Inverse nonequispaced FFT - Applications in MRI and numerical Stability

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Abstract

In magnetic resonance imaging (MRI), methods that use a non-Cartesian, e.g. spiral, grid in k-space are becoming increasingly important. This talk focuses on a recently proposed implicit discretisation scheme which generalises the standard approach based on gridding. While the latter succeeds for sufficiently uniform sampling sets and accurate estimated density compensation weights, the implicit method further improves the reconstruction quality when the sampling scheme or the weights are less regular. The convergence rate of the proposed iterative scheme is shown to depend on simple geometric quantities of the sampling set. This talk is based on recent research in [1, 2].

1 Introduction

In magnetic resonance imaging (MRI) the raw data is measured in k-space, the domain of spatial frequencies. Non-Cartesian sampling schemes, e.g., spiral or radial scans, have received much attention. In contrast to the use of the computationally efficient fast Fourier transform (FFT) for the reconstruction from Cartesian grids, the more general sampling trajectories need the recently developed nonequispaced FFTs [3], often referred to as gridding. On the other hand, iterative image reconstruction algorithms play an important role in modern tomographic systems and have been applied to data on spiral k-space trajectories [4].

We focus on the simplified signal equation

$$s(\boldsymbol{k}) = \int_{\mathbb{R}^3} p(\boldsymbol{r})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{r}\boldsymbol{k}} \mathrm{d}\boldsymbol{r} \tag{1}$$

and describe two different approaches. Let the available samples in k-space be contained in the shifted unit cube, i.e. $\boldsymbol{k} \in [-\frac{1}{2}, \frac{1}{2})^3$, and the field of view be restricted to $\Omega_{\boldsymbol{N}} \subset [-\frac{N_1}{2}, \frac{N_1}{2}) \times [-\frac{N_2}{2}, \frac{N_2}{2}) \times [-\frac{N_3}{2}, \frac{N_3}{2})$, where $\boldsymbol{N} = (N_1, N_2, N_3)^\top \in 2\mathbb{N}^3$. Then, the discretisation of integral (1) on equispaced points leads to

$$s(\boldsymbol{k}) \approx \tilde{s}(\boldsymbol{k}) := \sum_{\boldsymbol{r} \in I_{\boldsymbol{N}}^3} p(\boldsymbol{r})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{r}\boldsymbol{k}}\,, \tag{2}$$

where $I_{\boldsymbol{N}}^3 := \{-\frac{N_1}{2}, \ldots, \frac{N_1}{2} - 1\} \times \{-\frac{N_2}{2}, \ldots, \frac{N_2}{2} - 1\} \times \{-\frac{N_3}{2}, \ldots, \frac{N_3}{2} - 1\}$. Thus, the unknown object p is given *implicitly* by (2). The authors of [5] call this the inverse model.

A second possible discretisation uses the Fourier inversion theorem first, i.e.,

$$p(\boldsymbol{r}) = \int_{\mathbb{R}^3} s(\boldsymbol{k})\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{r}\boldsymbol{k}} \mathrm{d}\boldsymbol{k}\,. \tag{3}$$

The discretisation of the integral (3) leads to

$$p(\boldsymbol{r}) \approx \tilde{p}(\boldsymbol{r}) := \sum_{j=0}^{M-1} s(\boldsymbol{k}_j)\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{r}\boldsymbol{k}_j} w_j\,, \tag{4}$$

where w_j are weights, which compensate for local variations of the sampling density. Here, the unknown object $p \approx \tilde{p}$ can be computed *explicitly*.

The important difference between (2) and (4) is that the former is discretised in the image domain with pixels on a uniform grid and hence with unit weighting coefficients and the latter is an integral discretised in the k-space domain with non-uniform samples and specific weights.

2 Iterative solution

We reformulate problem (2) and (4) in matrix vector notation and denote the vector of the given values by $\boldsymbol{s} := (s(\boldsymbol{k}_j))_{j=0,\ldots,M-1} \in \mathbb{C}^M$, the reshaped vector of the unknown object by $\boldsymbol{p} := (p(\boldsymbol{r}))_{\boldsymbol{r} \in I_{\boldsymbol{N}}^3} \in \mathbb{C}^{N_1 \times N_2 \times N_3}$, the density compensation matrix by $\boldsymbol{W} := \operatorname{diag}(w_j)_{j=0,\ldots,M-1}$, and the nonequispaced Fourier matrix by

$$\boldsymbol{A} := \left(\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{r}\boldsymbol{k}_j}\right)_{j=0,\ldots,M-1;\, \boldsymbol{r} \in I_{\boldsymbol{N}}^3}\,, \tag{5}$$

whereas $\boldsymbol{A}^{\mathsf{H}}$ denotes its adjoint (conjugate transpose).

The gridding approximation (4) is easily computed by one matrix vector multiplication

$$\tilde{\boldsymbol{p}} = \boldsymbol{A}^{\mathsf{H}} \boldsymbol{W} \boldsymbol{s}. \tag{6}$$

The adjoint NFFT takes $\mathcal{O}(|I_{\boldsymbol{N}}^3| \log |I_{\boldsymbol{N}}^3| + M)$ floating point operations for this task.

Slightly more involved, the reconstruction problem (2) is solved by the method of least squares and hence, consists in solving the weighted normal equation of first kind

$$\boldsymbol{A}^{\mathsf{H}} \boldsymbol{W} \boldsymbol{A} \boldsymbol{p} = \boldsymbol{A}^{\mathsf{H}} \boldsymbol{W} \boldsymbol{s} \tag{7}$$

for the unknown vector $\boldsymbol{p}$. We include density compensation weights for the implicit discretisation since this is more natural with respect to the 'continuous residual' in k-space.

From the mathematical point of view, equation (7) is solved most efficiently by the conjugate gradients method. whereas the two multiplications with the (adjoint) nonequispaced Fourier matrix per iteration are computed by the NFFT. In summary, we suggest the following algorithm.

Algorithm 2.1
(Conjugate Gradients for Normal Equations)

Input:

$M \in \mathbb{N}$	*number of samples*
$N_1, N_2, N_3 \in \mathbb{N}$	*number of unknowns*
$(\boldsymbol{k}_j)_{j=0,\ldots,M-1}$	*sampling points*
$\boldsymbol{s} = (s)_{j=0,\ldots,M-1} \in \mathbb{C}^M$	*sampled values*
$\boldsymbol{W} = \operatorname{diag}(w_j)_{j=0,\ldots,M-1}$	*density weights*

$\boldsymbol{p}_0 = \boldsymbol{0}$
$\boldsymbol{r}_0 = \boldsymbol{s}$
$\tilde{\boldsymbol{p}}_0 = \tilde{\boldsymbol{z}}_0 = \boldsymbol{A}^{\mathsf{H}} \boldsymbol{W} \boldsymbol{r}_0$
for $l = 0, 1, \ldots$
  $\boldsymbol{v}_l = \boldsymbol{A} \tilde{\boldsymbol{p}}_l$
  $\alpha_l = \tilde{\boldsymbol{z}}_l^{\mathsf{H}} \tilde{\boldsymbol{z}}_l \,/\, \boldsymbol{v}_l^{\mathsf{H}} \boldsymbol{W} \boldsymbol{v}_l$
  $\boldsymbol{p}_{l+1} = \boldsymbol{p}_l + \alpha_l \tilde{\boldsymbol{p}}_l$
  $\boldsymbol{r}_{l+1} = \boldsymbol{r}_l - \alpha_l \boldsymbol{v}_l$
  $\tilde{\boldsymbol{z}}_{l+1} = \boldsymbol{A}^{\mathsf{H}} \boldsymbol{W} \boldsymbol{r}_{l+1}$
  $\beta_l = \tilde{\boldsymbol{z}}_{l+1}^{\mathsf{H}} \tilde{\boldsymbol{z}}_{l+1} \,/\, \tilde{\boldsymbol{z}}_l^{\mathsf{H}} \tilde{\boldsymbol{z}}_l$
  $\tilde{\boldsymbol{p}}_{l+1} = \tilde{\boldsymbol{z}}_{l+1} + \beta_l \tilde{\boldsymbol{p}}_l$
end for
Output: approximate solution $\boldsymbol{p}_l$

Remarkably, Algorithm 2.1 resembles a gridding solution after its first iteration. In subsequent iterations the residual $\|\boldsymbol{r}_l\|_{\boldsymbol{W}}$ is minimised

3 Numerical stability

The accuracy in the fast computation of the pure gridding approach (6) depends only on the NFFT parameter (oversampling factor, window-width) and can easily be chosen such that (6) is computed up to an error of 10^{-10}. However, this does not give any answer how good the discretisation (6) is. The actual reconstruction quality in gridding approaches heavily depends on the sampling geometry and chosen density compensation weights.

Closely related, the convergence rate of Algorithm 2.1 to the solution of (7) is also dependent on the sampling geometry. Two simple quantities to describe the regularity of a sampling set are given as follows. Taking periodicity into account, the distance of two points $\boldsymbol{k}, \boldsymbol{l} \in [-\frac{1}{2}, \frac{1}{2})^3$ is defined by

$$\operatorname{dist}(\boldsymbol{k}, \boldsymbol{l}) := \min_{\boldsymbol{j} \in \mathbb{Z}^d} \|(\boldsymbol{k} + \boldsymbol{j}) - \boldsymbol{l}\|_\infty .$$

Thus, the *mesh norm* and the *separation distance* of a sampling set are given by

$$\delta := 2 \max_{\boldsymbol{k} \in [-\frac{1}{2}, \frac{1}{2})^3} \min_{j=0,\ldots,M-1} \operatorname{dist}(\boldsymbol{k}_j, \boldsymbol{k}),$$
$$q := \min_{0 \le j < l < M} \operatorname{dist}(\boldsymbol{k}_j, \boldsymbol{k}_l) .$$

Recent results state the following qualitative results on the uniqueness of the reconstruction problem.

Theorem 3.1 *The nonequispaced Fourier matrix* $\boldsymbol{A}$*, cf.* (5)*, has full rank (allows for unique solution) if the sampling set is dense*

$$N_j < c\delta^{-1}, \qquad j = 1, 2, 3,$$

or the sampling set is separated

$$N_j > Cq^{-1}, \qquad j = 1, 2, 3,$$

where c, C *denote explicitly known constants.*

Moreover, under the conditions in Theorem 3.1 the singular values of the Fourier matrix $\boldsymbol{A}$ and thus the rate of convergence of the iterative reconstruction scheme have been estimated in [6, 1].

References

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