0 > 0$ depends on $c_{k,j}$, z_j, and k_j.

Proof. We employ the following integral formula (see [12, Formula 3.773(8)])

$$\int_0^\infty \frac{\cos(ax)}{(x^2+\beta^2)^{n+1/2}}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2^n\beta^n\Gamma(n+\frac{1}{2})} a^n\,K_n(a\beta) \quad \text{for } a > 0,\ \Re(\beta) > 0,\ n > -\tfrac{1}{2}, \qquad (2.19)$$

with the modified Bessel function of second kind K_n. Here we set $a = 2\pi v$, $\beta = \mathrm{i}z_j$ such that $\beta^2 = -z_j^2$, and $n + \frac{1}{2} = k$. Then

$$\int_0^\infty (x^2 - z_j^2)^{-k}\,\cos(2\pi x v)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2^{k-\frac{1}{2}}(\mathrm{i}z_j)^{k-\frac{1}{2}}\Gamma(k)}(2\pi v)^{k-\frac{1}{2}}\,K_{k-\frac{1}{2}}(2\pi\mathrm{i}vz_j)$$

$$= \left(\tfrac{\pi v}{\mathrm{i} z_j}\right)^{k-\frac{1}{2}} \tfrac{\sqrt{\pi}}{(k-1)!}\, K_{k-\frac{1}{2}}(2\pi \mathrm{i} v z_j).$$

This formula is a generalization of [22, p. 8, Formula 2.7]. For shortness, we set

$$\varphi_{k,j}(v) := \left(\tfrac{\pi v}{\mathrm{i} z_j}\right)^{k-\frac{1}{2}} \tfrac{\sqrt{\pi}}{(k-1)!}\, K_{k-\frac{1}{2}}(2\pi \mathrm{i} v z_j)\, \mathrm{e}^{2\pi s_0 v}\, (1+v)^{1-k}\,, \quad v > 0\,.$$

Using the asymptotic expansions (see [1, 9.6.9 and 9.7.2])

$$K_{k-\frac{1}{2}}(w) = \begin{cases} \frac{1}{2}\,\Gamma\big(k-\frac{1}{2}\big)\big(\frac{2}{w}\big)^{k-\frac{1}{2}}\big(1+\mathcal{O}(|w|^2)\big)\,, & w \to 0\,, \\ \sqrt{\frac{\pi}{2w}}\, \mathrm{e}^{-w}\big(1+\mathcal{O}(|w|^{-1})\big)\,, & w \to \infty\,, \end{cases}$$

with $w = 2\pi \mathrm{i} v z_j$, we see that finite limits of $\varphi_{k,j}(v)$ for $v \to +0$ and $v \to \infty$ exist. Hence each function $\varphi_{k,j}$ is continuous and bounded on $\mathbb{R}_+$. Estimating

$$\begin{aligned} |\hat{f}_{\cos}(v)|\, \mathrm{e}^{2\pi s_0 v}\, (1+v)^{1-k_{\max}} \quad &\leq \quad \sum_{j=1}^{J} \sum_{k=1}^{k_j} |c_{k,j}|\, |\varphi_{k,j}(v)|\, (1+v)^{k-k_{\max}} \\ &\leq \quad d_0 := \sup_{v \in \mathbb{R}_+} \sum_{j=1}^{J} \sum_{k=1}^{k_j} |c_{k,j}|\, |\varphi_{k,j}(v)|\,, \end{aligned}$$

we obtain the assertion of Theorem 2.8, since $(1+v)^{k-k_{\max}} \leq 1$ for all $v \in \mathbb{R}_+$. ■

Remark 2.9. *The function* $f(x) = \sum_{j=1}^{J} c_j (x^2 - z_j^2)^{-1}$ *with* $\Im(z_j) < 0$, $j = 1, \ldots, J$, *and* s_0 *in* (2.18) *satisfies the decay property* $\sup_{x \in \mathbb{R}_+} |f(x)|(1+x)^2 \leq c$ *with the constant*

$$c := \sum_{j=1}^{J} |c_j| \sup_{x \in \mathbb{R}_+} \tfrac{(1+x)^2}{|x^2 - z_j^2|}\,.$$

If all poles are purely imaginary, $z_j = -\mathrm{i} s_j$ *with* $s_j > 0$, *then* $\sup_{x \in \mathbb{R}_+} \frac{(1+x)^2}{x^2 + s_j^2} = 1 + s_j^{-2}$, *attained at* $x = s_j^2$, *so that* $c \leq \big(1 + \frac{1}{s_0^2}\big) \sum_{j=1}^{J} |c_j|$. *Using* $K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}$, *we obtain*

$$\hat{f}_{\cos}(v) = \sum_{j=1}^{J} c_j \left(\tfrac{\pi v}{\mathrm{i} z_j}\right)^{1/2} \sqrt{\pi}\, K_{\frac{1}{2}}(2\pi \mathrm{i} v z_j) = -\tfrac{\mathrm{i}\pi}{2} \sum_{j=1}^{J} \tfrac{c_j}{z_j}\, \mathrm{e}^{-2\pi \mathrm{i} v z_j}.$$

In particular,

$$|\hat{f}_{\cos}(v)| \leq \tfrac{\pi}{2} \sum_{j=1}^{J} \tfrac{|c_j|}{|z_j|}\, \mathrm{e}^{-2\pi v |\Im(z_j)|} \leq d_0\, \mathrm{e}^{-2\pi v s_0} \quad \textit{with} \quad d_0 \leq \tfrac{\pi}{2 s_0} \sum_{j=1}^{J} |c_j|,$$

where we used $|z_j| \geq |\Im(z_j)| \geq s_0$. *Thus* $\hat{f}_{\cos}(v)$ *possesses an exponential decay for* $v \in \mathbb{R}_+$ *as in Table* 2.1 *with* $d = d_0$, $\beta = 1$, *and* $s = 2\pi s_0$.

Corollary 2.10. *Let* $f(x) = \sum_{j=1}^{J} c_j (x^2 - z_j^2)^{-1}$ *as in Remark 2.9. Then, for all* $L,\ P \in 2\mathbb{N}$,

$$E_{P,L}^{\cos}(f) \ \le\ 4d_0\, L^{-1/2}\, \mathrm{e}^{-\pi s_0 P} \ +\ \tfrac{\pi^2}{2}\, c\, L^{-3/2} \tag{2.20}$$

with c *and* d_0 *as in Remark* 2.9 *and* s_0 *in (2.18). Moreover, if* $c_j \in \mathbb{R}_+$, $\Re(z_j) = 0$ *and* $2\big(|z_j| + \frac{1}{P}\big) \le L$ *for all* j*, then*

$$E_{P,L}^{\cos}(f) \ge c_0\, L^{-3/2} - 4d_0\, L^{-1/2}\, \mathrm{e}^{-\pi s_0 P} \tag{2.21}$$

with $c_0 = \sum_{j=1}^{J} c_j$*, showing that upper and lower bound exhibit the same decay* $L^{-3/2}$ *in* L *and the same geometric decay in* P*. The plateau reached for large* P *is therefore of the correct order* $L^{-3/2}$*, and only the constants differ.*

Proof. The proof of (2.20) follows directly by application of Theorem 2.4 together with Table 2.1. To show (2.21) we apply the inequality (2.14) in Corollary 2.7. By assumption f in (2.17) is real-valued and monotonically decreasing, and with $A := \frac{L}{2} + \frac{1}{P}$ we find

$$\begin{aligned}\int_A^{\infty} f(x)\ \mathrm{d}x &= \sum_{j=1}^{J} c_j \int_A^{\infty} (x^2 - z_j^2)^{-1}\ \mathrm{d}x = \sum_{j=1}^{J} \tfrac{c_j}{|z_j|} \operatorname{arccot}\big(\tfrac{A}{|z_j|}\big) \ge \sum_{j=1}^{J} \tfrac{c_j}{|z_j|}\, \tfrac{1}{1+\frac{A}{|z_j|}} \\ &= \sum_{j=1}^{J} \tfrac{c_j}{A+|z_j|} = \tfrac{2}{L} \sum_{j=1}^{J} \tfrac{c_j}{1+\frac{2}{L}\left(\frac{1}{P}+|z_j|\right)} \ \ge\ \tfrac{1}{L} \sum_{j=1}^{J} c_j\,,\end{aligned}$$

where we have used that $\operatorname{arccot}(x) \ge \frac{1}{x+1}$ for $x \ge 0$ and $\frac{2}{L}\big(\frac{1}{P} + |z_j|\big) \le 1$. ■

The first term in (2.20) decays geometrically with rate $\mathrm{e}^{-\pi s_0}$ whereas the second term decays only algebraically, since a rational function has merely polynomial decay on $\mathbb{R}_+$. With this knowledge it is meaningful to choose L essentially larger than P in such a setting. See Example 4.2 and Figure 4.1 for numerical illustration.

Next we show that for meromorphic functions, both error terms, $\delta_+(\hat{f}_{\cos}, P)$ and $\delta_+(f, L)$ can decay geometrically and moreover the error estimates are tight. We refer to [31, 32, 33] for further studies of analytic and meromorphic functions with geometric decay properties in time and frequency.

Theorem 2.11. *Let* $P \in 2\mathbb{N}$ *and* $P \ge 8$*. Then we find for* $f(x) := \operatorname{sech}(\pi x)$ *and* $L = P$ *the estimate*

$$\tfrac{1}{4\sqrt{P}}\, \mathrm{e}^{-\pi P/2} \ \le\ E_{P,P}^{\cos}(f) \ \le\ \tfrac{1.64}{\sqrt{P}}\, \mathrm{e}^{-\pi P/2}. \tag{2.22}$$

Proof. The function $f(x) := \operatorname{sech}(\pi x) = \frac{2}{e^{\pi x}+e^{-\pi x}}$ is meromorphic with the simple poles $\mathrm{i}\,(k+\frac{1}{2})$, $k \in \mathbb{Z}$, and similarly as in (2.18) we set $s_0 := \min_{k\in\mathbb{Z}} |\Im(\mathrm{i}\,(k+\frac{1}{2}))| = \frac{1}{2}$. The Fourier cosine transform is given by

$$\hat{f}_{\cos}(v) = \int_0^{\infty} \operatorname{sech}(\pi x)\ \cos(2\pi x v)\ \mathrm{d}x = \tfrac{1}{2}\ \operatorname{sech}(\pi v)\,, \quad v \in \mathbb{R}_+\,.$$

As in the proofs of Theorem 2.4 and Corollary 2.7, $\sqrt{P}\, E_{P,P}^{\cos}(f) = \max_{v\in[0,P/2]} |\sigma_2(v) - \sigma_1(v)|$, so that

$$\big|\sigma_1\big(\tfrac{P}{2}\big)\big| - \big|\sigma_2\big(\tfrac{P}{2}\big)\big| \le \sqrt{P}\, E_{P,P}^{\cos}(f) \le \max_{v\in[0,\,P/2]} \big(\big|\sigma_1(v)\big| + \big|\sigma_2(v)\big|\big). \tag{2.23}$$

We observe that $\operatorname{sech}(t) \in [\mathrm{e}^{-t},\, 2\mathrm{e}^{-t}]$ is monotonically decreasing for $t \geq 0$, such that

$$\left|\sigma_1\!\left(\tfrac{P}{2}\right)\right| = \hat{f}_{\cos}\!\left(\tfrac{P}{2}\right) + 2\sum_{k=1}^{\infty} \hat{f}_{\cos}\!\left(\tfrac{P}{2} + kP\right) \geq \tfrac{1}{2}\operatorname{sech}\tfrac{\pi P}{2} \;\geq\; \tfrac{1}{2}\,\mathrm{e}^{-\pi P/2}\,,$$

Since $\operatorname{sech}''(t) = \operatorname{sech}(t)\big(1 - 2\operatorname{sech}^2 t\big) > 0$ for $t > \operatorname{arsech}\frac{1}{\sqrt{2}} = 0.8814$, the function sech is convex on $[0.89, \infty)$. For $P \geq 8$, $k \in \mathbb{N}$ and $v \in [0, \frac{P}{2}]$ all arguments $\pi(kP \pm v)$ lie in $[4\pi, \infty)$, so that $v \mapsto \hat{f}_{\cos}(v + kP) + \hat{f}_{\cos}(kP - v)$ is convex and increasing in v on $[0, \frac{P}{2}]$, hence maximal at $v = \frac{P}{2}$. Summing over k gives $|\sigma_1(v)| \leq \sigma_1\!\left(\frac{P}{2}\right)$ and

$$\left|\sigma_1(v)\right| \leq \tfrac{1}{2}\operatorname{sech}\tfrac{\pi P}{2} + \sum_{k=1}^{\infty}\operatorname{sech}\left(\pi(\tfrac{P}{2} + kP)\right) \leq \mathrm{e}^{-P\pi/2} + 2\sum_{k=1}^{\infty}\mathrm{e}^{-\frac{P\pi}{2}(1+2k)} < \mathrm{e}^{-P\pi/2}\big(1 + 10^{-10}\big)\,.$$

On the other hand, $\sigma_2\!\left(\frac{P}{2}\right) = \frac{1}{P}\sum_{n>P^2/2}(-1)^n\operatorname{sech}\frac{\pi n}{P}$ is an alternating series with monotonically decreasing terms, whence $\left|\sigma_2\!\left(\frac{P}{2}\right)\right| \leq \frac{1}{P}\operatorname{sech}\frac{\pi P}{2} \leq \frac{2}{P}\mathrm{e}^{-\pi P/2}$. To obtain an upper bound for $\left|\sigma_2(v)\right|$ for all $v \in [0, \frac{P}{2}]$, we use $\operatorname{sech}(t) \leq 2\mathrm{e}^{-t}$ and estimate for every $v \in [0, \frac{P}{2}]$,

$$|\sigma_2(v)| \leq \tfrac{2}{P}\sum_{n>P^2/2}\mathrm{e}^{-\pi n/P} = \tfrac{2}{P}\,\tfrac{\mathrm{e}^{-\pi P/2}}{\mathrm{e}^{\pi/P}-1} = \left(\tfrac{2}{\pi}\,\tfrac{t}{\mathrm{e}^{t}-1}\Big|_{t=\pi/P}\right)\mathrm{e}^{-\pi P/2} \leq \tfrac{2}{\pi}\,\mathrm{e}^{-\pi P/2}\,,$$

since $t \mapsto \frac{t}{\mathrm{e}^t - 1}$ is decreasing with limit 1 at $t = 0$. Thus we obtain from (2.23) the estimate

$$\tfrac{1}{\sqrt{P}}\big(\tfrac{1}{2}\mathrm{e}^{-\pi P/2} - \tfrac{2}{P}\mathrm{e}^{-\pi P/2}\big) \leq E^{\cos}_{P,P}(f) \leq \tfrac{1}{\sqrt{P}}\big(\tfrac{2}{\pi}\mathrm{e}^{-\pi P/2} + \mathrm{e}^{-\pi P/2}(1 + 10^{-10})\big),$$

and the assertion follows for $P \geq 8$, since $\frac{1}{2} - \frac{2}{P} \geq \frac{1}{4}$ and $\frac{2}{\pi} + 1 + 10^{-10} < 1.64$. ■

For comparison, employing the notations of Table 2.1 we observe that both $f(x) := \operatorname{sech}(\pi x)$ and $\hat{f}_{\cos}(v) = \frac{1}{2}\operatorname{sech}(\pi v)$ have exponential decay with $\alpha = \beta = 1$ at the critical rate $r = s = 2\pi s_0 = \pi$ with the constants

$$c = \sup_{x \in \mathbb{R}_+}\operatorname{sech}(\pi x)\,\mathrm{e}^{\pi x} = 2\,, \qquad d = \sup_{v \in \mathbb{R}_+}\tfrac{1}{2}\operatorname{sech}(\pi v)\,\mathrm{e}^{\pi v} = 1\,.$$

Theorem 2.4 with Table 2.1 (second row twice) then yields

$$E^{\cos}_{P,L}(f) \leq \tfrac{2}{\sqrt{L}}\Big(2d\mathrm{e}^{-s\left(\frac{P}{2}\right)^{\beta}}\Big) + \sqrt{L}\Big(2c\mathrm{e}^{-r\left(\frac{L}{2}\right)^{\alpha}}\Big) = \tfrac{4}{\sqrt{L}}\,\mathrm{e}^{-\frac{\pi P}{2}} + 4\sqrt{L}\,\mathrm{e}^{-\frac{\pi L}{2}}\,. \tag{2.24}$$

For the balanced choice $L = P$ both terms are of order $\mathrm{e}^{-\pi P/2}$, see Example 4.3 and Figure 4.2 (left) for numerical illustration.

Remark 2.12. *The upper and the lower bound in Theorem* 2.11 *differ by the constant factor* $1.64/0.25 = 6.6$, *which is independent of* P. *Both bounds therefore exhibit the same decay* $P^{-1/2}\mathrm{e}^{-\pi P/2}$, *so that the rate in* (2.22) *is sharp for* $L = P$ *and neither of the two error parts* σ_1, σ_2 *dominates the other. The bound* (2.24) *shows that for* $L \neq P$ *one of the terms* $L^{-1/2}\mathrm{e}^{-\pi P/2}$, $L^{1/2}\mathrm{e}^{-\pi L/2}$ *becomes dominant, so that* $L = P$ *is the optimal balance.*

Remark 2.13. *1. Similar results as in Theorem* 2.8 *and Corollary* 2.10 *can be also derived for the Fourier sine integral of functions of the form* $x f(x)$ *with* $f(x)$ *in* (2.17).
2. Considering the meromorphic function $f(x) = \tanh(\pi x)\,\mathrm{sech}(\pi x)$ *with poles of order two at* $\mathrm{i}\,(k+\frac{1}{2})$*,* $k \in \mathbb{Z}$ *(thus again* $s_0 = \frac{1}{2}$*), the Fourier sine integral is given by*

$$\hat{f}_{\sin}(v) = v\,\mathrm{sech}(\pi v)$$

for $v \in \mathbb{R}_+$*. We find* $\sup_{x\in\mathbb{R}_+} |f(x)|\,\mathrm{e}^{\pi x} = 2$*, so that* f *satisfies the exponential decay condition of Table* 2.1 *with* $r = \pi$*,* $\alpha = 1$*,* $c = 2$*. For* $\hat{f}_{\sin}$*, however, the exponential row of Table* 2.1 *is* not *applicable at the critical rate* $s = \pi$*,* $\beta = 1$*. We have* $\sup_{v\in\mathbb{R}_+} |\hat{f}_{\sin}(v)|\,\mathrm{e}^{\pi v} = \sup_{v\in\mathbb{R}_+} \frac{2v}{1+\mathrm{e}^{-2\pi v}} = \infty$*, so that no finite constant* d *exists. Instead we estimate* $\delta_+(\hat{f}_{\sin}, P)$ *directly. For* $|v| \le \frac{P}{2}$ *and* $k \in \mathbb{N}$ *we have* $v + kP \ge (k-\frac{1}{2})P$ *and* $v + kP \le (k+\frac{1}{2})P$*, whence for* $P \ge 8$

$$\delta_+(\hat{f}_{\sin}, P) \le 2P \sum_{k=1}^{\infty} \big(k+\tfrac{1}{2}\big)\,\mathrm{e}^{-\pi(k-\frac{1}{2})P} \le 4\,P\,\mathrm{e}^{-\pi P/2}\,.$$

Thus $\hat{f}_{\sin}$ *decays geometrically at the critical rate up to the additional algebraic factor* P*, and* $E^{\sin}_{P,L}(f)$ *still converges geometrically for* $L = P$*, see Figure* 4.2 *(right). This extra factor* P *is exactly the reason why the sine case behaves differently from the cosine case in Example* 4.3.

3 Parametric quadrature to compute Chebyshev coefficients

It is well-known that a smooth function h can be well approximated on I using an expansion into Chebyshev polynomials,

$$\sum_{n=0}^{\infty}{}' a_n[h]\,T_n(x) := \tfrac{1}{2}a_0[h] + \sum_{n=1}^{\infty} a_n[h]\,T_n(x)\,, \quad x \in I\,, \tag{3.1}$$

where the *Chebyshev coefficients* $a_n[h]$ are defined in (1.4). Throughout this section, we assume that $h \in C(I)$ and $\sum_{n=0}^{\infty} |a_n[h]| < \infty$, then the Chebyshev series (3.1) converges absolutely and uniformly on I to h, see also [26, Chapter 6]). The decay condition $\sum_{n=0}^{\infty} |a_n[h]| < \infty$ is always satisfied for $h \in C^1(I)$, see e.g. [26, Theorem 6.11]. If we want to apply this expansion (3.1) numerically, we have to employ two approximation steps.

1. Efficient quadrature to compute the Chebyshev coefficients. Given the function values $h(x_j^{(P)})$ at the Chebyshev points $x_j^{(P)} := \cos\frac{\pi j}{P}$, $j = 0, \ldots, P$, we employ for $a_n[h]$ in (1.4) the quadrature formula

$$a_n^{(P)}[h] := \tfrac{2}{P} \sum_{j=0}^{P}{}'' h\big(x_j^{(P)}\big)\,T_n\big(x_j^{(P)}\big) = \tfrac{2}{P} \sum_{j=0}^{P}{}'' h\big(x_j^{(P)}\big)\,\cos\tfrac{\pi j n}{P}\,, \quad n = 0, 1, \ldots, P\,, \tag{3.2}$$

where the double prime indicates that the first and last terms are multiplied by $\frac{1}{2}$. The values $a_n^{(P)}[h]$, $n = 0, \ldots, P$, are the output of a DCT-I$(P+1)$, see Remark 2.6, and are computed with $\mathcal{O}(P \log P)$ arithmetical operations (see [25] or [26, Section 6.3]). If h is even resp. odd, then the coefficients $a_n[h]$ and $a_n^{(P)}[h]$ with odd resp. even index vanish. A very useful generalization of (3.2) was given by H. Wang and D. Huybrechs [35] (see Subsection 3.1).

2. Suitable truncation of the Chebyshev series. Having computed approximative coefficients $a_n^{(P)}[h]$, $n = 0, \ldots, P$, we fix a truncation number $L \le P$ and compute the truncated Chebyshev series in the form

$$t_{P,L}(x) := \tfrac{1}{2} a_0^{(P)}[h] + \sum_{n=1}^{L-1} a_n^{(P)}[h]\, T_n(x) + \varepsilon_L^{(P)} a_L^{(P)}[h] T_L(x), \quad \varepsilon_n^{(P)} := \begin{cases} \frac{1}{2} & n = P, \\ 1 & n < P. \end{cases} \tag{3.3}$$

We call $t_{P,L}$ the *Chebyshev sampling polynomial* of degree L. For $L = P$, the function $t_{P,P}$ is precisely the interpolation polynomial of degree P with $t_{P,P}\big(x_j^{(P)}\big) = h\big(x_j^{(P)}\big)$, $j = 0, \ldots, P$. Substituting $x = \cos(2\pi y)$ with $y \in \big[0, \frac{1}{2}\big]$ gives $T_n(x) = \cos(2\pi n y)$, so that $t_{P,L}$ can be evaluated at an equispaced grid using a fast DCT-I$(L+1)$ and at an *arbitrary nonequispaced grid* by an NDCT$(L+1)$, see also [28, Algorithm 2.3].

We study the *(parametric) quadrature error* and the *reconstruction error*

$$E_P^{\mathrm{Ch}}(h) := \max_{n=0,\ldots,P} \big| a_n[h] - \varepsilon_n^{(P)} a_n^{(P)}[h] \big| \,, \tag{3.4}$$

$$R_{P,L}^{\mathrm{Ch}}(h) := \| h - t_{P,L} \|_{C(I)} = \max_{x \in I} \big| h(x) - t_{P,L}(x) \big| \,, \tag{3.5}$$

and ask how they depend on the smoothness of h and on the choice of P and L.

3.1 Aliasing formula and decay rates of Chebyshev coefficients

We first introduce counterparts of the Poisson summation formulas in Lemma 2.1 and of the decay rate (2.7). The close connection between the Chebyshev coefficients $a_n[h]$ and the discrete coefficients $a_n^{(P)}[h]$ is described by the *aliasing formula for Chebyshev coefficients*.

Lemma 3.1. *Let $P \in \mathbb{N} \setminus \{1\}$ be fixed. If $h \in C(I)$ satisfies $\sum_{n=0}^{\infty} |a_n[h]| < \infty$, then for every $n = 0, \ldots, P$ the aliasing formula*

$$a_n^{(P)}[h] = a_n[h] + \sum_{k=1}^{\infty} \big(a_{n+2kP}[h] + a_{2kP-n}[h] \big) \,, \tag{3.6}$$

is satisfied, where both series converge absolutely. In particular, for $n = 0$ and $n = P$ we have

$$\begin{aligned} a_0^{(P)}[h] &= a_0[h] + 2 \sum_{k=1}^{\infty} a_{2kP}[h] \,, \\ a_P^{(P)}[h] &= 2 \sum_{k=1}^{\infty} a_{(2k-1)P}[h] \,. \end{aligned}$$

Proof. By assumption, the Chebyshev series (3.1) converges absolutely and uniformly on I to h (see [26, Chapter 6]), such that on the $(P+1)$-point Chebyshev grid we have

$$h\big(x_j^{(P)}\big) = \tfrac{1}{2}\, a_0[h] + \sum_{k=1}^{\infty} a_k[h] \, \cos \tfrac{jk\pi}{P} \,, \quad j = 0, \ldots, P \,.$$

Inserting this into (3.2) and using $2 \cos\varphi \cos\psi = \cos(\varphi - \psi) + \cos(\varphi + \psi)$, we obtain

$$a_n^{(P)}[h] = \tfrac{1}{P}\, a_0[h] \sum_{j=0}^{P}{}'' \cos \tfrac{\pi j n}{P} + \tfrac{1}{P} \sum_{k=1}^{\infty} a_k[h] \sum_{j=0}^{P}{}'' \Big(\cos \tfrac{\pi j (k-n)}{P} + \cos \tfrac{\pi j (k+n)}{P} \Big) \,.$$

We apply the relation

$$\tfrac{1}{P} \sum_{j=0}^{P}{}'' \cos \tfrac{\pi j m}{P} = \begin{cases} 1 & m \in 2P\,\mathbb{Z} \,, \\ 0 & m \in \mathbb{Z} \setminus 2P\,\mathbb{Z} \,, \end{cases}$$

for $m = n$, $m = k - n$, and $m = k + n$, and obtain exactly the summation indices k of the form $k = n + 2mP$ or $k = 2mP - n$ with $m \in \mathbb{N}$, this yields (3.6). ■

Formula (3.6) directly implies representations of the *aliasing remainder*. We obtain

$$r_n^{(P)}[h] := a_n^{(P)}[h] - a_n[h] = \sum_{k=1}^{\infty} \big(a_{n+2kP}[h] + a_{2kP-n}[h]\big)\,, \qquad n = 0, \ldots, P, \tag{3.7}$$

and in particular $r_0^{(P)}[h] = 2\sum_{k=1}^{\infty} a_{2kP}[h]$, and $r_P^{(P)}[h] = a_P[h] + 2\sum_{k=1}^{\infty} a_{(2k+1)P}[h]$. Every $r_n^{(P)}[h]$ with $n \le P-1$ involves only Chebyshev coefficients $a_m[h]$ with indices $m > P$. This motivates us to define the *Chebyshev decay rate of h with respect to the truncation parameter* $N \in \mathbb{N}$,

$$\delta_{\mathrm{Ch}}(h, N) := \sum_{n=N+1}^{\infty} |a_n[h]|\,, \tag{3.8}$$

the analogue of the one-sided decay rate (2.7).

We now derive the Chebyshev decay rate from smoothness properties of h, in the polynomial and in the exponential case.

Lemma 3.2 ([26, Theorem 6.16])**.** *Let $r \in \mathbb{N}_0$ and $h \in C^{r+1}(I)$. Then*

$$|a_n[h]| \le \frac{2\,\|h^{(r+1)}\|_{C(I)}}{n\,(n-1)\cdots(n-r)}\,, \quad n > r\,. \tag{3.9}$$

If in addition $r \ge 1$, then

$$\delta_{\mathrm{Ch}}(h, N) \le \frac{2\,\|h^{(r+1)}\|_{C(I)}}{r\,(N-r)^r}\,, \quad N > r\,. \tag{3.10}$$

Proof. For the bound of $|a_n[h]|$ in (3.9) we refer to [26, Theorem 6.16]. Let $r \ge 1$. Summing over $n \ge N+1$ and using an integral comparison yields

$$\delta_{\mathrm{Ch}}(h, N) \le 2\,\|h^{(r+1)}\|_{C(I)} \sum_{n=N+1}^{\infty} \frac{1}{(n-r)^{r+1}} \le 2\,\|h^{(r+1)}\|_{C(I)} \int_N^{\infty} \frac{\mathrm{d}t}{(t-r)^{r+1}} = \frac{2\,\|h^{(r+1)}\|_{C(I)}}{r\,(N-r)^r}\,.$$

■

For $h \in C^\infty(I)$, D. Elliott [6] gave the estimate $\big|a_n[h]\big| \le \frac{1}{2^{n-1}\,n!} \max_{x\in I} \big|h^{(n)}(x)\big|$, $n \in \mathbb{N}_0$. Sharper geometric rates are available for real-analytic functions h, which can be analytically continued to a neighborhood of I in the complex plane. For this reason it is very useful to represent the related Chebyshev coefficients $a_n[h]$ as contour integrals (see [7, 35]). Let E_{ρ_0} with $\rho_0 > 1$ be the *Bernstein ellipse*

$$E_{\rho_0} := \big\{z = \tfrac{1}{2}\,\rho_0\,\mathrm{e}^{\mathrm{i}\,\theta} + \tfrac{1}{2}\,\rho_0^{-1}\,\mathrm{e}^{-\mathrm{i}\,\theta} : \theta \in [0,\, 2\pi)\big\}\,,$$

with center 0, foci ±1, and semi-axes $\frac{1}{2}\left(\rho_0 \pm \rho_0^{-1}\right)$. By $\mathcal{E}_{\rho_0}$ we denote the open *Bernstein region* bounded by E_{ρ_0} being a complex neighborhood of I.

From the recursion $T_{n+1}(z) = 2z\,T_n(z) - T_{n-1}(z)$, $T_0(z) = 1$, $T_1(z) = z$, one obtains by induction $T_n\big(\frac{1}{2}z + \frac{1}{2}z^{-1}\big) = \frac{1}{2}z^n + \frac{1}{2}z^{-n}$ for $z \in \mathbb{C}\setminus\{0\}$ and $n \in \mathbb{N}_0$.

Assume that h can be extended analytically to $\mathcal{E}_{\rho_0}$. Since the Joukowski map $z \mapsto \frac{1}{2}(z + z^{-1})$ maps $|z| = \rho$ onto E_ρ and is invariant under $z \mapsto z^{-1}$, the function $h\big(\frac{1}{2}z + \frac{1}{2}z^{-1}\big)$ is analytic precisely on the *annulus* $\rho_0^{-1} < |z| < \rho_0$ and has there a Laurent expansion

$$h\big(\tfrac{1}{2}z + \tfrac{1}{2}z^{-1}\big) = \sum_{n\in\mathbb{Z}} c_n z^n$$

with coefficients

$$c_n = \frac{1}{2\pi\mathrm{i}} \int_{C_\rho} h\big(\tfrac{1}{2}\,z + \tfrac{1}{2}\,z^{-1}\big)\, z^{-n-1}\,\mathrm{d}z = \frac{1}{2\pi\rho^n} \int_0^{2\pi} h\big(\tfrac{1}{2}\,\rho\,\mathrm{e}^{\mathrm{i}\theta} + \tfrac{1}{2}\rho^{-1}\,\mathrm{e}^{-\mathrm{i}\theta}\big)\mathrm{e}^{-in\theta}\,\mathrm{d}\theta, \quad n \in \mathbb{Z},$$

where C_ρ denotes the circle $\{z \in \mathbb{C} : |z| = \rho\}$. In particular, $c_n = c_{-n}$. Comparing it with the Chebyshev expansion yields the representation of $a_n[h]$ as a *contour integral* (see [30, 35])

$$a_n[h] = 2c_n = \tfrac{1}{\pi\rho^n} \int_0^{2\pi} h\big(\tfrac{1}{2}\,\rho \mathrm{e}^{\mathrm{i}\theta} + \tfrac{1}{2}\,\rho^{-1}\mathrm{e}^{-\mathrm{i}\theta}\big)\,\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta\,, \quad n \in \mathbb{N}_0\,, \tag{3.11}$$

for $1 < \rho < \rho_0$, and hence the bound $|a_n[h]| \le 2\,\rho^{-n} \max_{z\in E_\rho} |h(z)|$, see e.g. [30, Theorem 3.8] or [34, Theorem 8.1]. With $\sigma := \log\rho > 0$ and $c := 2\,\max\{|h(z)| : z \in E_\rho\}$ we obtain

$$|a_n[h]| \le c\,\mathrm{e}^{-\sigma n}\,, \quad n \in \mathbb{N}_0\,. \tag{3.12}$$

Lemma 3.3. *Assume that the Chebyshev coefficients of $h \in C(I)$ satisfy the condition* (3.12) *with some $c > 0$ and $\sigma > 0$. Then the Chebyshev decay rate satisfies*

$$\delta_{\mathrm{Ch}}(h, N) \le \tfrac{c}{\mathrm{e}^{\sigma}-1}\,\mathrm{e}^{-\sigma N}\,, \quad N \in \mathbb{N}\,.$$

Proof. Using (3.12), the estimate follows directly from

$$\delta_{\mathrm{Ch}}(h, N) \le c \sum_{n=N+1}^{\infty} \mathrm{e}^{-\sigma n} = \tfrac{c\,\mathrm{e}^{-\sigma(N+1)}}{1-\mathrm{e}^{-\sigma}} = \tfrac{c}{\mathrm{e}^{\sigma}-1}\,\mathrm{e}^{-\sigma N}.$$

■

Applying an M-point trapezoidal rule to the contour integral in (3.11), the Chebyshev coefficients can alternatively be computed with conveniently chosen $\rho \in (1, \rho_0)$ as

$$a_n^{(M,\rho)}[h] := \tfrac{2}{M\rho^n} \sum_{j=0}^{M-1} h\big(\tfrac{1}{2}\,\rho\,\mathrm{e}^{2\pi\mathrm{i}\,j/M} + \tfrac{1}{2}\,\rho^{-1}\,\mathrm{e}^{-2\pi\mathrm{i}\,j/M}\big)\,\mathrm{e}^{-2\pi\mathrm{i}\,jn/M}$$

stably and with high accuracy by FFT (see [35]). In contrast to (3.2), this sum uses samples of h on the Bernstein ellipse E_ρ rather than on I. In the limit case $\rho = 1$ with $M = 2P$, the quadrature formulas $a_n^{(M,\rho)}[h]$ coincide with (3.2), i.e., $a_n^{(2P,1)}[h] = a_n^{(P)}[h]$ for $n = 0, \ldots, P$.

3.2 Chebyshev quadrature error and reconstruction error

Using the aliasing formula (3.6) and the Chebyshev decay rate, we can now analyse the errors $E_P^{\mathrm{Ch}}(h)$ in (3.4) and $R_{P,L}^{\mathrm{Ch}}(h)$ in (3.5).

Theorem 3.4. *Let $P \in \mathbb{N}\setminus\{1\}$ and $L \le P$ be fixed and let $h \in C(I)$ satisfy $\sum_{n=0}^{\infty} |a_n[h]| < \infty$. Then the parametric quadrature error* (3.4) *satisfies*

$$E_P^{\mathrm{Ch}}(h) \le 2\,\delta_{\mathrm{Ch}}(h, P)\,.$$

Furthermore, the reconstruction error $R_{P,L}^{\mathrm{Ch}}(h) = \|h - t_{P,L}\|_{C(I)}$ in (3.5) *satisfies*

$$R_{P,L}^{\mathrm{Ch}}(h) \le \delta_{\mathrm{Ch}}(h, L) + \delta_{\mathrm{Ch}}(h, P). \tag{3.13}$$

In the special case $L = P$ we have $R_{P,P}^{\mathrm{Ch}}(h) \le 2\,\delta_{\mathrm{Ch}}(h, P)$.

Proof. 1. The stated bound for $E_P^{\mathrm{Ch}}(h)$ directly follows from (3.7) by the triangle inequality. For $n \le P-1$, the occurring indices satisfy $n+2kP \ge 2P > P$ and $2kP-n \ge 2P-(P-1) = P+1$, so that each of the two sums is bounded by $\delta_{\mathrm{Ch}}(h, P)$. For $n = P$ we have $\frac{1}{2}\,a_P^{(P)}[h] - a_P[h] = \sum_{k=1}^{\infty} a_{(2k+1)P}[h]$, whose indices exceed P as well.

2. We estimate the reconstruction error $R_{P,L}^{\mathrm{Ch}}(h)$. We consider first the case $L < P$. Inserting $a_n^{(P)}[h] = a_n[h] + r_n^{(P)}[h]$ into (3.3) yields the error decomposition

$$h(x) - t_{P,L}(x) = \underbrace{\sum_{n=L+1}^{\infty} a_n[h]\, T_n(x)}_{\text{degree truncation}} - \underbrace{\sideset{}{'}\sum_{n=0}^{L} r_n^{(P)}[h]\, T_n(x)}_{\text{sampling aliasing}}\,, \tag{3.14}$$

where the prime indicates that the first term of the second sum is multiplied by $\frac{1}{2}$.

The degree-truncation term is bounded by $\sum_{n=L+1}^{\infty} |a_n[h]| = \delta_{\mathrm{Ch}}(h, L)$ since $|T_n(x)| \le 1$ for $x \in I$. For the sampling aliasing term each remainder $r_n^{(P)}[h]$, $n = 0, \ldots, L$, is a sum of coefficients $a_m[h]$ with $m = n + 2kP \ge 2P$ and $m = 2kP - n \ge 2P - L \ge P + 1$, $k \in \mathbb{N}$. Since $|T_n(x)| \le 1$ on I, the sampling aliasing term is bounded by $\frac{1}{2}\left|r_0^{(P)}[h]\right| + \sum_{n=1}^{L} \left|r_n^{(P)}[h]\right|$, and by (3.7) this is at most

$$\sum_{k=1}^{\infty} \left|a_{2kP}[h]\right| + \sum_{n=1}^{L}\sum_{k=1}^{\infty} \Big(\left|a_{n+2kP}[h]\right| + \left|a_{2kP-n}[h]\right|\Big)\,,$$

where all occurring indices are pairwise disjoint. Consequently

$$\sideset{}{'}\sum_{n=0}^{L} \left|r_n^{(P)}[h]\right| \le \sum_{m=P+1}^{\infty} |a_m[h]| = \delta_{\mathrm{Ch}}(h, P)\,.$$

For $L = P$ we obtain similarly with $\tilde{r}_P^{(P)}[h] := \frac{1}{2}\, a_P^{(P)}[h] - a_P[h]$ that

$$h(x) - t_{P,P}(x) = \sum_{n=P+1}^{\infty} a_n[h]\, T_n(x) - \sideset{}{'}\sum_{n=0}^{P-1} r_n^{(P)}[h]\, T_n(x) - \tilde{r}_P^{(P)}[h]\, T_P(x)\,, \tag{3.15}$$

again with the prime indicating that the term $n = 0$ is halved. The same argument applies to $r_n^{(P)}[h]$, $n = 0, \ldots, P-1$, where now $m = 2kP - n \ge 2P - (P-1) = P + 1$, the extra term obeys $\left|\tilde{r}_P^{(P)}[h]\right| \le \sum_{k=1}^{\infty} \left|a_{(2k+1)P}[h]\right|$, where the indices in the sum are odd multiples of P and hence are disjoint from the previous ones, which lie in $(2kP - P, 2kP + P)$. Thus the sampling aliasing term is again bounded by $\delta_{\mathrm{Ch}}(h, P)$ and the error estimates follow. ∎

Remark 3.5. *1. The estimate $E_P^{\mathrm{Ch}}(h) \le 2\,\delta_{\mathrm{Ch}}(h, P)$ of Theorem 3.4 is the exact analogue of Theorem 2.4. In all three settings the quadrature error is bounded by twice a transform-side tail, $\delta_+(\hat{f}_{\cos}, P)$, $\delta_+(\hat{f}_{\sin}, P)$ and $\delta_{\mathrm{Ch}}(h, P)$, respectively. The additional function-side tail $\sqrt{L}\,\delta_+(f, L)$ of* (2.11) *stems from the truncation of the sampling sum at $n = \frac{LP}{2}$ and has no counterpart in the Chebyshev case, where the finite sum* (3.2) *uses all available samples. Consequently no scaling factor $L^{-1/2}$ is needed in* (3.4).
For examples of Chebyshev expansions of special functions we refer to [26, Table A.2].

2. The case distinction $\varepsilon_L^{(P)}$ in (3.3) *is essential for $L < P$. Halving the top coefficient $a_L^{(P)}[h]$ in this case as well would add the uncorrected term $\frac{1}{2}a_L[h]\, T_L(x)$ to* (3.14)*, and $a_L[h]$ belongs neither to the degree-truncation tail $\delta_{\mathrm{Ch}}(h, L)$ nor to the aliasing tail $\delta_{\mathrm{Ch}}(h, P)$, so that* (3.13) *would fail for rapidly decaying coefficients. For $L = P$, by contrast, halving is required. It yields the interpolating polynomial and, through $a_P[h] - \frac{1}{2}a_P^{(P)}[h] = -\sum_{k\ge1} a_{(2k+1)P}[h]$, replaces the otherwise unbounded contribution $a_P[h]$ by genuine aliasing of indices larger than P.*

Corollary 3.6. *Let $P \in \mathbb{N} \setminus \{1\}$ and $L \in \mathbb{N}$ with $L \le P$ be fixed.*

1. *For some $r \in \mathbb{N}$ with $r < L$, let $h \in C^{r+1}(I)$ be given. Then*

$$R_{P,L}^{\mathrm{Ch}}(h) \le \tfrac{2}{r}\|h^{(r+1)}\|_{C(I)}\big(\tfrac{1}{(L-r)^r} + \tfrac{1}{(P-r)^r}\big), \quad E_P^{\mathrm{Ch}}(h) \le \tfrac{4}{r}\|h^{(r+1)}\|_{C(I)}(P-r)^{-r}. \tag{3.16}$$

   For $L = P$ the Chebyshev interpolation polynomial $t_{P,P}$ satisfies $R_{P,P}^{\mathrm{Ch}}(h) = \mathcal{O}(P^{-r})$ as $P \to \infty$.

2. *Let $h \in C(I)$ satisfy (3.12) with some $c > 0$ and $\sigma > 0$. Then*

$$R_{P,L}^{\mathrm{Ch}}(h) \le \tfrac{c}{\mathrm{e}^{\sigma}-1}\big(\mathrm{e}^{-\sigma L} + \mathrm{e}^{-\sigma P}\big), \qquad E_P^{\mathrm{Ch}}(h) \le \tfrac{2c}{\mathrm{e}^{\sigma}-1}\,\mathrm{e}^{-\sigma P}. \tag{3.17}$$

   For $L = P$ the Chebyshev interpolation polynomial $t_{P,P}$ fulfills $R_{P,P}^{\mathrm{Ch}}(h) \le \frac{2c}{\mathrm{e}^{\sigma}-1}\,\mathrm{e}^{-\sigma P}$.

Proof. The estimates (3.16) and (3.17) follow directly from Theorem 3.4 together with Lemmas 3.2 and 3.3. ■

We observe that in both cases considered in Corollary 3.6, the smallest reconstruction error estimate is achieved for $L = P$.

3.3 Chebyshev quadrature and reconstruction error for rational functions

We now consider rational functions, which provide an exponential decay of the Chebyshev coefficients. We investigate how tight the obtained error estimates are.

First we compute the Chebyshev coefficients explicitly for a rational function

$$h(x) = \sum_{j=1}^{J} \frac{c_j}{x-z_j}, \qquad x \in I, \tag{3.18}$$

with pairwise distinct simple poles $z_1, \ldots, z_J \in \mathbb{C} \setminus I$ and $c_1, \ldots, c_J \in \mathbb{C} \setminus \{0\}$. Note that in [7], the representation of $a_n[h]$ (for $J = 1$) has been derived differently using contour integration.

Theorem 3.7. *Let h be given as in (3.18). Assume that $w_j := z_j + s_j$ with $s_j := \sqrt{z_j^2 - 1}$, where the branch of the square root s_j is chosen in such a way that $|w_j| > 1$. Then for $\rho_0 := \min\{|w_j| : j = 1, \ldots, J\} > 1$ we have*

$$a_n[h] = -2 \sum_{j=1}^{J} \frac{c_j}{s_j}\, w_j^{-n}, \quad |a_n[h]| \le \gamma_0\, \rho_0^{-n}, \quad \gamma_0 := 2 \sum_{j=1}^{J} \frac{|c_j|}{|s_j|}, \tag{3.19}$$

for all $n \in \mathbb{N}_0$. Consequently, the quadrature error $E_P^{\mathrm{Ch}}(h)$ and the reconstruction error $R_{P,L}^{\mathrm{Ch}}(h)$ satisfy

$$E_P^{\mathrm{Ch}}(h) \le \frac{2\gamma_0}{\rho_0 - 1}\, \rho_0^{-P}, \quad R_{P,L}^{\mathrm{Ch}}(h) \le \frac{\gamma_0}{\rho_0 - 1}\big(\rho_0^{-L} + \rho_0^{-P}\big).$$

Proof. We consider only $h(x) := \frac{1}{x-z_j}$ with $z_j \in \mathbb{C} \setminus I$ for arbitrary $j \in \{1, \ldots, J\}$. Let $w_j = z_j + s_j$ with $s_j = \sqrt{z_j^2 - 1}$ and $|w_j| > 1$. We study the Chebyshev expansion (3.1) with $a_n := \frac{-2}{s_j}\, w_j^{-n}$, $n \in \mathbb{N}_0$ and prove that this expansion converges to h, thereby showing $a_n[h] := a_n$ for $n \in \mathbb{N}_0$.

Since $|a_n| = \frac{2}{|s_j|}\, |w_j|^{-n}$ with $|w_j| > 1$, we have $\sum_{n=0}^{\infty} |a_n| < \infty$, i.e., the Chebyshev series converges absolutely and uniformly on I to a function $g \in C(I)$. Using $x\, T_n(x) = \frac{1}{2}\big(T_{n-1}(x) + T_{n+1}(x)\big)$, we obtain

$$a_n[(x - z_j)\, g(x)] = \tfrac{1}{2}\big(a_{n-1}[g] + a_{n+1}[g]\big) - z_j\, a_n[g], \quad n \in \mathbb{N}_0,$$

with $a_{-1}[g] := a_1[g]$. For $n = 0$ we conclude with $(z_j + s_j)^{-1} = z_j - s_j$

$$a_0[(x - z_j)\, g(x)] = a_1[g] - z_j\, a_0[g] = \tfrac{-2}{s_j}\,(z_j - s_j) + z_j\,\tfrac{2}{s_j} = 2\,,$$

and for $n \in \mathbb{N}$,

$$a_n[(x - z_j)\, g(x)] = -\tfrac{w_j^{1-n}}{s_j}\,\big(1 + (z_j - s_j)^2 - 2z_j\,(z_j - s_j)\big) = 0\,.$$

Hence $(x - z_j)\, g(x) = 1$ for all $x \in I$, i.e., $g = h$. The representation of $a_n[h]$ in (3.19) follows by linearity. The corresponding estimate for $|a_n[h]|$ in (3.19) can be directly concluded. Finally, the estimates of $E_P^{\mathrm{Ch}}(h)$ and $R_{P,L}^{\mathrm{Ch}}(h)$ follow from Corollary 3.6. ■

Remark 3.8. *Observe that by definition of ρ_0, there exists one w_j with $\rho_0 = |w_j|$, and therefore the decay ρ_0^{-n} in* (3.19) *cannot be improved. In other words, ρ_0 is the parameter of the greatest Bernstein region $\mathcal{E}_{\rho_0}$ in which h is analytic, since $\sup_{n\in\mathbb{N}_0} |a_n[h]|\,\rho^n = \infty$ for every $\rho > \rho_0$. Furthermore, $E_P^{\mathrm{Ch}}(h)$ has optimal rate. Indeed, for the term $h_j(x) = \frac{c_j}{x - z_j}$ with $\rho_0 = |w_j|$ we obtain from* (3.7) *and* (3.19)*, for $0 \le n \le P - 1$,*

$$r_n^{(P)}[h_j] = \tfrac{-2c_j}{s_j}\,\big(w_j^n + w_j^{-n}\big)\,\frac{w_j^{-2P}}{1 - w_j^{-2P}}\,,$$

which is largest in modulus for $n = P - 1$, whereas the term $n = P$ of (3.4)*, namely $\big|a_P[h_j] - \frac{1}{2}a_P^{(P)}[h_j]\big| = \big|\sum_{k\ge1} a_{(2k+1)P}[h_j]\big|$, is of order ρ_0^{-3P}. Note that the weight $\varepsilon_P^{(P)} = \frac{1}{2}$ must not be omitted here. Hence the maximum in* (3.4) *is attained at $n = P - 1$ for large P and*

$$E_P^{\mathrm{Ch}}(h_j) \;\ge\; \big|\tfrac{2c_j}{s_j}\big|\,\frac{1 - \rho_0^{-2P+2}}{1 + \rho_0^{-2P}}\,\rho_0^{-P-1}\,,$$

which has decay rate ρ_0^{-P} and matches Theorem 3.7 *up to the constant factor $\frac{2\rho_0}{\rho_0 - 1}$.*

Finally, we consider the Chebyshev expansion of a rational function with poles of higher order. Again, we can explicitly derive the Chebyshev coefficients and show that the obtained estimates for the corresponding parametric quadrature errors are tight.

Theorem 3.9 (Chebyshev expansion of a power of the Cauchy kernel)**.** *Let $k \in \mathbb{N}$ be given. For fixed $z_0 \in \mathbb{C} \setminus I$, let $s_0 := \sqrt{z_0^2 - 1}$ and $w_0 := z_0 + s_0$, where the branch of the square root s_0 is chosen in such a way that $|w_0| > 1$.*
Then the Chebyshev coefficients of the kth power of the Cauchy kernel $h_k(x) := \frac{1}{(x - z_0)^k}$, $x \in I$, are given as

$$a_n[h_k] = \frac{2}{\pi}\int_0^{\pi} \frac{\cos(n\theta)}{(\cos\theta - z_0)^k}\,\mathrm{d}\theta = \frac{2\,(-1)^k}{(k-1)!\,s_0^k}\,q_k(n)\,w_0^{-n}\,, \quad n \in \mathbb{N}_0\,,\ k \in \mathbb{N}\,, \tag{3.20}$$

with a monic polynomial $q_k(n)$ of degree $k - 1$, which satisfies the recurrence relation

$$-(k-1)\,s_0\,q_{k-1}(n) = \tfrac{w_0}{2}\,q_k(n-1) + \tfrac{1}{2w_0}\,q_k(n+1) - z_0\,q_k(n)\,, \quad q_1(n) = 1\,, \tag{3.21}$$

with the initial condition

$$-(k-1)\,s_0\,q_{k-1}(0) = w_0^{-1}\,q_k(1) - z_0\,q_k(0)\,, \quad k \in \mathbb{N} \setminus \{1\}\,. \tag{3.22}$$

Further the Chebyshev coefficients $a_n[h_k]$ can be estimated by

$$\big|a_n[h_k]\big| \le c_k\,(n+1)^{k-1}\,|w_0|^{-n}\,, \quad n \in \mathbb{N}_0\,, \tag{3.23}$$

with some $c_k > 0$.
Let $P \in \mathbb{N} \setminus \{1\}$ *and* $L \in \mathbb{N}$ *with* $L \le P$ *and* $(L+1)\,\ln(|w_0|) > k-1$ *be fixed. Then the quadrature error* $E_P^{\mathrm{Ch}}(h_k)$ *and the reconstruction error* $R_{P,L}^{\mathrm{Ch}}(h_k)$ *satisfy*

$$E_P^{\mathrm{Ch}}(h_k) \le \tfrac{2c_k\,(P+1)^k|w_0|^{-P}}{(P+1)\,\ln(|w_0|)+1-k}\,, \quad R_{P,L}^{\mathrm{Ch}}(h_k) \le \tfrac{c_k\,(P+1)^k|w_0|^{-P}}{(P+1)\,\ln(|w_0|)+1-k} + \tfrac{c_k(L+1)^k|w_0|^{-L}}{(L+1)\,\ln(|w_0|)+1-k}\,.$$

Proof. 1. We show (3.20) by induction with respect to $k \in \mathbb{N}$. Obviously, (3.20) is true in the case $k=1$ with $q_1(n)=1$ by Theorem 3.7. Assume now that (3.20) holds for some $k \in \mathbb{N}$. Let g be the function with the Chebyshev coefficients

$$a_n[g] = \tfrac{2\,(-1)^{k+1}}{k!\,s_0^{k+1}}\,q_{k+1}(n)\,w_0^{-n}\,, \quad n \in \mathbb{N}_0\,.$$

Since $\sum_{n\ge 0}|a_n[g]| < \infty$, the corresponding Chebyshev series (3.1) converges absolutely and uniformly on I to some $g \in C(I)$. For $g_1(x) := (x-z_0)\,g(x)$, the recurrence $x\,T_n(x) = \frac{1}{2}\big(T_{n-1}(x)+T_{n+1}(x)\big)$ gives

$$a_n[g_1] = \tfrac{1}{2}a_{n-1}[g] + \tfrac{1}{2}a_{n+1}[g] - z_0\,a_n[g], \qquad n \in \mathbb{N}_0,$$

with $a_{-1}[g] = a_1[g]$. For $n=0$ it follows by (3.22) that

$$a_0[g_1] = a_1[g] - z_0\,a_0[g] = \tfrac{2(-1)^{k+1}}{k!s_0^{k+1}}\,\big(w_0^{-1}q_{k+1}(1) - z_0q_{k+1}(0)\big) = \tfrac{2(-1)^k}{(k-1)!s_0^k}\,q_k(0) = a_0[h_k].$$

For arbitrary $n \in \mathbb{N}$ we obtain by (3.21) that

$$\begin{aligned} a_n[g_1] &= \tfrac{1}{2}\,a_{n-1}[g] + \tfrac{1}{2}\,a_{n+1}[g] - z_0\,a_n[g] \\ &= \tfrac{2\,(-1)^{k+1}}{k!\,s_0^{k+1}}\,\big(\tfrac{w_0}{2}\,q_{k+1}(n-1) + \tfrac{w_0^{-1}}{2}\,q_{k+1}(n+1) - z_0\,q_{k+1}(n)\big)\,w_0^{-n}\,, \\ &= \tfrac{2\,(-1)^k}{(k-1)!\,s_0^k}\,q_k(n)\,w_0^{-n} = a_n[h_k]\,. \end{aligned}$$

Hence $g_1 = h_k$ on I, i.e. $g = h_{k+1}$.
2. Formula (3.20) yields $|a_n[h_k]| = \frac{2}{(k-1)!\,|s_0|^k}\,|q_k(n)|\,|w_0|^{-n}$. Since $q_k(n)$ is a monic polynomial of degree $k-1$, the supremum $c_k := \sup_{n\in\mathbb{N}_0}\frac{|a_n[h_k]|}{(n+1)^{k-1}\,|w_0|^{-n}}$ is finite and (3.23) holds. Note that the function $x^{k-1}\,|w_0|^{-x}$, $x \in [P+1,\infty)$, is decreasing since

$$\tfrac{\mathrm{d}}{\mathrm{d}x}\big(x^{k-1}|w_0|^{-x}\big) = x^{k-2}|w_0|^{-x}\big(k-1-x\ln(|w_0|)\big) \le 0$$

for $x \ge P+1$, which follows from the assumption $(P+1)\,\ln(|w_0|) \ge (L+1)\,\ln(|w_0|) > k-1$. Thus we can estimate

$$\begin{aligned} \delta_{\mathrm{Ch}}(h_k,P) &= \sum_{n=P+1}^{\infty}|a_n[h_k]| \le c_k\sum_{n=P+1}^{\infty}(n+1)^{k-1}|w_0|^{-n} = c_k|w_0|\sum_{n=P+2}^{\infty}n^{k-1}|w_0|^{-n} \\ &\le c_k|w_0|\int_{P+1}^{\infty}x^{k-1}\,|w_0|^{-x}\,\mathrm{d}x = \tfrac{c_k|w_0|}{(\ln(|w_0|))^k}\int_{(P+1)\ln(|w_0|)}^{\infty}u^{k-1}\,\mathrm{e}^{-u}\,\mathrm{d}u \\ &= \tfrac{c_k|w_0|}{(\ln(|w_0|))^k}\,\Gamma\big(k,(P+1)\ln(|w_0|)\big)\,. \end{aligned}$$

Using [24, Proposition 2.7], the upper incomplete gamma function can be estimated for $w_1 := (P+1)\ln(|w_0|) > k-1$ by

$$\Gamma(k,w_1) := \textstyle\int_{w_1}^{\infty}u^{k-1}\,\mathrm{e}^{-u}\,\mathrm{d}u \le \tfrac{w_1^k}{w_1+1-k}\,\mathrm{e}^{-w_1}\,.$$

Thus we obtain with $|w_0|\,\mathrm{e}^{-w_1} = |w_0|^{-P}$ that

$$\delta_{\mathrm{Ch}}(h_k, P) \le \tfrac{c_k\,(P+1)^k\,|w_0|^{-P}}{(P+1)\,\ln(|w_0|)+1-k}\,,$$

where we have $(P+1)\,\ln(|w_0|) + 1 - k > 0$ by assumption. The estimates for the quadrature error and the reconstruction error now follow from Theorem 3.4. ■

Remark 3.10. *The polynomials q_k can be explicitly determined from* (3.21) *and* (3.22). *We obtain for $k = 2, \ldots, 5$,*

$$\begin{aligned}
q_2(n) &= n + \tfrac{z_0}{s_0}\,,\\
q_3(n) &= n^2 + \tfrac{3z_0}{s_0}\,n + 2 + \tfrac{3}{s_0^2}\,,\\
q_4(n) &= n^3 + \tfrac{6z_0}{s_0}\,n^2 + \big(11 + \tfrac{15}{s_0^2}\big)\,n + \tfrac{6z_0}{s_0} + \tfrac{15z_0}{s_0^3}\,,\\
q_5(n) &= n^4 + \tfrac{10z_0}{s_0}\,n^3 + \big(35 + \tfrac{45}{s_0^2}\big)\,n^2 + \big(\tfrac{50z_0}{s_0} + \tfrac{105z_0}{s_0^3}\big)\,n + 24 + \tfrac{120}{s_0^2} + \tfrac{105}{s_0^4}\,.
\end{aligned}$$

Since q_k is monic of degree $k-1$, formula (3.20) *yields the exact asymptotics*

$$\big|a_n[h_k]\big| \sim \tfrac{2}{(k-1)!\,|s_0|^k}\,n^{k-1}\,|w_0|^{-n}\,, \qquad n \to \infty\,.$$

Hence the exponent $k-1$ in (3.23) *cannot be lowered and $c_k \ge \frac{2}{(k-1)!\,|s_0|^k}$, so that the estimates of Theorem* 3.9 *are sharp in their n- and P-dependence. A different representation of the Chebyshev coefficients for rational functions with poles of higher order, based on associated Legendre functions, has been given in* [7, Section 4]. *Note that our explicit representation* (3.20) *with* (3.21) *and* (3.22) *is very user friendly.*

4 Numerical examples

We illustrate our error estimates of Theorem 2.4 and Theorem 3.4 by concrete examples.

4.1 Numerical experiments for the Fourier cosine and sine transform

We consider a compactly supported function as well as rational and meromorphic functions.

Example 4.1. ***Fourier cosine transform of a compactly supported function.*** *Let $f(x) := 1 - x$ for $x \in [0,\,1]$ and $f(x) := 0$ for $x > 1$. Then we find*

$$\hat{f}_{\cos}(v) = \int_0^1 (1-x)\,\cos(2\pi v x)\,\mathrm{d}x = \tfrac{1}{2}\,\big(\tfrac{\sin(\pi v)}{\pi v}\big)^2 = \tfrac{1}{2}\,\big(\mathrm{sinc}(\pi v)\big)^2\,, \quad v \in \mathbb{R}_+\,.$$

Since $\mathrm{supp}\,f = [0,\,1]$, we conclude from Table 2.1 *(last line) that $\delta_+(f, L) = 0$ for $L \ge 2$. Further, $\hat{f}_{\cos} \in C(\mathbb{R}_+)$ has polynomial decay with $b = 2$ (see Table* 2.1*, first line). Considering $\hat{f}_{\cos}(v)\,(1+v)^2$ separately on $[0,\,\frac{1}{2}]$ and $[\frac{1}{2},\,\infty)$ shows that $|\hat{f}_{\cos}(v)|\,(1+v)^2 \le d = \frac{9}{4}$ for all $v \in \mathbb{R}_+$. Thus, for any $L \ge 2$ it holds*

$$E_{P,L}^{\cos}(f) \le 2d\,(2^2-1)\,\zeta(2)\,L^{-1/2}\,P^{-2} = \tfrac{9\,\pi^2}{4\sqrt{L}}\,P^{-2}\,.$$

Hence for fixed $L \ge 2$ the error decays like P^{-2} as $P \to \infty$. Taking $L = 2$, we use samples $f(\frac{n}{P})$ for $n = 0, \ldots, P$ to compute $c_{P,2}(v)$ in (1.2) *and the error $\frac{1}{\sqrt{2}}|\hat{f}_{\cos}(v) - c_{P,2}(v)|$ with $\hat{f}_{\cos}(v) = \frac{1}{2}\,\big(\mathrm{sinc}(\pi v)\big)^2$ for $v \in [0,\,\frac{P}{2}]$. Table* 4.1 *confirms this: the errors are reduced by a factor tending to* 4 *whenever P is doubled, i.e. they decay like P^{-2}, while the bound overestimates them by a factor between* 78 *and* 98.

P	4	8	16	32
$E^{\cos}_{P,2}(f)$	$1.00 \cdot 10^{-2}$	$2.82 \cdot 10^{-3}$	$7.58 \cdot 10^{-4}$	$1.97 \cdot 10^{-4}$
bound $\frac{9\pi^2}{4\sqrt{L}} P^{-2}$	$9.81 \cdot 10^{-1}$	$2.45 \cdot 10^{-1}$	$6.13 \cdot 10^{-2}$	$1.53 \cdot 10^{-2}$
ratio bound/error	98.0	86.9	80.9	77.8

Table 4.1: Scaled parametric quadrature error $E^{\cos}_{P,2}(f)$ for the compactly supported function f of Example 4.1 and the bound of Theorem 2.4 combined with Table 2.1.

Example 4.2. ***Fourier cosine transform of a rational function.*** *We consider* $f(x) = \left(x^2 + s_0^2\right)^{-1}$ *with* $s_0 = \frac{1}{10}$ *to illustrate Corollary* 2.10 *and Remark* 2.9. *The rational function* f *has the simple poles* $\pm \mathrm{i} s_0$. *Within the framework* (2.17) *we take* $z_1 = -\mathrm{i} s_0$ *in the open lower half plane, so that* $f(x) = (x^2 - z_1^2)^{-1}$ *and* $c_1 = 1$. *By* (2.19),

$$\hat{f}_{\cos}(v) = \frac{\pi}{2s_0}\, \mathrm{e}^{-2\pi s_0 v} = 5\pi\, \mathrm{e}^{-\pi v/5}\,, \quad v \in \mathbb{R}_+\,.$$

Thus, by Table 2.1 *(second line),* $|\hat{f}_{\cos}(v)|\,\mathrm{e}^{sv} \le d$ *with* $s = \frac{\pi}{5}$, $d = 5\pi$, *while* $c = \sup_{x \in \mathbb{R}_+} |f(x)|(1+x)^2 = 1 + s_0^{-2} = 101$, *attained at* $x = s_0^2$, *by Remark* 2.9.

Figure 4.1 *shows* $E^{\cos}_{P,L}(f)$ *for* $L \in \{256, 1024, 4096\}$. *For each fixed* L, *the error is first reduced by the factor* 1.88 *per step of* 2 *in* P, *matching the predicted* $\mathrm{e}^{2\pi s_0} = \mathrm{e}^{\pi/5} \approx 1.87$ *of* (2.20), *and then settles on the plateau caused by the second, purely algebraic term of* (2.20). *The measured plateau values* $4.88 \cdot 10^{-4}$, $6.10 \cdot 10^{-5}$, $7.63 \cdot 10^{-6}$ *agree to three digits with* $2L^{-3/2}$ *and are independent of* P. *This is exactly the lower bound* (2.21), *whose first term equals* $L^{-1/2} \int_{L/2+1/P}^{\infty} f(x)\, \mathrm{d}x \sim 2L^{-3/2}$. *The upper bound* (2.20) *gives the same rate* $L^{-3/2}$, *with the larger constant* $\frac{\pi^2}{2} c \approx 498$. *Thus the transform side is sharp geometric and the function side is the bottleneck.*

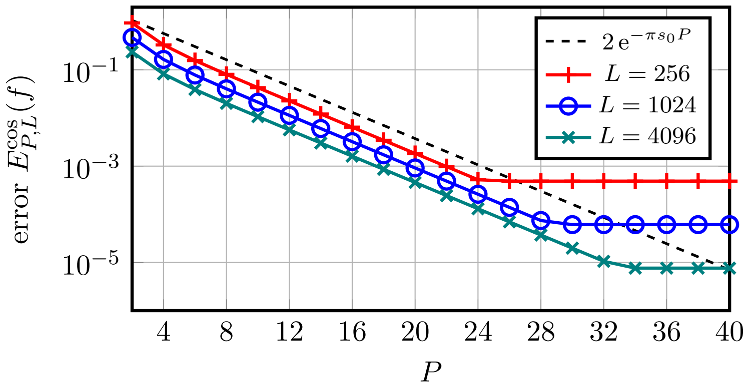
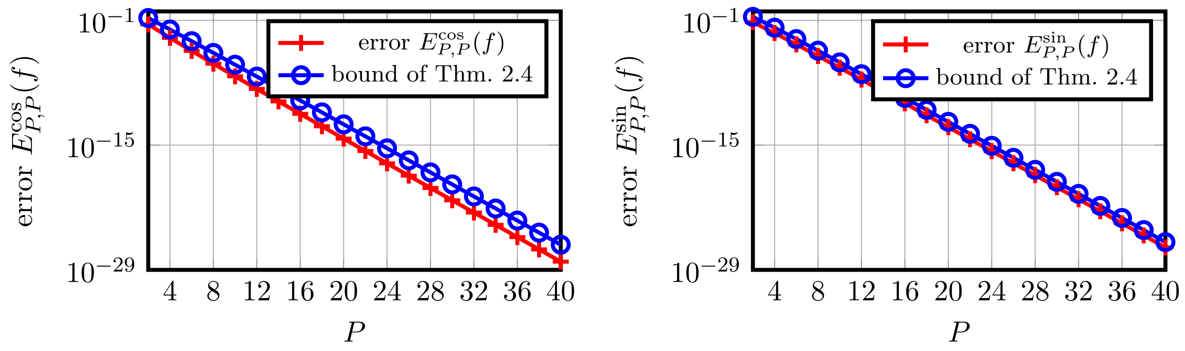


Figure 4.1: Error $E^{\cos}_{P,L}(f)$ for the rational function $f(x) = (x^2 + \frac{1}{100})^{-1}$ of Example 4.2: geometric decay in P at the sharp rate $\mathrm{e}^{-\pi s_0}$ and plateau $2\,L^{-3/2}$ for sufficiently large P.

Example 4.3. ***Fourier cosine/sine transform for meromorphic functions.*** *For the self-dual pair* $f(x) = \operatorname{sech}(\pi x)$, $\hat{f}_{\cos}(v) = \frac{1}{2}\operatorname{sech}(\pi v)$ *considered in Theorem* 2.11 *and for the odd pair* $f(x) = \tanh(\pi x)\operatorname{sech}(\pi x)$, $\hat{f}_{\sin}(v) = v \operatorname{sech}(\pi v)$ *of Remark* 2.13, *both functions decay at the geometric rate* $2\pi s_0 = \pi$. *With the balanced choice* $L = P$, *Figure* 4.2 *shows geometric convergence over more than* 25 *decades: the errors are reduced by a factor tending to* $\mathrm{e}^{\pi} \approx 23.1$ *per step of* 2 *in* P, *in agreement with the rate* $\mathrm{e}^{-\pi P/2}$ *of* (2.24) *(the observed factors decrease monotonically from* 33.2 *at* $P = 2$ *to* 23.8 *at* $P = 38$ *and approach* e^{π} *from above, the deviation being the factor* $\sqrt{(P+2)/P}$ *caused by the scaling* $L^{-1/2}$ *in* (1.3)*). Since the errors* *fall far below the double precision unit roundoff, they are evaluated in the cancellation-free form* $\max_v |\sigma_2(v) - \sigma_1(v)|$ *of the proof of Theorem* 2.4*, with the common factor* $\mathrm{e}^{-\pi P/2}$ *split off. In both panels the plotted bound is that of Theorem* 2.4 *with the decay rates evaluated directly from* (2.7)*. For the sine case this is essential, since by Remark* 2.13 *the exponential row of Table* 2.1 *does not apply to* $\hat{f}_{\sin}$ *at the critical rate* $s = \pi$*. In the sine case the bound exceeds the true error by the constant factor* 4.0*, which can be read off exactly. The maximum in* (1.3) *is attained at* $v = \frac{P}{2}$*, where* $s_{P,P}$ *vanishes, so that* $E^{\sin}_{P,P}(f) = \frac{1}{\sqrt{P}}\big|\hat{f}_{\sin}\big(\frac{P}{2}\big)\big|$*; moreover* $\delta_+(\hat{f}_{\sin}, P) = \big|\hat{f}_{\sin}\big(\frac{P}{2}\big)\big|$ *and* $\delta_+(f, P) \approx \frac{2}{P}\big|\hat{f}_{\sin}\big(\frac{P}{2}\big)\big|$*, so that the bound* (2.12) *equals* $\frac{4}{\sqrt{P}}\big|\hat{f}_{\sin}\big(\frac{P}{2}\big)\big|$*. In the cosine case the ratio grows essentially like* $2P$*, from* 5.1 *at* $P = 2$ *to* 80.1 *at* $P = 40$*, because the estimate* $\max_v |\sigma_2(v)| \le L\,\delta_+(f, L)$ *in the proof of Theorem* 2.4 *loses a factor of order* P*. The true value is* $\max_v |\sigma_2(v)| \le \frac{2}{\pi}\mathrm{e}^{-\pi P/2}$ *by the proof of Theorem* 2.11*, whereas* $P\,\delta_+(f, P) \approx 2P\,\mathrm{e}^{-\pi P/2}$*.*



(a) $f(x) = \mathrm{sech}(\pi x)$ (b) $f(x) = \tanh(\pi x)\,\mathrm{sech}(\pi x)$

Figure 4.2: Geometric convergence at the critical rate for the balanced choice $L = P$, Example 4.3.

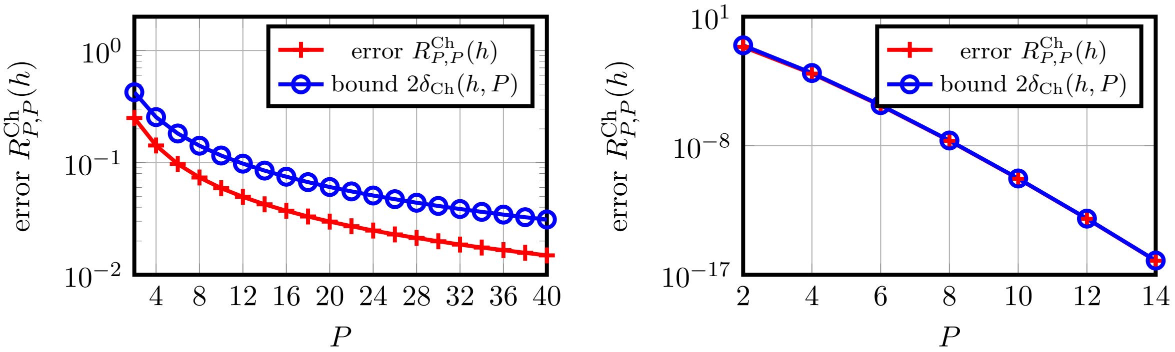
4.2 Chebyshev sampling polynomials

Example 4.4. ***Polynomial decay of Chebyshev coefficients.*** *The Chebyshev coefficients of* $h(x) = |x|$ *are of the form* $a_{2k}[h] = \frac{4(-1)^{k+1}}{\pi(4k^2-1)}$ *and* $a_{2k+1}[h] = 0$*,* $k \in \mathbb{N}_0$*, see* [26, Table A.2]*. Since only even indices contribute, the Chebyshev decay rate for even* $P = 2m$ *evaluates exactly by telescoping,*

$$\delta_{\mathrm{Ch}}(h, P) = \frac{4}{\pi}\sum_{k=m+1}^{\infty}\frac{1}{4k^2-1} = \frac{2}{\pi(P+1)}\,.$$

Theorem 3.4 *with* $L = P$ *yields* $R^{\mathrm{Ch}}_{P,P}(h) \le \frac{4}{\pi(P+1)}$*, i.e., algebraic convergence of order* P^{-1}*. Note that Corollary* 3.6 *(1) is* not *applicable here. Via Lemma* 3.2 *it requires* $h \in C^{r+1}(I)$ *and the constant* $\frac{4}{r}\|h^{(r+1)}\|_{C(I)}$*, whereas* $h(x) = |x|$ *is not even in* $C^1(I)$ *and* $\|h''\|_{C(I)}$ *does not exist. The bound above is obtained from the exact evaluation of* $\delta_{\mathrm{Ch}}(h, P)$ *instead, which is available because the Chebyshev coefficients of* $|x|$ *are known in closed form. This example thus illustrates that the estimate of Theorem* 3.4 *remains useful for functions of very low regularity, for which the smoothness-based Corollary* 3.6 *gives nothing. Figure* 4.3*(left) shows that the bound* $2\,\delta_{\mathrm{Ch}}(h, P)$ *is about twice the true error.*

Example 4.5. ***Super exponential decay of Chebyshev coefficients.*** *We consider* $h(x) = \mathrm{e}^x$ *with* $a_n[h] = 2I_n(1)$ *(see* [26, Table A.2]*), i.e.,* $a_0[h] \approx 2.532$*,* $a_1[h] \approx 1.130$*,* $a_2[h] \approx 0.271$*,* $a_3[h] \approx 0.0443$*,* $a_4[h] \approx 5.474 \times 10^{-3}$*,* $a_5[h] \approx 5.429 \times 10^{-4}$*. By* $I_n(1) \le \frac{\mathrm{e}}{2^n\,n!}$*, the coefficients decay super-exponentially, so that the Chebyshev decay rate* $\delta_{\mathrm{Ch}}(h, P) = 2\sum_{n=P+1}^{\infty} I_n(1)$ *decreases*



(a) $h(x) = |x|$: polynomial decay P^{-1}.   (b) $h(x) = \mathrm{e}^x$: super-exponential decay.

Figure 4.3: Error $R^{\mathrm{Ch}}_{P,P}(h)$ and bound $2\,\delta_{\mathrm{Ch}}(h,P)$ for Example 4.4 (left) and Example 4.5 (right).

many orders of magnitude per unit increase in P, see Figure 4.3 (right), and (3.17) applies with any $\sigma > 0$. Here the common range $P = 2, 4, \ldots$ of the other examples has to be truncated at $P = 14$, where the error already reaches the level of the double precision unit roundoff.

Example 4.6. ***Geometric decay of Chebyshev coefficients.*** *The Runge function $h(x) = \frac{1}{1+x^2}$ is real-analytic with simple poles $z_{1,2} = \pm\mathrm{i}$ and $w_{1,2} = \pm\mathrm{i}(1+\sqrt{2})$, so that $\rho_0 = 1+\sqrt{2}$. We obtain*

$$a_n[h] = \tfrac{1}{\sqrt{2}}\left(w_1^{-n} + w_2^{-n}\right) = \sqrt{2}\,(1+\sqrt{2})^{-n}\cos\tfrac{n\pi}{2}\,, \quad n \in \mathbb{N}_0\,,$$

i.e., $a_{2k}[h] = (-1)^k\sqrt{2}q^k$ and $a_{2k+1}[h] = 0$ for $k \in \mathbb{N}_0$ with $q = (1+\sqrt{2})^{-2} = 3-2\sqrt{2} \approx 0.1716$, see [26, Table A.2]*, yielding the sharp estimate $\left|a_n[h]\right| \le \sqrt{2}\,\rho_0^{-n}$. The Chebyshev decay rate for even $P = 2m$,*

$$\delta_{\mathrm{Ch}}(h,P) = \tfrac{\sqrt{2}\,q^{m+1}}{1-q} = \tfrac{\sqrt{2}\,q}{1-q}\,q^{P/2}\,,$$

decreases geometrically: increasing P by 2 multiplies $q^{P/2}$ by $q \approx 0.1716$, while $q^{1/2} = \rho_0^{-1} \approx 0.4142$ is the factor per unit *increase in P. The data underlying Figure 4.4 confirm the factor ≈ 0.16 per step of 2 in P, e.g. $8.579 \cdot 10^{-2} \to 1.336 \cdot 10^{-2}$ from $P = 2$ to $P = 4$. With $\sigma = \log\rho_0 = -\frac{1}{2}\log q \approx 0.881$, Corollary 3.6 (2) with $L = P$ and $c = \sqrt{2}$ provides $R^{\mathrm{Ch}}_{P,P}(h) \le \frac{2\sqrt{2}}{\mathrm{e}^{\sigma}-1}\,\mathrm{e}^{-\sigma P} = 2\,\rho_0^{-P}$, since $\mathrm{e}^{\sigma} - 1 = \rho_0 - 1 = \sqrt{2}$. Note that the constants in Corollary 3.6 (2) are c and σ, whereas Theorem 3.7 is formulated with γ_0 and ρ_0. Figure 4.4 confirms the geometric convergence on the same range $P = 2, 4, \ldots, 40$ as in Figures 4.1–4.3; for $P \ge 36$ the error is obtained from the coefficient representation (3.15) instead of forming the difference $h - t_{P,P}$ directly.*

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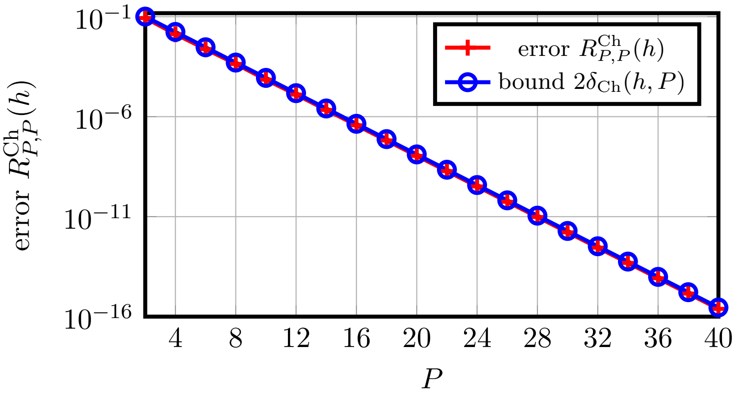

Figure 4.4: Error $R^{\mathrm{Ch}}_{P,P}(h)$ and bound $2\,\delta_{\mathrm{Ch}}(h,P)$ for Example 4.6 with $h(x) = \frac{1}{1+x^2}$.

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