On regularized Shannon sampling formulas with localized sampling

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In this paper we present new regularized Shannon sampling formulas which use localized sampling with special window functions, namely Gaussian, B–spline, and sinh-type window functions. In contrast to the classical Shannon sampling series, the regularized Shannon sampling formulas possess an exponential decay and are numerically robust in the presence of noise. Several numerical experiments illustrate the theoretical results.

Key words: Regularized Shannon sampling formulas, Whittaker–Kotelnikov–Shannon sampling theorem, bandlimited function, window functions, error estimates, numerical robustness.

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1 Introduction

The classical Whittaker–Kotelnikov–Shannon sampling theorem plays a fundamental role in signal processing, since this result describes the close relation between a bandlimited function and its equidistant samples. This sampling theorem states that any function $f \in L^2(\mathbb{R})$ with bandwidth $\le \frac{N}{2}$, i. e., the support of the Fourier transform

$$\hat{f}(v) := \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} x v}\, \mathrm{d}x\,, \quad v \in \mathbb{R}\,,$$

is contained in $\big[-\frac{N}{2},\, \frac{N}{2}\big]$, can be recovered from its samples $f\big(\frac{\ell}{L}\big)$, $\ell \in \mathbb{Z}$, with $L \ge N$ and it holds

$$f(x) = \sum_{\ell \in \mathbb{Z}} f\big(\tfrac{\ell}{L}\big)\, \mathrm{sinc}\Big(L\pi\, \big(x - \tfrac{\ell}{L}\big)\Big)\,, \quad x \in \mathbb{R}\,, \tag{1.1}$$

with the sinc function

$$\mathrm{sinc}\, x := \begin{cases} \frac{\sin x}{x} & x \in \mathbb{R} \setminus \{0\}\,, \\ 1 & x = 0\,. \end{cases}$$

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Unfortunately, the practical use of this sampling theorem is limited, since it requires infinitely many samples which is impossible in practice. Further the sinc function decays very slowly such that the *Shannon sampling series*

$$\sum_{\ell \in \mathbb{Z}} f\big(\tfrac{\ell}{L}\big) \operatorname{sinc}\Big(L\pi \left(x - \tfrac{\ell}{L}\right)\Big)\,, \quad x \in \mathbb{R}\,,$$

has rather poor convergence. Moreover, in the presence of noise or quantization in the samples $f\big(\frac{\ell}{L}\big)$, $\ell \in \mathbb{Z}$, the convergence of Shannon sampling series may even break down completely (see [3]).

To overcome these drawbacks, one can use the following three techniques (see [12, 5, 7, 17]):
1. The function $\operatorname{sinc}(L\pi\,\cdot)$ is *regularized by a truncated window function*

$$\varphi_m(x) := \varphi(x)\,\mathbf{1}_{[-m/L,\,m/L]}(x)\,, \quad x \in \mathbb{R}\,,$$

where the window function $\varphi : \mathbb{R} \to [0,\,1]$ belongs to the set $\Phi_{m,L}$ (as defined in Section 3) and where $\mathbf{1}_{[-m/L,\,m/L]}$ denotes the indicator function of the interval $\big[-\frac{m}{L},\,\frac{m}{L}\big]$ with some $m \in \mathbb{N} \setminus \{1\}$. Then we recover a function $f \in L^2(\mathbb{R})$ with bandwidth $\le \frac{N}{2}$ by the *regularized Shannon sampling formula*

$$(R_{\varphi,m}f)(x) := \sum_{\ell \in \mathbb{Z}} f\left(\tfrac{\ell}{L}\right)\, \operatorname{sinc}\left(L\pi\left(x - \tfrac{\ell}{L}\right)\right)\, \varphi_m\left(x - \tfrac{\ell}{L}\right)\,,$$

where $L \ge N$. Obviously, this is an *interpolating approximation* of f, since it holds

$$(R_{\varphi,m}f)\left(\tfrac{k}{L}\right) = f\left(\tfrac{k}{L}\right)\,, \quad k \in \mathbb{Z}\,.$$

2. The use of the truncated window function φ_m with compact support $\big[-\frac{m}{L},\,\frac{m}{L}\big]$ leads to *localized sampling* of f, i. e., the computation of $(R_{\varphi,m}f)(x)$ for $x \in \mathbb{Z} \setminus \frac{1}{L}\,\mathbb{Z}$ requires only $2m$ samples $f\big(\frac{k}{L}\big)$, where $k \in \mathbb{Z}$ fulfills the condition $|k - Lx| \le m$. If f has bandwidth $\le N/2$ and if $L \ge N$, then the reconstruction of f on the interval $[0,\,1]$ requires only $2m + L$ samples $f\big(\frac{\ell}{L}\big)$ with $\ell = -m,\, 1-m,\, \ldots,\, m+L$.
3. In many applications, one usually employs *oversampling*, i. e., a function $f \in L^2(\mathbb{R})$ of bandwidth $\le N/2$ is sampled on a finer grid $\frac{1}{L}\,\mathbb{Z}$ with $L > N$.

This concept of regularized Shannon sampling formulas with localized sampling and oversampling has already been studied by various authors, e. g. in [12, 15] and references therein for the Gaussian window function. An improvement of the theoretical error bounds for the Gaussian window function was made by [5], whereas oversampling was neglected in this work. Rather, the special case $L = N = 1$ was studied. The case of erroneous sampling for the Gaussian window function was examined in [14]. Generalizations of the Gaussian regularized Shannon sampling formula to holomorphic functions f were introduced by [16] using contour integration and by [18] for the approximation of derivatives of f. A survey of different approaches for window functions can be found in [13]. Furthermore, in [17] the problem was approached in Fourier space. Oversampling then is equivalent to continuing the Fourier transform of the sinc function on the larger interval $[-L/2, L/2]$. Here the aim is to find a regularization function whose Fourier transform is smooth. However, the resulting function does not have an explicit representation and therefore cannot be directly used in spatial domain. Nevertheless, the complexity and efficiency of the received methods was not the main focus of the aforementioned approaches. On the contrary, we now propose new window functions φ such that smaller truncation parameters m are sufficient for achieving high accuracy, therefore yielding short sums being evaluable very fast.

In this paper we present new regularized Shannon sampling formulas with localized sampling. We derive new estimates of the uniform approximation error

$$\max_{x\in\mathbb{R}} \big|f(x) - (R_{\varphi,m}f)(x)\big|\,,$$

where we apply several window functions φ, such as rectangular, Gaussian, B–spline, and sinh-type window functions. It is shown that the uniform approximation error decays exponentially with respect to m, if $\varphi \in \Phi_{m,L}$ is the Gaussian, B–spline, or sinh-type window function. Otherwise, if $\varphi \in \Phi_{m,L}$ is chosen as the rectangular window function, then the uniform approximation error of the regularized Shannon sampling formula has a poor decay of order $m^{-1/2}$. Further we show that the regularized Shannon sampling formulas are numerically robust for noisy samples, i. e., if $\varphi \in \Phi_{m,L}$ is the Gaussian, B–spline, or sinh-type window function, then the uniform perturbation error only grows as $m^{1/2}$.

In our approach we need the Fourier transform of the product of $\mathrm{sinc}(L\pi\,\cdot)$ and the window function φ. Since the sinc function belongs to $L^2(\mathbb{R})$, but not to $L^1(\mathbb{R})$, we present the convolution property of the Fourier transform for $L^2(\mathbb{R})$ functions in the preliminary Section 2 (see Lemma 2.2). In Section 3 we consider regularized Shannon sampling formulas for an arbitrary window function $\varphi \in \Phi_{m,L}$. Here the main results are Theorem 3.2 (with a unified approach to error estimates for regularized Shannon sampling formulas) and Theorem 3.4 (on the numerical robustness of regularized Shannon sampling formulas). In Section 4 we consider the Gaussian window function (as in [12, 5]). In Theorem 4.3 it is shown that the uniform approximation error decays exponentially with respect to m. In Section 5 we use the B–spline window function and prove in Theorem 5.5 that the uniform approximation error decays exponentially with respect to m. In Section 6 we discuss the sinh-type window function. Then in Theorem 6.3 it is proved that the uniform approximation error decays exponentially with respect to m. Several numerical experiments illustrate the theoretical results. Finally, in the concluding Section 7, we compare the proposed window functions and show the superiority of the new proposed sinh-type window function.

2 Convolution property of the Fourier transform

Let $C_0(\mathbb{R})$ denote the Banach space of continuous functions $f\colon \mathbb{R} \to \mathbb{C}$ vanishing as $|x| \to \infty$ with norm

$$\|\hat{f}\|_{C_0(\mathbb{R})} \coloneqq \max_{t\in\mathbb{R}} |\hat{f}(t)|\,.$$

As known, the Fourier transform defined by

$$\hat{f}(v) \coloneqq \int_{\mathbb{R}} f(t)\,\mathrm{e}^{-2\pi\mathrm{i}vt}\,\mathrm{d}t\,, \quad v \in \mathbb{R}\,, \tag{2.1}$$

is a continuous mapping from $L^1(\mathbb{R})$ into $C_0(\mathbb{R})$ with

$$\|\hat{f}\|_{C_0(\mathbb{R})} \le \|f\|_{L^1(\mathbb{R})} \coloneqq \int_{\mathbb{R}} |f(t)|\,\mathrm{d}t\,.$$

Here we are interested in the Hilbert space $L^2(\mathbb{R})$ with inner product and norm

$$\langle f, g\rangle_{L^2(\mathbb{R})} := \int_{\mathbb{R}} f(t)\, \overline{g(t)}\, \mathrm{d}t\,, \quad \|f\|_{L^2(\mathbb{R})} := \big(\langle f, f\rangle_{L^2(\mathbb{R})}\big)^{1/2}\,.$$

By the theorem of Plancherel, the Fourier transform is also an invertible, continuous mapping from $L^2(\mathbb{R})$ onto itself with $\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}$.

For $f,\, g \in L^1(\mathbb{R})$, the convolution property of the Fourier transform reads as follows

$$(f * g)\hat{} = \hat{f}\, \hat{g} \in C_0(\mathbb{R})\,, \tag{2.2}$$

where the convolution is defined as

$$(f * g)(x) := \int_{\mathbb{R}} f(x-t)\, g(t)\, \mathrm{d}t\,, \quad x \in \mathbb{R}\,.$$

However, for any $f,\, g \in L^2(\mathbb{R})$ the convolution property of the Fourier transform is not true in the form (2.2), since by Young's inequality $f * g \in C_0(\mathbb{R})$, by Hölder's inequality $\hat{f}\, \hat{g} \in L^1(\mathbb{R})$ and since the Fourier transform does not map $C_0(\mathbb{R})$ into $L^1(\mathbb{R})$. Thus, the convolution property of the Fourier transform in $L^2(\mathbb{R})$ has the following form:

Lemma 2.1. *For all $f,\, g \in L^2(\mathbb{R})$ it holds*

$$f * g = (\hat{f}\, \hat{g})\check{} \in C_0(\mathbb{R})\,, \tag{2.3}$$

where $\check{h}$ denotes the inverse Fourier transform of $h \in L^1(\mathbb{R})$ defined as

$$\check{h}(t) := \int_{\mathbb{R}} h(v)\, \mathrm{e}^{2\pi \mathrm{i} v t}\, \mathrm{d}v\,, \quad t \in \mathbb{R}\,.$$

Proof. For arbitrary $f,\, g \in L^2(\mathbb{R})$ it holds $\hat{f},\, \hat{g} \in L^2(\mathbb{R})$. Since the Schwartz space $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, there exist sequences $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ in $\mathcal{S}(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^2(\mathbb{R})} = \lim_{n\to\infty} \|g_n - g\|_{L^2(\mathbb{R})} = 0\,. \tag{2.4}$$

Since the Fourier transform is a continuous mapping on $L^2(\mathbb{R})$, it follows that

$$\lim_{n\to\infty} \|\hat{f}_n - \hat{f}\|_{L^2(\mathbb{R})} = \lim_{n\to\infty} \|\hat{g}_n - \hat{g}\|_{L^2(\mathbb{R})} = 0\,. \tag{2.5}$$

If we write

$$(f * g) - (f_n * g_n) = (f - f_n) * g + f_n * (g - g_n)\,,$$

we see by the triangle inequality and Young's inequality that

$$\|(f * g) - (f_n * g_n)\|_{C_0(\mathbb{R})} \le \|f - f_n\|_{L^2(\mathbb{R})}\, \|g\|_{L^2(\mathbb{R})} + \|f_n\|_{L^2(\mathbb{R})}\, \|g - g_n\|_{L^2(\mathbb{R})}$$

and hence by (2.4)

$$\lim_{n\to\infty} \|(f * g) - (f_n * g_n)\|_{C_0(\mathbb{R})} = 0\,. \tag{2.6}$$

If we write

$$\hat{f}\, \hat{g} - \hat{f}_n\, \hat{g}_n = (\hat{f} - \hat{f}_n)\, \hat{g} + \hat{f}_n\, (\hat{g} - \hat{g}_n)\,,$$

we see by the triangle inequality and Hölder's inequality that

$$\|\hat f\,\hat g - \hat f_n\,\hat g_n\|_{L^1(\mathbb{R})} \le \|\hat f - \hat f_n\|_{L^2(\mathbb{R})}\,\|\hat g\|_{L^2(\mathbb{R})} + \|\hat f_n\|_{L^2(\mathbb{R})}\,\|\hat g - \hat g_n\|_{L^2(\mathbb{R})}$$

and hence by (2.5)

$$\lim_{n\to\infty} \|\hat f\,\hat g - \hat f_n\,\hat g_n\|_{L^1(\mathbb{R})} = 0\,. \tag{2.7}$$

By the convolution property of the Fourier transform in $\mathcal{S}(\mathbb{R})$, we have for $f_n,\, g_n \in \mathcal{S}(\mathbb{R})$ that

$$(f_n * g_n)\hat{} = \hat f_n\,\hat g_n\,.$$

Note that $f_n * g_n \in \mathcal{S}(\mathbb{R})$ and $\hat f_n\,\hat g_n \in \mathcal{S}(\mathbb{R})$ (see [9, p. 175]). Since the Fourier transform on $\mathcal{S}(\mathbb{R})$ is invertible (see [9, p. 175]), it holds

$$f_n * g_n = (\hat f_n\,\hat g_n)\check{}\,. \tag{2.8}$$

Moreover, since the inverse Fourier transform is a continuous mapping from $L^1(\mathbb{R})$ into $C_0(\mathbb{R})$, it holds by [9, pp. 66–67] that

$$\|(\hat f\,\hat g)\check{} - (\hat f_n\,\hat g_n)\check{}\|_{C_0(\mathbb{R})} \le \|\hat f\,\hat g - \hat f_n\,\hat g_n\|_{L^1(\mathbb{R})}\,.$$

From (2.7) it follows that

$$\lim_{n\to\infty} \|(\hat f\,\hat g)\check{} - (\hat f_n\,\hat g_n)\check{}\|_{C_0(\mathbb{R})} = 0\,. \tag{2.9}$$

Thus, by (2.8) we conclude that

$$\begin{aligned}\|f * g - (\hat f\,\hat g)\check{}\|_{C_0(\mathbb{R})} &\le \|f * g - f_n * g_n\|_{C_0(\mathbb{R})} + \|f_n * g_n - (\hat f\,\hat g)\check{}\|_{C_0(\mathbb{R})}\\ &= \|f * g - f_n * g_n\|_{C_0(\mathbb{R})} + \|(\hat f_n\,\hat g_n)\check{} - (\hat f\,\hat g)\check{}\|_{C_0(\mathbb{R})}\,.\end{aligned}$$

For $n \to \infty$ the right hand side of above estimate converges to zero by (2.6) and (2.9). This implies (2.3). ■

We obtain the following equivalent formulation of the convolution property in $L^2(\mathbb{R})$, if we replace $f \in L^2(\mathbb{R})$ by $\hat f \in L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$ by $\hat g \in L^2(\mathbb{R})$ in (2.3).

Lemma 2.2. *For all $f,\, g \in L^2(\mathbb{R})$ it holds*

$$\hat f * \hat g = (f\,g)\check{} \in C_0(\mathbb{R})\,. \tag{2.10}$$

Proof. For any $f,\, g \in L^2(\mathbb{R})$ it holds

$$\hat{\hat f} = f(-\,\cdot)\,, \quad \hat{\hat g} = g(-\,\cdot)\,.$$

Note that by Hölder's inequality it holds $f\,g \in L^1(\mathbb{R})$. Then by above Lemma 2.1 it follows that

$$\begin{aligned}(\hat f * \hat g)(t) &= \big(\hat{\hat f}\,\hat{\hat g}\big)\check{}(t) = \big(f(-\,\cdot)\,g(-\,\cdot)\big)\check{}(t)\\ &= \int_{\mathbb{R}} f(-v)\,g(-v)\,\mathrm{e}^{2\pi\mathrm{i}vt}\,\mathrm{d}v = \int_{\mathbb{R}} f(v)\,g(v)\,\mathrm{e}^{-2\pi\mathrm{i}vt}\,\mathrm{d}v = (f\,g)\hat{}(t)\,.\end{aligned}$$

This completes the proof. ■

Note that Lemma 2.2 improves a corresponding result in [2, p. 209]. There it was remarked that for $f,\, g \in L^2(\mathbb{R})$ it holds $(f\,g)\hat{} = \hat f * \hat g \in L^\infty(\mathbb{R})$, but by Lemma 2.2 the function $\hat f * \hat g$ belongs to $C_0(\mathbb{R}) \subset L^\infty(\mathbb{R})$.

3 Regularized Shannon sampling formulas with localized sampling

Let

$$\mathcal{B}_\delta(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subseteq [-\delta,\, \delta]\}$$

be the *Paley–Wiener space*. The functions of $\mathcal{B}_\delta(\mathbb{R})$ are called *bandlimited to* $[-\delta,\, \delta]$, where $\delta > 0$ is the so-called *bandwidth*. By definition, the Paley–Wiener space $\mathcal{B}_\delta(\mathbb{R})$ consists of equivalence classes of almost equal functions. Each of these equivalence classes contains a smooth function, since by inverse Fourier transform it holds for each $r \in \mathbb{N}_0$ that

$$f^{(r)}(x) = \int_{-\delta}^{\delta} \hat{f}(v)\, (2\pi \mathrm{i} v)^r\, \mathrm{e}^{2\pi \mathrm{i} v x}\, \mathrm{d}v\,,$$

i. e., $f^{(r)} \in C_0(\mathbb{R})$, because $(2\pi\mathrm{i}\cdot)^r\, \hat{f} \in L^1([-\delta,\, \delta])$. In the following we will always select the smooth representation of an equivalence class in $\mathcal{B}_\delta(\mathbb{R})$.

In this paper, we consider bandlimited functions $f \in \mathcal{B}_\delta(\mathbb{R})$ with $\delta \in (0,\, N/2)$, where $N \in \mathbb{N}$ is fixed. For $L := N(1+\lambda)$ with $\lambda \geq 0$, and any $m \in \mathbb{N} \setminus \{1\}$ with $2m \ll L$, we introduce the set $\Phi_{m,L}$ of all window functions $\varphi : \mathbb{R} \to [0,\, 1]$ with the following properties:

- Each window function φ belongs to $L^2(\mathbb{R})$ and is even, positive on $(-m/L,\, m/L)$ and continuous on $\mathbb{R} \setminus \{-m/L,\, m/L\}$.
- Each restricted window function $\varphi|_{[0,\,\infty)}$ is decreasing with $\varphi(0) = 1$.
- For each window function φ, its Fourier transform

$$\hat{\varphi}(v) := \int_{\mathbb{R}} \varphi(t)\, \mathrm{e}^{-2\pi \mathrm{i} v t}\, \mathrm{d}t = 2 \int_0^\infty \varphi(t)\, \cos(2\pi\, vt)\, \mathrm{d}t$$

is explicitly known.

Examples of such window functions are the *rectangular window function*

$$\varphi_{\mathrm{rect}}(x) := \mathbf{1}_{[-m/L,\, m/L]}(x)\,, \quad x \in \mathbb{R}\,,$$

where $\mathbf{1}_{[-m/L,\, m/L]}$ is the indicator function of the interval $[-m/L,\, m/L]$, the modified B-*spline window function*

$$\varphi_{\mathrm{B}}(x) := \frac{1}{M_{2s}(0)}\, M_{2s}\Big(\frac{Lxs}{m}\Big)\,, \quad x \in \mathbb{R}\,,$$

where M_{2s} is the centered cardinal B–spline of even order $2s$, the *Gaussian window function*

$$\varphi_{\mathrm{Gauss}}(x) := \mathrm{e}^{-x^2/(2\sigma^2)}\,, \quad x \in \mathbb{R}\,,$$

with some $\sigma > 0$, and the sinh-*type window function*

$$\varphi_{\sinh}(x) := \frac{1}{\sinh \beta}\, \sinh\big(\beta\, \sqrt{1-(Lx/m)^2}\big)\, \mathbf{1}_{[-m/L,\, m/L]}(x)\,, \quad x \in \mathbb{R}\,,$$

with certain $\beta > 0$. All these window functions are well-studied in the context of the nonequispaced fast Fourier transform (NFFT), see e. g. [10] and references therein.

Let $\varphi \in \Phi_{m,L}$ be a given window function. By

$$\varphi_m(x) := \varphi(x)\, \mathbf{1}_{[-m/L,\, m/L]}(x)\,, \quad x \in \mathbb{R}\,, \tag{3.1}$$

we denote the *truncated window function* of $\varphi \in \Phi_{m,L}$. We shall use the *regularized Shannon sampling formula with localized sampling*

$$(R_{\varphi,m}f)(t) := \sum_{\ell\in\mathbb{Z}} f\big(\tfrac{\ell}{L}\big)\,\mathrm{sinc}\big(L\pi\,\big(t-\tfrac{\ell}{L}\big)\big)\,\varphi_m\big(t-\tfrac{\ell}{L}\big)\,, \quad t\in\mathbb{R}\,, \tag{3.2}$$

to rapidly reconstruct the values $f(t)$ for $t \in \mathbb{R}$ from given sampling data $f\big(\frac{\ell}{L}\big)$, $\ell \in \mathbb{Z}$, with high accuracy. Obviously, $R_{\varphi,m}f$ interpolates f on $\frac{1}{L}\,\mathbb{Z}$. For $t \in \big(0,\,\frac{1}{L}\big)$ the regularized Shannon sampling formula reads as follows

$$(R_{\varphi,m}f)(t) = \sum_{\ell\in\mathcal{I}_{2m}^{\mathrm{r}}} f\big(\tfrac{\ell}{L}\big)\,\mathrm{sinc}\big(L\pi\,\big(t-\tfrac{\ell}{L}\big)\big)\,\varphi_m\big(t-\tfrac{\ell}{L}\big)$$

with the index set $\mathcal{I}_{2m}^{\mathrm{r}} := \{-m+1,\,-m+2,\,\ldots,\,m\}$.

It is known that $\{\mathrm{sinc}\big(L\pi\,\big(\,\cdot\,-\frac{\ell}{L}\big)\big) : \ell\in\mathbb{Z}\}$ is an orthogonal system in $L^2(\mathbb{R})$, since by the shifting property the Fourier transform of $\mathrm{sinc}\big(L\pi\,\big(\,\cdot\,-\frac{\ell}{L}\big)\big)$ is equal to

$$\frac{1}{L}\,\mathrm{e}^{-2\pi\mathrm{i}\ell v/L}\,\mathbf{1}_{[-L/2,\,L/2]}(v)\,, \quad v\in\mathbb{R}\,,$$

and since by the Parseval identity it holds for all $\ell,\,k\in\mathbb{Z}$ that

$$\big\langle \mathrm{sinc}\big(L\pi\,\big(\,\cdot\,-\tfrac{\ell}{L}\big)\big),\,\mathrm{sinc}\big(L\pi\,\big(\,\cdot\,-\tfrac{k}{L}\big)\big)\big\rangle_{L^2(\mathbb{R})} = \frac{1}{L^2}\int_{-L/2}^{L/2} \mathrm{e}^{2\pi\mathrm{i}(k-\ell)\,v/L}\,\mathrm{d}v = \frac{1}{L}\,\delta_{\ell-k}$$

with the Kronecker symbol $\delta_{\ell-k}$. Hence, the system $\{\mathrm{sinc}\big(L\pi\,\big(\,\cdot\,-\frac{\ell}{L}\big)\big) : \ell\in\mathbb{Z}\}$ forms an orthogonal basis of $\mathcal{B}_{N/2}(\mathbb{R})$ with

$$\big\langle \mathrm{sinc}\big(L\pi\,\big(\,\cdot\,-\tfrac{\ell}{L}\big)\big),\,\mathrm{sinc}\big(L\pi\,\big(\,\cdot\,-\tfrac{k}{L}\big)\big)\big\rangle_{L^2(\mathbb{R})} = \tfrac{1}{L}\,\delta_{\ell-k} \tag{3.3}$$

for all $\ell,\,k\in\mathbb{Z}$. From (1.1) and (3.3) it follows that for any $f\in\mathcal{B}_\delta(\mathbb{R})\subset\mathcal{B}_{L/2}(\mathbb{R})$ with $\delta\in(0,\,N/2)$ and $L\geq N$ it holds

$$\|f\|_{L^2(\mathbb{R})}^2 = \tfrac{1}{L}\sum_{\ell\in\mathbb{Z}}\big|f\big(\tfrac{\ell}{L}\big)\big|^2\,. \tag{3.4}$$

Firstly, we consider the regularized Shannon sampling formula (3.2) with the simple rectangular window function $\varphi = \varphi_{\mathrm{rect}}$, i. e., for some $m\in\mathbb{N}\setminus\{1\}$ we form the *rectangular regularized Shannon sampling formula with localized sampling*

$$(R_{\mathrm{rect},m}f)(t) := \sum_{\ell\in\mathbb{Z}} f\big(\tfrac{\ell}{L}\big)\,\mathrm{sinc}\big(L\pi\,\big(t-\tfrac{\ell}{L}\big)\big)\,\mathbf{1}_{[-m/L,\,m/L]}\big(t-\tfrac{\ell}{L}\big)\,, \quad t\in\mathbb{R}\,. \tag{3.5}$$

Obviously, the rectangular regularized Shannon sampling formula (3.5) interpolates f on the grid $\frac{1}{L}\,\mathbb{Z}$, i. e., for all $m\in\mathbb{N}\setminus\{1\}$, the interpolation property

$$f\big(\tfrac{\ell}{L}\big) = (R_{\mathrm{rect},m}f)\big(\tfrac{\ell}{L}\big)\,, \quad \ell\in\mathbb{Z}\,, \tag{3.6}$$

is fulfilled. Especially for $t\in\big(0,\,\frac{1}{L}\big)$, we obtain the rectangular regularized Shannon sampling formula

$$(R_{\mathrm{rect},m}f)(t) = \sum_{\ell\in\mathcal{I}_{2m}^{\mathrm{r}}} f\big(\tfrac{\ell}{L}\big)\,\mathrm{sinc}\big(L\pi\,\big(t-\tfrac{\ell}{L}\big)\big)$$

as an approximation to $f(t)$. On any interval $\big(\frac{k}{L},\, \frac{k+1}{L}\big)$ with $k \in \mathbb{Z}$ the rectangular regularized Shannon sampling formula reads as follows

$$(R_{\mathrm{rect},m}f)\big(t + \tfrac{k}{L}\big) = \sum_{\ell \in \mathcal{I}_{2m}^{\mathrm{r}}} f\big(\tfrac{\ell+k}{L}\big)\, \mathrm{sinc}\big(L\pi\,(t - \tfrac{\ell}{L})\big)\,, \quad t \in \big(0,\, \tfrac{1}{L}\big)\,. \tag{3.7}$$

Since the sinc function decays slowly at infinity, (3.5) is not a good approximation to f on $\mathbb{R}$. The convergence rate of the sequence $\big(f - R_{\mathrm{rect},m}f\big)_{m=1}^{\infty}$ is only $\mathcal{O}(m^{-1/2})$ for sufficiently large m. The following lemma is a consequence of a result in [7].

Lemma 3.1. *Let $f \in \mathcal{B}_{N/2}(\mathbb{R})$ with fixed $N \in \mathbb{N}$, $L := N(1+\lambda)$ with $\lambda \ge 0$ and $m \in \mathbb{N} \setminus \{1\}$ be given. Then it holds*

$$\|f - R_{\mathrm{rect},m}f\|_{C_0(\mathbb{R})} \le \frac{L}{\pi}\sqrt{\frac{2}{m} + \frac{1}{m^2}}\, \|f\|_{L^2(\mathbb{R})}\,.$$

Proof. Since $R_{\mathrm{rect},m}f$ possesses similar representations (3.7) on each interval $\big(\frac{k}{L},\, \frac{k+1}{L}\big)$, $k \in \mathbb{Z}$, we consider $f(t) - (R_{m,\mathrm{rect}}f)(t)$ only for $t \in \big[0,\, \frac{1}{L}\big]$ and show that

$$\max_{t \in [0,\,1/L]} |f(t) - (R_{\mathrm{rect},m}f)(t)| \le \frac{L}{\pi}\sqrt{\frac{2}{m} + \frac{1}{m^2}}\, \|f\|_{L^2(\mathbb{R})}\,. \tag{3.8}$$

For $f \in \mathcal{B}_{N/2}(\mathbb{R})$ it holds the equality (3.4). The Whittaker–Kotelnikov–Shannon sampling theorem (see (1.1)) implies that

$$f(t) - (R_{\mathrm{rect},m}f)(t) = \sum_{\ell \in \mathbb{Z}\setminus\mathcal{I}_{2m}^{\mathrm{r}}} f\big(\tfrac{\ell}{L}\big)\, \mathrm{sinc}\big(L\pi\,(t - \tfrac{\ell}{L})\big)\,.$$

We introduce the auxiliary function

$$h_m(t) := \sum_{\ell \in \mathbb{Z}\setminus\mathcal{I}_{2m}^{\mathrm{r}}} \big[\mathrm{sinc}\big(L\pi\,(t - \tfrac{\ell}{L})\big)\big]^2 \ge 0\,, \quad t \in \big[0,\, \tfrac{1}{L}\big]\,.$$

Then by the Cauchy–Schwarz inequality and (3.4) it follows that

$$\begin{aligned} \big|f(t) - (R_{\mathrm{rect},m}f)(t)\big| &= \Big|\sum_{\ell \in \mathbb{Z}\setminus\mathcal{I}_{2m}^{\mathrm{r}}} f\big(\tfrac{\ell}{L}\big)\, \mathrm{sinc}\big(L\pi\,(t - \tfrac{\ell}{L})\big)\Big| \\ &\le \Big(\sum_{\ell \in \mathbb{Z}\setminus\mathcal{I}_{2m}^{\mathrm{r}}} \big|f\big(\tfrac{\ell}{L}\big)\big|^2\Big)^{1/2} \Big(\sum_{\ell \in \mathbb{Z}\setminus\mathcal{I}_{2m}^{\mathrm{r}}} \big[\mathrm{sinc}\big(L\pi\,(t - \tfrac{\ell}{L})\big)\big]^2\Big)^{1/2} \\ &\le L\,\sqrt{h_m(t)}\, \|f\|_{L^2(\mathbb{R})}\,. \end{aligned} \tag{3.9}$$

By the integral test for convergence of series we estimate the function

$$\begin{aligned} h_m(t) &= \frac{(\sin(L\pi t))^2}{\pi^2} \sum_{\ell \in \mathbb{Z}\setminus\mathcal{I}_{2m}^{\mathrm{r}}} \frac{1}{(Lt-\ell)^2} \le \frac{1}{\pi^2} \sum_{k \in \mathbb{Z}\setminus\mathcal{I}_{2m}^{\mathrm{r}}} \frac{1}{k^2} \\ &\le \frac{1}{\pi^2}\left(\frac{1}{m^2} + 2\int_m^{\infty} \frac{1}{t^2}\,\mathrm{d}t\right) = \frac{1}{\pi^2}\left(\frac{1}{m^2} + \frac{2}{m}\right) \end{aligned} \tag{3.10}$$

for $t \in \big(0,\, \frac{1}{L}\big)$. Then from (3.9) and (3.10) it follows for each $t \in \big(0,\, \frac{1}{L}\big)$ that

$$\big|f(t) - (R_{\mathrm{rect},m} f)(t)\big| \le \frac{L}{\pi}\, \sqrt{\frac{2}{m} + \frac{1}{m^2}}\, \|f\|_{L^2(\mathbb{R})}\,.$$

This inequality is also true for $t = 0$ and $t = \frac{1}{L}$ by (3.6). Hence, this implies the estimate (3.8).

By the same technique, the above estimate of the approximation error

$$\max_{t \in [k/L,\,(k+1)/L]} \big|f(t) - (R_{\mathrm{rect},m} f)(t)\big|$$

can be shown for each $k \in \mathbb{Z}$. This completes the proof. ■

In view of the slow convergence of the sequence $\big(R_{\mathrm{rect},m} f(t)\big)_{m=1}^{\infty}$, it has been proposed to modify the rectangular regularized Shannon sampling sum (3.5) by multiplying the sinc function with a more convenient truncated window function $\varphi \in \Phi_{m,L}$ (see [12] and [5]). For any $m \in \mathbb{N} \setminus \{1\}$ the regularized Shannon sampling formula with localized sampling is given by

$$(R_{\varphi,m} f)(t) = \sum_{\ell \in \mathbb{Z}} f\big(\tfrac{\ell}{L}\big)\, \mathrm{sinc}\big(L\pi\, \big(t - \tfrac{\ell}{L}\big)\big)\, \varphi_m\big(t - \tfrac{\ell}{L}\big)\,, \quad t \in \mathbb{R}\,, \tag{3.11}$$

with the truncated window function $\varphi_m(x) = \varphi(x)\, \mathbf{1}_{[-m/L,\, m/L]}(x)$. Note that it holds the interpolation property

$$f\big(\tfrac{\ell}{L}\big) = (R_{\varphi,m} f)\big(\tfrac{\ell}{L}\big)\,, \quad \ell \in \mathbb{Z}\,. \tag{3.12}$$

We reconstruct f by $R_{\varphi,m} f$ for each open interval $\big(\frac{k}{L},\, \frac{k+1}{L}\big)$, $k \in \mathbb{Z}$. Especially for $t \in \big(0,\, \frac{1}{L}\big)$, we obtain the regularized Shannon sampling formula

$$(R_{\varphi,m} f)(t) = \sum_{\ell \in \mathcal{I}_{2m}^{\mathrm{r}}} f\big(\tfrac{\ell}{L}\big)\, \psi\big(t - \tfrac{\ell}{L}\big)\,,$$

where

$$\psi(x) \coloneqq \mathrm{sinc}(L\pi x)\, \varphi(x) \tag{3.13}$$

is the *regularized* sinc *function.* For the reconstruction of f on any interval $\big(\frac{k}{L},\, \frac{k+1}{L}\big)$ with $k \in \mathbb{Z}$, we use

$$(R_{\varphi,m} f)\big(t + \tfrac{k}{L}\big) = \sum_{\ell \in \mathcal{I}_{2m}^{\mathrm{r}}} f\big(\tfrac{\ell+k}{L}\big)\, \psi\big(t - \tfrac{\ell}{L}\big)\,, \quad t \in \big(0,\, \tfrac{1}{L}\big)\,. \tag{3.14}$$

Now we estimate the uniform approximation error

$$\|f - R_{\varphi,m} f\|_{C_0(\mathbb{R})} \coloneqq \max_{t \in \mathbb{R}} \big|f(t) - (R_{\varphi,m} f)(t)\big| \tag{3.15}$$

of the regularized Shannon sampling formula.

Theorem 3.2. *Let $f \in \mathcal{B}_\delta(\mathbb{R})$ with $\delta = \tau N$, $\tau \in (0,\, 1/2)$, $N \in \mathbb{N}$, $L = N(1+\lambda)$ with $\lambda \geq 0$ and $m \in \mathbb{N} \setminus \{1\}$ be given. Further let $\varphi \in \Phi_{m,L}$ with the truncated window function* (3.1) *be given.*
Then the regularized Shannon sampling formula (3.11) *with localized sampling satisfies*

$$\|f - R_{\varphi,m} f\|_{C_0(\mathbb{R})} \leq \big(E_1(m,\delta,L) + E_2(m,\delta,L)\big)\, \|f\|_{L^2(\mathbb{R})}\,, \tag{3.16}$$

where the corresponding error constants are defined by

$$E_1(m,\delta,L) \coloneqq \sqrt{2\delta}\, \max_{v \in [-\delta,\delta]} \left| 1 - \int_{v-\frac{L}{2}}^{v+\frac{L}{2}} \hat{\varphi}(u)\, \mathrm{d}u \right|\,, \tag{3.17}$$

$$E_2(m,\delta,L) \coloneqq \frac{\sqrt{2L}}{\pi m} \left(\varphi^2\big(\tfrac{m}{L}\big) + L \int_{\frac{m}{L}}^{\infty} \varphi^2(t)\, \mathrm{d}t \right)^{1/2}\,. \tag{3.18}$$

Proof. Initially, we only consider the error on the interval $\big[0,\, \frac{1}{L}\big]$. Here we split the approximation error

$$f(t) - (R_{\varphi,m} f)(t) = e_1(t) + e_{2,0}(t)\,, \quad t \in \big[0,\, \tfrac{1}{L}\big]\,,$$

into the regularization error

$$e_1(t) \coloneqq f(t) - \sum_{\ell \in \mathbb{Z}} f\big(\tfrac{\ell}{L}\big)\, \psi\big(t - \tfrac{\ell}{L}\big)\,, \quad t \in \mathbb{R}\,, \tag{3.19}$$

and the truncation error

$$e_{2,0}(t) \coloneqq \sum_{\ell \in \mathbb{Z}} f\big(\tfrac{\ell}{L}\big)\, \psi\big(t - \tfrac{\ell}{L}\big) - (R_m f)(t)\,, \quad t \in \big[0,\, \tfrac{1}{L}\big]\,, \tag{3.20}$$

where ψ denotes the regularized sinc function (3.13).

We start with the regularization error (3.19). By Lemma 2.2, the Fourier transform of ψ reads as

$$\hat{\psi}(v) = \frac{1}{L} \int_{\mathbb{R}} \mathbf{1}_{[-L/2,\, L/2]}(v-u)\, \hat{\varphi}(u)\, \mathrm{d}u = \frac{1}{L} \int_{v-L/2}^{v+L/2} \hat{\varphi}(u)\, \mathrm{d}u\,.$$

Hence, using the shifting property of the Fourier transform, the Fourier transform of $\psi\big(\cdot - \frac{\ell}{L}\big)$ reads as

$$\frac{1}{L}\, \mathrm{e}^{-2\pi \mathrm{i} v \ell/L} \int_{v-L/2}^{v+L/2} \hat{\varphi}(u)\, \mathrm{d}u\,.$$

Therefore, the Fourier transform of the regularization error e_1 has the form

$$\hat{e}_1(v) = \hat{f}(v) - \left(\sum_{\ell \in \mathbb{Z}} f\big(\tfrac{\ell}{L}\big)\, \frac{1}{L}\, \mathrm{e}^{-2\pi \mathrm{i} v \ell/L} \right) \int_{v-L/2}^{v+L/2} \hat{\varphi}(u)\, \mathrm{d}u\,. \tag{3.21}$$

By the assumption $f \in \mathcal{B}_\delta(\mathbb{R})$ with $\delta \in (0,\, N/2)$ and $L \geq N$, it holds

$$\mathrm{supp}\, \hat{f} \subseteq [-\delta,\, \delta] \subset [-L/2,\, L/2]$$

and hence the restricted function $\hat{f}|_{[-L/2,\,L/2]}$ belongs to $L^2([-L/2,\,L/2])$. Thus, this function possesses the L-periodic Fourier expansion

$$\hat{f}(v) = \sum_{k\in\mathbb{Z}} c_k(\hat{f})\,\mathrm{e}^{-2\pi\mathrm{i}kv/L}\,, \quad v \in [-L/2,\,L/2]\,,$$

with the Fourier coefficients

$$c_k(\hat{f}) = \frac{1}{L}\int_{-L/2}^{L/2} \hat{f}(u)\,\mathrm{e}^{2\pi\mathrm{i}ku/L}\,\mathrm{d}u = \frac{1}{L}\,f\big(\tfrac{k}{L}\big)\,, \quad k \in \mathbb{Z}\,.$$

Then $\hat{f}$ can be represented as

$$\hat{f}(v) = \hat{f}(v)\,\mathbf{1}_{[-\delta,\,\delta]}(v) = \Big(\sum_{k\in\mathbb{Z}} \tfrac{1}{L}\,f\big(\tfrac{k}{L}\big)\,\mathrm{e}^{-2\pi\mathrm{i}kv/L}\Big)\,\mathbf{1}_{[-\delta,\,\delta]}(v)\,, \quad v \in \mathbb{R}\,. \tag{3.22}$$

Introducing the function

$$\eta(v) := \mathbf{1}_{[-\delta,\,\delta]}(v) - \int_{v-L/2}^{v+L/2} \hat{\varphi}(u)\,\mathrm{d}u\,, \quad v \in \mathbb{R}\,, \tag{3.23}$$

we see by inserting (3.22) into (3.21) that $\hat{e}_1(v) = \hat{f}(v)\,\eta(v)$ and thereby $|\hat{e}_1(v)| \le |\hat{f}(v)|\,|\eta(v)|$. Thus, by inverse Fourier transform we get

$$\begin{aligned}|e_1(t)| &= \Big|\int_{\mathbb{R}} \hat{e}_1(v)\,\mathrm{e}^{2\pi\mathrm{i}tv}\,\mathrm{d}v\Big| \le \int_{\mathbb{R}} |\hat{e}_1(v)|\,\mathrm{d}v \le \int_{-\delta}^{\delta} |\hat{f}(v)|\,|\eta(v)|\,\mathrm{d}v \\ &\le \max_{v\in[-\delta,\delta]} |\eta(v)| \int_{-\delta}^{\delta} |\hat{f}(v)|\,\mathrm{d}v\,.\end{aligned}$$

Using Cauchy–Schwarz inequality and Parseval identity, we see that

$$\int_{-\delta}^{\delta} |\hat{f}(v)|\,\mathrm{d}v \;\le\; \Big(\int_{-\delta}^{\delta} 1^2\,\mathrm{d}v\Big)^{1/2} \Big(\int_{-\delta}^{\delta} |\hat{f}(v)|^2\,\mathrm{d}v\Big)^{1/2} = \sqrt{2\delta}\,\|\hat{f}\|_{L^2(\mathbb{R})} = \sqrt{2\delta}\,\|f\|_{L^2(\mathbb{R})}.$$

In summary, using the error constant (3.17) this yields

$$\|e_1\|_{C_0(\mathbb{R})} \le E_1(m,\delta,L)\,\|f\|_{L^2(\mathbb{R})}\,.$$

Now we estimate the truncation error. By (3.20) it holds for $t \in \big(0,\,\frac{1}{L}\big)$ that

$$e_{2,0}(t) = \sum_{\ell\in\mathbb{Z}} f\big(\tfrac{\ell}{L}\big)\,\psi\big(t-\tfrac{\ell}{L}\big)\,\big[1-\mathbf{1}_{[-m/L,m/L]}\big(t-\tfrac{\ell}{L}\big)\big] = \sum_{\ell\in\mathbb{Z}\setminus\mathcal{I}_{2m}^{\mathrm{r}}} f\big(\tfrac{\ell}{L}\big)\,\psi\big(t-\tfrac{\ell}{L}\big)\,.$$

Using (3.13) and the non-negativity of φ, we receive

$$|e_{2,0}(t)| \le \sum_{\ell\in\mathbb{Z}\setminus\mathcal{I}_{2m}^{\mathrm{r}}} \big|f\big(\tfrac{\ell}{L}\big)\big|\,\big|\mathrm{sinc}\big(L\pi\,\big(t-\tfrac{\ell}{L}\big)\big)\big|\,\varphi\big(t-\tfrac{\ell}{L}\big)\,.$$

For $t \in \big(0,\,\frac{1}{L}\big)$ and $\ell \in \mathbb{Z}\setminus\mathcal{I}_{2m}^{\mathrm{r}}$ we obtain

$$\big|\mathrm{sinc}\big(L\pi\,\big(t-\tfrac{\ell}{L}\big)\big)\big| \le \frac{1}{\pi\,|Lt-\ell|} \le \frac{1}{\pi m}$$

and hence

$$|e_{2,0}(t)| \le \frac{1}{\pi m} \sum_{\ell \in \mathbb{Z}\setminus \mathcal{I}_{2m}^{\mathrm{r}}} \big|f\big(\tfrac{\ell}{L}\big)\big|\,\varphi\big(t-\tfrac{\ell}{L}\big)\,.$$

Then the Cauchy–Schwarz inequality implies that

$$|e_{2,0}(t)| \le \frac{1}{\pi m} \bigg(\sum_{\ell \in \mathbb{Z}\setminus \mathcal{I}_{2m}^{\mathrm{r}}} \big|f\big(\tfrac{\ell}{L}\big)\big|^2\bigg)^{1/2} \bigg(\sum_{\ell \in \mathbb{Z}\setminus \mathcal{I}_{2m}^{\mathrm{r}}} \varphi^2\big(t-\tfrac{\ell}{L}\big)\bigg)^{1/2}.$$

By (3.4) it holds

$$\bigg(\sum_{\ell \in \mathbb{Z}\setminus \mathcal{I}_{2m}^{\mathrm{r}}} \big|f\big(\tfrac{\ell}{L}\big)\big|^2\bigg)^{1/2} \le \sqrt{L}\,\|f\|_{L^2(\mathbb{R})}\,.$$

Since $\varphi|_{[0,\,\infty)}$ decreases monotonously by assumption, for $t \in \big(0,\,\frac{1}{L}\big)$ we can estimate the series as follows

$$\begin{aligned}\sum_{\ell \in \mathbb{Z}\setminus \mathcal{I}_{2m}^{\mathrm{r}}} \varphi^2\big(t-\tfrac{\ell}{L}\big) &= \bigg(\sum_{\ell=-\infty}^{-m} + \sum_{\ell=m+1}^{\infty}\bigg) \varphi^2\big(t-\tfrac{\ell}{L}\big) = \sum_{\ell=m}^{\infty} \varphi^2\big(t+\tfrac{\ell}{L}\big) + \sum_{\ell=m+1}^{\infty} \varphi^2\big(t-\tfrac{\ell}{L}\big) \\ &\le \sum_{\ell=m}^{\infty} \varphi^2\big(\tfrac{\ell}{L}\big) + \sum_{\ell=m+1}^{\infty} \varphi^2\big(\tfrac{1}{L}-\tfrac{\ell}{L}\big) = 2\sum_{\ell=m}^{\infty} \varphi^2\big(\tfrac{\ell}{L}\big)\,.\end{aligned}$$

Using the integral test for convergence of series, we obtain that

$$\sum_{\ell=m}^{\infty} \varphi^2\big(\tfrac{\ell}{L}\big) = \varphi^2\big(\tfrac{m}{L}\big) + \sum_{\ell=m+1}^{\infty} \varphi^2\big(\tfrac{\ell}{L}\big) < \varphi^2\big(\tfrac{m}{L}\big) + \int_m^{\infty} \varphi^2\big(\tfrac{t}{L}\big)\,\mathrm{d}t = \varphi^2\big(\tfrac{m}{L}\big) + L\int_{m/L}^{\infty} \varphi^2(t)\,\mathrm{d}t\,.$$

By the interpolation property of $R_{\varphi,m}f$, it holds $e_{2,0}(0) = e_{2,0}\big(\frac{1}{L}\big) = 0$. Hence, we obtain by (3.18) that

$$\max_{t\in[0,1/L]} |e_{2,0}(t)| \le \frac{\sqrt{L}}{\pi m}\,\|f\|_{L^2(\mathbb{R})} \bigg(2\varphi^2\big(\tfrac{m}{L}\big) + 2L\int_{m/L}^{\infty} \varphi^2(t)\,\mathrm{d}t\bigg)^{1/2} = E_2(m,\delta,L)\,\|f\|_{L^2(\mathbb{R})}\,.$$

By the same technique, this error estimate can be shown for each interval $\big[\frac{k}{L},\,\frac{k+1}{L}\big]$ with $k\in\mathbb{Z}$. On the open interval $\big(\frac{k}{L},\,\frac{k+1}{L}\big)$ with $k\in\mathbb{Z}$, we split the error by (3.14) in the form

$$f\big(t+\tfrac{k}{L}\big) - (R_{\varphi,m}f)\big(t+\tfrac{k}{L}\big) = e_1\big(t+\tfrac{k}{L}\big) + e_{2,k}(t)\,, \quad t\in\big(0,\,\tfrac{1}{L}\big)\,,$$

with

$$\begin{aligned} e_1\big(t+\tfrac{k}{L}\big) &= f\big(t+\tfrac{k}{L}\big) - \sum_{\ell\in\mathbb{Z}} f\big(\tfrac{\ell}{L}\big)\,\psi\big(t-\tfrac{\ell-k}{L}\big) = f\big(t+\tfrac{k}{L}\big) - \sum_{\ell\in\mathbb{Z}} f\big(\tfrac{\ell+k}{L}\big)\,\psi\big(t-\tfrac{\ell}{L}\big)\,, \\ e_{2,k}(t) &:= \sum_{\ell \in \mathbb{Z}\setminus \mathcal{I}_{2m}^{\mathrm{r}}} f\big(\tfrac{\ell+k}{L}\big)\,\psi\big(t-\tfrac{\ell}{L}\big)\,.\end{aligned}$$

As above shown, it holds

$$\big\|e_1\big(\,\cdot+\tfrac{k}{L}\big)\big\|_{C_0(\mathbb{R})} = \|e_1\|_{C_0(\mathbb{R})}\,,$$

$$|e_{2,k}(t)| \;\le\; E_2(m,\delta,L)\,\|f\|_{L^2(\mathbb{R})}\,, \quad t \in \big(0,\,\tfrac{1}{L}\big)\,.$$

By the interpolation property of $R_{\varphi,m}f$, it holds $e_{2,0}\big(\frac{k}{L}\big) = e_{2,0}\big(\frac{k+1}{L}\big) = 0$ for each $k \in \mathbb{Z}$. Hence, it follows that

$$\begin{aligned}\max_{t\in[k/L,\,(k+1)/L]} |f(t) - (R_{\varphi,m}f)(t)| \;&\le\; \|e_1\|_{C_0(\mathbb{R})} + \max_{t\in[0,\,1/L]} |e_{2,k}(t)| \\ &\le\; \big(E_1(m,\delta,L) + E_2(m,\delta,L)\big)\,\|f\|_{L^2(\mathbb{R})}\,.\end{aligned}$$

This completes the proof. ■

Remark 3.3. Theorem 3.2 can be simplified, if the window function $\varphi \in \Phi_{m,L}$ is continuous on $\mathbb{R}$ and vanishes on $\mathbb{R} \setminus \big[-\frac{m}{L},\,\frac{m}{L}\big]$. Then the truncation errors $e_{2,k}(t)$ are equal to zero for all $t \in \big(0,\,\frac{1}{L}\big)$ and $k \in \mathbb{Z}$, such that $E_2(m,\delta,L) = 0$. For such window functions $\varphi \in \Phi_{m,L}$ we obtain the simple error estimate

$$\|f - R_{\varphi,m}f\|_{C_0(\mathbb{R})} \le E_1(m,\delta,L)\,\|f\|_{L^2(\mathbb{R})}\,.$$

We remark that this is the case for the B–spline as well as the sinh-type window function, but not for the Gaussian window function since φ_{Gauss} does not vanish on $\mathbb{R} \setminus \big[-\frac{m}{L},\,\frac{m}{L}\big]$. Also the rectangular window function does not fit into this setting since φ_{rect} is not continuous on $\mathbb{R}$.
□

If the samples $f\big(\frac{\ell}{L}\big)$, $\ell \in \mathbb{Z}$, of a bandlimited function $f \in \mathcal{B}_\delta(\mathbb{R})$ are not known exactly, i. e., only erroneous samples $\tilde f_\ell := f\big(\frac{\ell}{L}\big) + \varepsilon_\ell$ with $|\varepsilon_\ell| \le \varepsilon$, $\ell \in \mathbb{Z}$, are known, then the corresponding Shannon sampling series may differ appreciably from f (see [3]). In contrast to the Shannon sampling series, the regularized Shannon sampling formula is numerically robust. Here we denote the regularized Shannon sampling formula with erroneous samples $\tilde f_\ell$ by

$$(R_{\varphi,m}\tilde f)(t) = \sum_{\ell\in\mathbb{Z}} \tilde f_\ell \,\mathrm{sinc}\big(L\pi\,(t - \tfrac{\ell}{L})\big)\,\varphi_m\big(t - \tfrac{\ell}{L}\big)\,, \quad t \in \mathbb{R}\,. \tag{3.24}$$

Theorem 3.4. *Let $f \in \mathcal{B}_\delta(\mathbb{R})$ with $\delta = \tau N$, $\tau \in (0,\,1/2)$, $N \in \mathbb{N}$, $L = N(1+\lambda)$ with $\lambda \ge 0$ and $m \in \mathbb{N} \setminus \{1\}$ be given. Further let $\varphi \in \Phi_{m,L}$ with the truncated window function* (3.1) *as well as $\tilde f_\ell = f(\ell/L) + \varepsilon_\ell$, where $|\varepsilon_\ell| \le \varepsilon$ for all $\ell \in \mathbb{Z}$, be given.*
Then the regularized Shannon sampling sum (3.11) *with localized sampling is numerically robust and satisfies*

$$\|R_{\varphi,m}\tilde f - R_{\varphi,m}f\|_{C_0(\mathbb{R})} \;\le\; \varepsilon\,\big(2 + L\,\hat\varphi(0)\big)\,, \tag{3.25}$$

$$\|f - R_{\varphi,m}\tilde f\|_{C_0(\mathbb{R})} \;\le\; \|f - R_{\varphi,m}f\|_{C_0(\mathbb{R})} + \varepsilon\,\big(2 + L\,\hat\varphi(0)\big)\,. \tag{3.26}$$

Proof. Initially, we only consider the error on the interval $\big[0,\,\frac{1}{L}\big]$. By (3.11) it holds

$$\begin{aligned}\tilde e_0(t) &:= (R_{\varphi,m}\tilde f)(t) - (R_{\varphi,m}f)(t) \\ &= \sum_{\ell\in\mathbb{Z}} \Big(\tilde f_\ell - f\,\big(\tfrac{\ell}{L}\big)\Big)\,\psi\big(t - \tfrac{\ell}{L}\big) = \sum_{\ell\in\mathbb{Z}} \varepsilon_\ell\,\psi\big(t - \tfrac{\ell}{L}\big)\,, \quad t \in \big(0,\,\tfrac{1}{L}\big)\,.\end{aligned}$$

Using (3.13), the non-negativity of φ and $|\varepsilon_\ell| \le \varepsilon$, we receive

$$|\tilde e_0(t)| \le \sum_{\ell\in\mathcal{I}_{2m}^{\mathrm{r}}} |\varepsilon_\ell|\,\big|\mathrm{sinc}\big(L\pi\,(t - \tfrac{\ell}{L})\big)\big|\,\varphi\big(t - \tfrac{\ell}{L}\big) \le \varepsilon \sum_{\ell\in\mathcal{I}_{2m}^{\mathrm{r}}} \varphi\big(t - \tfrac{\ell}{L}\big)\,.$$

Since $\varphi|_{[0,\infty)}$ decreases monotonously by assumption, we can estimate the sum for $t \in \big(0,\, \frac{1}{L}\big)$ as follows

$$\sum_{\ell \in \mathcal{I}_{2m}^{\mathrm{r}}} \varphi\big(t - \tfrac{\ell}{L}\big) = \Big(\sum_{\ell=-m+1}^{0} + \sum_{\ell=1}^{m}\Big)\, \varphi\big(t - \tfrac{\ell}{L}\big) = \sum_{\ell=0}^{m-1} \varphi\big(t + \tfrac{\ell}{L}\big) + \sum_{\ell=1}^{m} \varphi\big(t - \tfrac{\ell}{L}\big)$$
$$\le \sum_{\ell=0}^{m-1} \varphi\big(\tfrac{\ell}{L}\big) + \sum_{\ell=1}^{m} \varphi\big(\tfrac{1}{L} - \tfrac{\ell}{L}\big) = 2 \sum_{\ell=0}^{m-1} \varphi\big(\tfrac{\ell}{L}\big)\,.$$

Using the integral test for convergence of series, we obtain that

$$\sum_{\ell=0}^{m-1} \varphi\big(\tfrac{\ell}{L}\big) < \varphi(0) + \int_0^{m-1} \varphi\big(\tfrac{t}{L}\big)\, \mathrm{d}t = \varphi(0) + L \int_0^{(m-1)/L} \varphi(t)\, \mathrm{d}t\,.$$

By the definition of the Fourier transform (2.1) it holds for $\varphi \in \Phi_{m,L}$ that

$$\hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(t)\, \mathrm{d}t \ge \int_{-m/L}^{m/L} \varphi(t)\, \mathrm{d}t = 2 \int_0^{m/L} \varphi(t)\, \mathrm{d}t \ge 2 \int_0^{(m-1)/L} \varphi(t)\, \mathrm{d}t\,,$$

and therefore

$$|\tilde{e}_0(t)| \le 2\,\varepsilon \sum_{\ell=0}^{m-1} \varphi\big(\tfrac{\ell}{L}\big) \le 2\,\varepsilon\big(\varphi(0) + \tfrac{L}{2}\,\hat{\varphi}(0)\big) = \varepsilon\big(2\,\varphi(0) + L\,\hat{\varphi}(0)\big)\,, \quad t \in \big(0,\, \tfrac{1}{L}\big)\,.$$

Additionally, by the interpolation property (3.12), it holds $|\tilde{e}_0(0)| = |\varepsilon_0| \le \varepsilon$ as well as $\big|\tilde{e}_0\big(\frac{1}{L}\big)\big| = |\varepsilon_1| \le \varepsilon$. Hence, by $\varphi \in \Phi_{m,L}$ we have $\varphi(0) = 1$ and therefore we obtain that

$$\max_{t \in [0, 1/L]} |\tilde{e}_0(t)| \le \varepsilon\big(2 + L\,\hat{\varphi}(0)\big)\,.$$

By the same technique, this error estimate can be shown for each interval $\big[\frac{k}{L},\, \frac{k+1}{L}\big]$ with $k \in \mathbb{Z}$. On the open interval $\big(\frac{k}{L},\, \frac{k+1}{L}\big)$ with $k \in \mathbb{Z}$, we use (3.14) to denote the error in the form

$$\tilde{e}_k(t) := \big(R_{\varphi,m}\tilde{f}\big)\big(t + \tfrac{k}{L}\big) - \big(R_{\varphi,m} f\big)\big(t + \tfrac{k}{L}\big) = \sum_{\ell \in \mathcal{I}_{2m}^{\mathrm{r}}} \varepsilon_{\ell+k}\, \psi\big(t - \tfrac{\ell}{L}\big)\,, \quad t \in \big(0,\, \tfrac{1}{L}\big)\,.$$

As above shown, it holds

$$|\tilde{e}_k(t)| \le \varepsilon\big(2\,\varphi(0) + L\,\hat{\varphi}(0)\big)\,, \quad t \in \big(0,\, \tfrac{1}{L}\big)\,.$$

By the interpolation property (3.12), it holds

$$\big|\tilde{e}_0\big(\tfrac{k}{L}\big)\big| = |\varepsilon_k| \le \varepsilon\,, \quad \big|\tilde{e}_0\big(\tfrac{k+1}{L}\big)\big| = |\varepsilon_{k+1}| \le \varepsilon$$

for each $k \in \mathbb{Z}$. Hence, by $\varphi \in \Phi_{m,L}$ we have $\varphi(0) = 1$ and therefore we obtain that

$$\max_{t \in [k/L,\, (k+1)/L]} |(R_{\varphi,m}\tilde{f})(t) - (R_{\varphi,m} f)(t)| \le \varepsilon\big(2 + L\,\hat{\varphi}(0)\big)\,.$$

From the triangle inequality it follows (3.26). This completes the proof. ■

Now it merely remains to estimate the error constants $E_j(m, \delta, L)$, $j = 1, 2$, for the different window functions, which shall be done in the following sections.

4 Gaussian regularized Shannon sampling formula

Firstly, we consider the Gaussian window function

$$\varphi_{\mathrm{Gauss}}(x) \coloneqq \mathrm{e}^{-x^2/(2\sigma^2)}\,, \quad x \in \mathbb{R}\,, \tag{4.1}$$

with some $\sigma > 0$ and introduce the *Gaussian regularized* sinc *function*

$$\psi_{\mathrm{Gauss}}(x) \coloneqq \mathrm{sinc}(L\pi x)\,\mathrm{e}^{-x^2/(2\sigma^2)}\,, \quad x \in \mathbb{R}\,. \tag{4.2}$$

Lemma 4.1. *Let $L \in \mathbb{N}$ and $\sigma > 0$ be given. Then the Fourier transform of the Gaussian regularized* sinc *function* (4.2) *reads as follows*

$$\begin{aligned} \hat{\psi}_{\mathrm{Gauss}}(v) &= \frac{1}{L\sqrt{\pi}} \int_{\sqrt{2}\pi\sigma(v-L/2)}^{\sqrt{2}\pi\sigma(v+L/2)} \mathrm{e}^{-t^2}\,\mathrm{d}t \\ &= \frac{1}{2L}\left[\mathrm{erf}\big(\sqrt{2}\pi\sigma\,(v+L/2)\big) - \mathrm{erf}\big(\sqrt{2}\pi\sigma\,(v-L/2)\big)\right], \end{aligned} \tag{4.3}$$

where

$$\mathrm{erf}\,x \coloneqq \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t\,, \quad x \in \mathbb{R}\,,$$

denotes the error function*. The function $\hat{\psi}_{\mathrm{Gauss}}$ is even, smooth, and positive on $\mathbb{R}$. Further $\hat{\psi}_{\mathrm{Gauss}}$ decreases on $[0,\,\infty)$ and it holds*

$$\max_{v\in\mathbb{R}} \hat{\psi}_{\mathrm{Gauss}}(v) = \frac{1}{L}\,\mathrm{erf}(\sqrt{2}\pi\sigma L/2) < \frac{1}{L}\,.$$

Proof. We apply the equality (2.10) for the sinc function

$$f(x) \coloneqq \mathrm{sinc}(L\pi x)\,, \quad x \in \mathbb{R}\,,$$

and the Gaussian window function

$$g(x) \coloneqq \mathrm{e}^{-x^2/(2\sigma^2)}\,, \quad x \in \mathbb{R}\,,$$

with certain $\sigma > 0$. These functions possess the Fourier transforms

$$\begin{aligned} \hat{f}(v) &= \frac{1}{L}\,\mathbf{1}_{[-L/2,\,L/2]}(v)\,, \quad v \in \mathbb{R}\,, \\ \hat{g}(v) &= \sqrt{2\pi}\,\sigma\,\mathrm{e}^{-2\pi^2\sigma^2 v^2}\,, \quad v \in \mathbb{R}\,. \end{aligned} \tag{4.4}$$

Thus, we obtain

$$(\hat{f} * \hat{g})(v) = \int_{\mathbb{R}} \hat{f}(v-u)\,\hat{g}(u)\,\mathrm{d}u = \frac{\sqrt{2\pi}\,\sigma}{L} \int_{v-L/2}^{v+L/2} \mathrm{e}^{-2\pi^2\sigma^2u^2}\,\mathrm{d}u$$

and by substitution $t = \sqrt{2}\pi\sigma u$

$$(\hat{f} * \hat{g})(v) = \frac{1}{L\sqrt{\pi}} \int_{\sqrt{2}\pi\sigma(v-L/2)}^{\sqrt{2}\pi\sigma(v+L/2)} \mathrm{e}^{-t^2}\,\mathrm{d}t$$

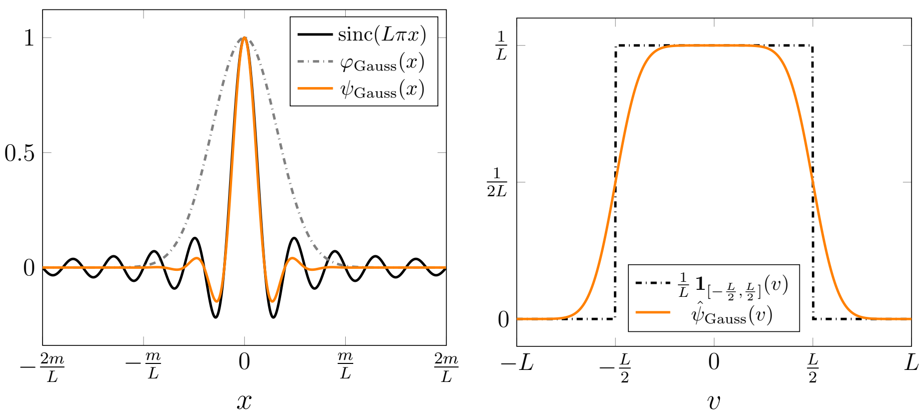
$$= \frac{1}{2L}\left[\operatorname{erf}\left(\sqrt{2}\pi\sigma\,(v+L/2)\right) - \operatorname{erf}\left(\sqrt{2}\pi\sigma\,(v-L/2)\right)\right].$$

On the other hand, it holds

$$(f\,g)\hat{}\,(v) = \int_{\mathbb{R}} \operatorname{sinc}(L\pi x)\, \mathrm{e}^{-x^2/(2\sigma^2)}\, \mathrm{e}^{-2\pi \mathrm{i} v x}\, \mathrm{d}x\,.$$

Hence, from (2.10) it follows the equality (4.3). By (4.3) we see that $\hat{\psi}_{\text{Gauss}}$ is even, smooth, and positive on $\mathbb{R}$. Since the error function increases on $\mathbb{R}$, the function (4.3) decreases on $[0,\,\infty)$. ■

A visualization of the Gaussian regularized sinc function (4.2) and its Fourier transform (4.3) can be found in Figure 4.1. We remark that the function (4.2) belongs to $C^{\infty}(\mathbb{R})$, but it is not bandlimited.



(a) ψ_{Gauss} from (4.2)

(b) $\hat{\psi}_{\text{Gauss}}$ from (4.3)

Figure 4.1: The Gaussian regularized sinc function ψ_{Gauss} as well as its Fourier transform $\hat{\psi}_{\text{Gauss}}$ with $m = 5$ and $\sigma = \frac{1}{L}\sqrt{\frac{8}{\pi}}$.

Lemma 4.2. *The function* (4.2) *is essentially bandlimited on the larger interval* $\left[-\frac{L}{2}\,(1+\varepsilon),\, \frac{L}{2}\,(1+\varepsilon)\right]$ *with certain* $\varepsilon \in (0,\,1)$*, i. e., it holds for all* $v \in \mathbb{R} \setminus \left[-\frac{L}{2}\,(1+\varepsilon),\, \frac{L}{2}\,(1+\varepsilon)\right]$ *that*

$$0 < \hat{\psi}_{\text{Gauss}}(v) \le \frac{1}{\sqrt{2\pi}\,L^2\pi\sigma\varepsilon}\,\mathrm{e}^{-\pi^2\sigma^2L^2\varepsilon^2/2}\,.$$

For $v \in \left[-\frac{L}{2}\,(1-\varepsilon),\, \frac{L}{2}\,(1-\varepsilon)\right]$ *it holds*

$$0 < \frac{1}{L} - \hat{\psi}_{\text{Gauss}}(v) \le \frac{2}{\sqrt{2\pi}L^2\pi\sigma\varepsilon}\,\mathrm{e}^{-\pi^2\sigma^2L^2\varepsilon^2/2}\,.$$

Proof. For $v \in \mathbb{R} \setminus \left[-\frac{L}{2}\,(1+\varepsilon),\, \frac{L}{2}\,(1+\varepsilon)\right]$, we can estimate the Fourier transform (4.3) in the following form

$$0 < \hat{\psi}_{\text{Gauss}}(v) = \frac{1}{L\sqrt{\pi}} \int_{\sqrt{2}\pi\sigma(v-L/2)}^{\sqrt{2}\pi\sigma(v+L/2)} \mathrm{e}^{-t^2}\,\mathrm{d}t \le \frac{1}{L\sqrt{\pi}} \int_{\sqrt{2}\pi\sigma(|v|-L/2)}^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t\,.$$

By [1, p. 298, Formula 7.1.13], for $x \ge 0$ it holds the inequality

$$\frac{1}{x+\sqrt{x^2+2}}\,\mathrm{e}^{-x^2} \le \int_x^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t \le \frac{1}{x+\sqrt{x^2+4/\pi}}\,\mathrm{e}^{-x^2}\,,$$

which can be simplified to

$$\int_x^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t \le \frac{1}{2x}\,\mathrm{e}^{-x^2}\,, \quad x>0\,. \tag{4.5}$$

Note that the upper bound in (4.5) decreases for $x>0$. Using the inequality (4.5), it follows for $v \in \mathbb{R}\setminus\big[-\frac{L}{2}\,(1+\varepsilon),\, \frac{L}{2}\,(1+\varepsilon)\big]$ that

$$0 < \hat{\psi}_{\mathrm{Gauss}}(v) \le \frac{1}{2\sqrt{2\pi}\,L\pi\sigma\,(|v|-L/2)}\,\mathrm{e}^{-2\pi^2\sigma^2\,(|v|-L/2)^2} \le \frac{1}{\sqrt{2\pi}\,L^2\pi\sigma\varepsilon}\,\mathrm{e}^{-\pi^2\sigma^2L^2\varepsilon^2/2}\,.$$

Thus, for fixed $\sigma>0$ and convenient $\varepsilon>0$, the Fourier transform $\hat{\psi}_{\mathrm{Gauss}}$ is negligible for $|v| \ge \frac{L}{2}\,(1+\varepsilon)$.

Later we will choose $\sigma = \sqrt{\frac{m}{\pi L\,(L-2\delta)}}$ with $m \in \mathbb{N}\setminus\{1\}$ and $\delta = \tau N$, where $0<\tau<\frac{1}{2}$. Hence, it holds $\sigma = \frac{1}{N}\sqrt{\frac{m}{\pi\,(1+\lambda)(1+\lambda-2\tau)}}$. Then for $v \in \mathbb{R}\setminus\big[-\frac{L}{2}\,(1+\varepsilon),\, \frac{L}{2}\,(1+\varepsilon)\big]$ we obtain

$$0 < \hat{\psi}_{\mathrm{Gauss}}(v) \le \frac{\sqrt{1+\lambda-2\tau}}{\sqrt{2m}\,\pi N\varepsilon\sqrt{(1+\lambda)^3}}\,\mathrm{e}^{-\pi\,m(1+\lambda)\,\varepsilon^2/(2+2\lambda-4\tau)}\,.$$

For $v \in \big[-\frac{L}{2}\,(1-\varepsilon),\, \frac{L}{2}\,(1-\varepsilon)\big]$ we consider

$$\begin{aligned}
\frac{1}{L} - \hat{\psi}_{\mathrm{Gauss}}(v) &= \frac{1}{L\sqrt{\pi}}\Big[\int_{\mathbb{R}} \mathrm{e}^{-t^2}\,\mathrm{d}t - \int_{\sqrt{2}\pi\sigma(v-L/2)}^{\sqrt{2}\pi\sigma(v+L/2)} \mathrm{e}^{-t^2}\,\mathrm{d}t\Big] \\
&= \frac{1}{L\sqrt{\pi}}\Big[\int_{-\infty}^{\sqrt{2}\pi\sigma(v-L/2)} \mathrm{e}^{-t^2}\,\mathrm{d}t + \int_{\sqrt{2}\pi\sigma(v+L/2)}^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t\Big] \\
&= \frac{1}{L\sqrt{\pi}}\Big[\int_{\sqrt{2}\pi\sigma(L/2-v)}^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t + \int_{\sqrt{2}\pi\sigma(v+L/2)}^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t\Big] > 0\,.
\end{aligned}$$

Hence, it holds

$$0 < \frac{1}{L} - \hat{\psi}_{\mathrm{Gauss}}(v) \le \frac{2}{L\sqrt{\pi}}\int_{\sqrt{2}\pi\sigma(L/2-|v|)}^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t\,.$$

Using (4.5) and $L/2-|v| \ge L\varepsilon/2$, it follows that

$$\begin{aligned}
0 < \frac{1}{L} - \hat{\psi}_{\mathrm{Gauss}}(v) &\le \frac{1}{\sqrt{2\pi}L\pi\sigma\,(L/2-|v|)}\,\mathrm{e}^{-2\pi^2\sigma^2\,(L/2-|v|)^2} \\
&\le \frac{2}{\sqrt{2\pi}L^2\pi\sigma\varepsilon}\,\mathrm{e}^{-\pi^2\sigma^2L^2\varepsilon^2/2}\,.
\end{aligned}$$

This completes the proof. ■

Now we show that for the Gaussian regularized sinc function (4.2) the uniform approximation error (3.15) decays exponentially with respect to m.

Theorem 4.3. *Let $f \in \mathcal{B}_\delta(\mathbb{R})$ with $\delta \in (0,\, N/2)$, $N \in \mathbb{N}$, $L = N(1+\lambda)$ with $\lambda \ge 0$, and $m \in \mathbb{N} \setminus \{1\}$ be given.*
Then the regularized Shannon sampling formula (3.2) *with the Gaussian window function* (4.1) *and $\sigma = \sqrt{\frac{m}{\pi L\,(L-2\delta)}}$ satisfies the error estimate*

$$\|f - R_{\mathrm{Gauss},m} f\|_{C_0(\mathbb{R})} \le \frac{2\sqrt{\pi\delta L} + L(m+1)/\sqrt{m}}{\pi\,\sqrt{m\pi(L-2\delta)}}\, \mathrm{e}^{-\pi m(L/2-\delta)/L}\, \|f\|_{L^2(\mathbb{R})}\,. \tag{4.6}$$

Proof (cf. [12] and [5]). By Theorem 3.2 we have to compute the error constants $E_j(m,\delta,L)$, $j = 1, 2$, for the Gaussian window function (4.1). First we study the regularization error constant (3.17). By (4.3) we recognize that the auxiliary function (3.23) is given by

$$\eta(v) = \mathbf{1}_{[-\delta,\,\delta]}(v) - \frac{1}{\sqrt{\pi}} \int_{\sqrt{2}\pi\sigma(v-L/2)}^{\sqrt{2}\pi\sigma(v+L/2)} \mathrm{e}^{-t^2}\, \mathrm{d}t\,.$$

For $v \in [-\delta,\, \delta]$, the function η can be evaluated as

$$\begin{aligned}
\eta(v) &= \frac{1}{\sqrt{\pi}} \left[\int_{\mathbb{R}} \mathrm{e}^{-t^2}\, \mathrm{d}t - \int_{\sqrt{2}\pi\sigma(v-L/2)}^{\sqrt{2}\pi\sigma(v+L/2)} \mathrm{e}^{-t^2}\, \mathrm{d}t \right] \\
&= \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^{\sqrt{2}\pi\sigma(v-L/2)} \mathrm{e}^{-t^2}\, \mathrm{d}t + \int_{\sqrt{2}\pi\sigma(v+L/2)}^{\infty} \mathrm{e}^{-t^2}\, \mathrm{d}t \right] \\
&= \frac{1}{\sqrt{\pi}} \left[\int_{\sqrt{2}\pi\sigma(L/2-v)}^{\infty} \mathrm{e}^{-t^2}\, \mathrm{d}t + \int_{\sqrt{2}\pi\sigma(v+L/2)}^{\infty} \mathrm{e}^{-t^2}\, \mathrm{d}t \right].
\end{aligned}$$

Using (4.5), $\eta(v)$ can be estimated by

$$\eta(v) < \frac{1}{\sqrt{\pi}} \left(\frac{\mathrm{e}^{-2\pi^2\sigma^2(L/2-v)^2}}{2\,\sqrt{2}\pi\sigma(L/2-v)} + \frac{\mathrm{e}^{-2\pi^2\sigma^2(L/2+v)^2}}{2\,\sqrt{2}\pi\sigma(L/2+v)} \right).$$

Since the function $\frac{1}{x}\, \mathrm{e}^{-\sigma^2 x^2/2}$ decreases for $x > 0$, and $L/2 - v,\; L/2 + v \in [L/2-\delta,\, L/2+\delta]$ by $v \in [-\delta,\, \delta]$ with $0 < \delta < L/2$, we conclude that

$$\eta(v) < \frac{\mathrm{e}^{-2\pi^2\sigma^2(L/2-v)^2}}{\sqrt{2\pi}\,\pi\sigma(L/2-v)} \le \frac{\mathrm{e}^{-2\pi^2\sigma^2(L/2-\delta)^2}}{\sqrt{2\pi}\,\sigma(L\pi-\delta)}\,.$$

Hence, by (3.17) and (3.23) we receive

$$E_1(m,\delta,L) \le \frac{\sqrt{\delta}}{\sqrt{\pi}\,\pi\sigma(L/2-\delta)}\, \mathrm{e}^{-2\pi^2\sigma^2(L/2-\delta)^2}\,. \tag{4.7}$$

Now we examine the truncation error constant (3.18). Here it holds

$$\varphi_{\mathrm{Gauss}}^2(\tfrac{m}{L}) + L \int_{m/L}^{\infty} \varphi_{\mathrm{Gauss}}^2(t)\, \mathrm{d}t = \mathrm{e}^{-m^2/(L^2\sigma^2)} + L\sigma \int_{m/(L\sigma)}^{\infty} \mathrm{e}^{-t^2}\, \mathrm{d}t\,.$$

From (4.5) it follows

$$\mathrm{e}^{-m^2/(L^2\sigma^2)} + L\sigma \int_{m/(L\sigma)}^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t \le \frac{2m + L^2\sigma^2}{2m}\,\mathrm{e}^{-m^2/(L^2\sigma^2)}\,.$$

Thus, by (3.17) we obtain

$$E_2(m, \delta, L) \le \frac{\sqrt{2L}}{\pi m}\,\sqrt{\frac{2m + L^2\sigma^2}{2m}}\,\mathrm{e}^{-m^2/(2L^2\sigma^2)}\,. \tag{4.8}$$

For the special parameter $\sigma = \sqrt{\frac{m}{\pi L\,(L-2\delta)}}$, both error terms (4.7) and (4.8) have the same exponential decay such that

$$E_1(m, \delta, L) \le \frac{2}{\pi}\,\sqrt{\frac{\delta L}{m\,(L - 2\delta)}}\,\mathrm{e}^{-\pi m(L/2-\delta)/L}\,,$$

$$E_2(m, \delta, L) \le \frac{1}{\pi}\,\sqrt{\frac{L(2\pi(L - 2\delta) + 1)/m}{m\pi(L - 2\delta)}}\,\mathrm{e}^{-\pi m(L/2-\delta)/L}\,.$$

For $\delta \in (0, N/2)$ and $m \in \mathbb{N} \setminus \{1\}$ it additionally holds

$$\sqrt{2\pi(L - 2\delta) + 1} \le \sqrt{L}\,\sqrt{2\pi + 1} \le \sqrt{L}(m + 1)\,.$$

This completes the proof. ■

We remark that Theorem 4.3 improves the corresponding results in [12] and [5] in such a way that there it was stated an exponential decay in $(m - 1)$, while this rate could be improved in Theorem 4.3 to m.

Example 4.4. We aim to visualize the error bound from Theorem 4.3. For a given function $f \in \mathcal{B}_\delta(\mathbb{R})$ with $\delta = \tau N \in (0, N/2)$ and $L = N(1 + \lambda)$, where $0 < \tau < \frac{1}{2}$ and $\lambda \ge 0$, we consider the approximation error

$$e_{m,\tau,\lambda}(f) := \max_{t\in[-1,\,1]} |f(t) - (R_{\varphi,m}f)(t)|\,. \tag{4.9}$$

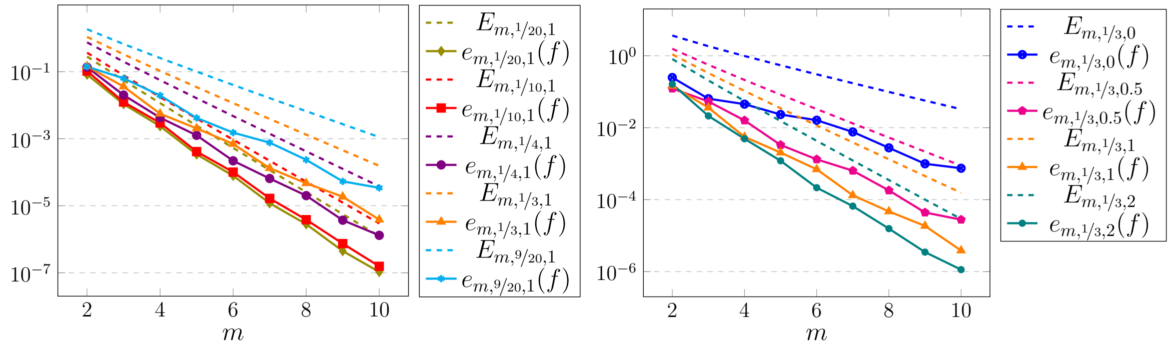
For $\varphi = \varphi_{\mathrm{Gauss}}$ we show that by (4.6) it holds $e_{m,\tau,\lambda}(f) \le E_{m,\tau,\lambda}\,\|f\|_{L^2(\mathbb{R})}$ where

$$E_1(m, \delta, L) + E_2(m, \delta, L) \le E_{m,\tau,\lambda} := \frac{2\sqrt{\pi\delta L} + L(m + 1)/\sqrt{m}}{\pi\,\sqrt{m\pi(L - 2\delta)}}\,\mathrm{e}^{-\pi m(L/2-\delta)/L}\,, \tag{4.10}$$

with $\sigma = \sqrt{\frac{m}{\pi L\,(L-2\delta)}}$. The error (4.9) shall here be approximated by evaluating a given function f and its approximation $R_{\varphi,m}f$ at $S = 10^5$ equidistant points $t_s \in [-1,\,1]$, $s \in \mathcal{I}_S$. By the definition of the regularized Shannon sampling formula in (3.2) it can be seen that for $t \in [-1, 1]$ we have

$$(R_{\varphi,m}f)(t) = \sum_{\ell=-L-m}^{L+m} f(\tfrac{\ell}{L})\,\psi(t - \tfrac{\ell}{L})\,.$$

Here we study the function $f(t) = \sqrt{2\delta}\,\mathrm{sinc}(2\delta\pi t)$, $t \in \mathbb{R}$, such that it holds $\|f\|_{L^2(\mathbb{R})} = 1$. We fix $N = 128$ and consider the evolution for different values $m \in \mathbb{N} \setminus \{1\}$, i.e., we are still free to make a choice for the parameters τ and λ. In a first experiment we fix $\lambda = 1$ and choose different values for $\tau < \frac{1}{2}$, namely we consider $\tau \in \{1/20,\, 1/10,\, 1/4,\, 1/3,\, 9/20\}$. The corresponding results are depicted in Figure 4.2 (a). We recognize that the smaller the factor τ can be chosen, the better the error results are. As a second experiment we fix $\tau = \frac{1}{3}$, but now choose different $\lambda \in \{0, 0.5, 1, 2\}$. The associated results are displayed in Figure 4.2 (b). It can clearly be seen that the higher the oversampling parameter λ is chosen, the better the error results get. We remark that for larger choices of N, the line plots in Figure 4.2 would only be shifted slightly upwards, such that for all N we receive almost the same error results. □



(a) $\lambda = 1$ and various $\tau < \frac{1}{2}$ (b) $\tau = \frac{1}{3}$ and various $\lambda \geq 0$

Figure 4.2: Maximum approximation error (4.9) and error constant (4.10) using φ_{Gauss} in (4.1) and $\sigma = \sqrt{\frac{m}{\pi L\,(L-2\delta)}}$ for the function $f(x) = \sqrt{2\delta}\,\mathrm{sinc}(2\delta\pi x)$ with $N = 128$, $m = 2, 3, \ldots, 10$, as well as $\tau \in \{1/20,\, 1/10,\, 1/4,\, 1/3,\, 9/20\}$, $\delta = \tau N$, and $\lambda \in \{0, 0.5, 1, 2\}$, respectively.

Now we show that for the regularized Shannon sampling formula with the Gaussian window function (4.1) the uniform perturbation error (3.25) only grows as $\mathcal{O}(\sqrt{m})$. We remark that a similar result can also be found in [14].

Theorem 4.5. *Let $f \in \mathcal{B}_\delta(\mathbb{R})$ with $\delta = \tau N$, $\tau \in (0,\, 1/2)$, $N \in \mathbb{N}$, $L = N(1+\lambda)$ with $\lambda \geq 0$ and $m \in \mathbb{N} \setminus \{1\}$ be given. Further let $R_{\mathrm{Gauss},m}\tilde{f}$ be as in (3.24) with the noisy samples $\tilde{f}_\ell = f\big(\frac{\ell}{L}\big) + \varepsilon_\ell$, where $|\varepsilon_\ell| \leq \varepsilon$ for all $\ell \in \mathbb{Z}$.*
Then the regularized Shannon sampling formula (3.2) with the Gaussian window function (4.1) and $\sigma = \sqrt{\frac{m}{\pi L\,(L-2\delta)}}$ is numerically robust and satisfies

$$\|R_{\mathrm{Gauss},m}\tilde{f} - R_{\mathrm{Gauss},m}f\|_{C_0(\mathbb{R})} \leq \varepsilon\left(2 + \sqrt{\frac{2+2\lambda}{\lambda+1-2\tau}}\,\sqrt{m}\right). \tag{4.11}$$

Proof. By Theorem 3.4 we only have to compute $\hat{\varphi}_{\mathrm{Gauss}}(0)$ for the Gaussian window function (4.1). By (4.4) we recognize that

$$\hat{\varphi}_{\mathrm{Gauss}}(0) = \sqrt{2\pi}\,\sigma = \sqrt{\frac{2m}{L\,(L-2\delta)}} = \frac{1}{L}\sqrt{\frac{2+2\lambda}{\lambda+1-2\tau}}\,\sqrt{m}$$

such that (3.25) yields the assertion. ■

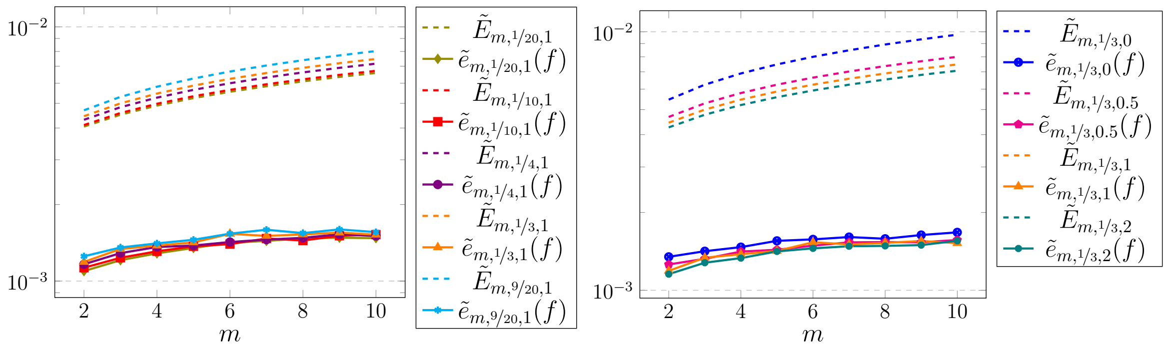
Example 4.6. Now we aim to visualize the error bound from Theorem 4.5. Similar to Example 4.4, we consider the perturbation error

$$\tilde{e}_{m,\tau,\lambda}(f) := \max_{t\in[-1,\,1]} |(R_{\varphi,m}\tilde{f})(t) - (R_{\varphi,m}f)(t)|\,. \tag{4.12}$$

For $\varphi = \varphi_{\text{Gauss}}$ we show that by (4.11) it holds $\tilde{e}_{m,\tau,\lambda}(f) \le \tilde{E}_{m,\tau,\lambda}$, where

$$\tilde{E}_{m,\tau,\lambda} := \varepsilon \left(2 + \sqrt{\frac{2+2\lambda}{\lambda+1-2\tau}}\,\sqrt{m} \right) . \tag{4.13}$$

We conduct the same experiments as in Example 4.4 and introduce a maximum perturbation of $\varepsilon = 10^{-3}$ as well as uniformly distributed random numbers ε_ℓ in $(-\varepsilon, \varepsilon)$. Due to the randomness we perform the experiments 100 times and then take the maximum error over all runs. The associated results are displayed in Figure 4.3. □



(a) $\lambda = 1$ and various $\tau < \frac{1}{2}$ (b) $\tau = \frac{1}{3}$ and various $\lambda \ge 0$

Figure 4.3: Maximum perturbation error (4.12) over 100 runs and error constant (4.13) using φ_{Gauss} in (4.1) and $\sigma = \sqrt{\frac{m}{\pi L\,(L-2\delta)}}$ for the function $f(x) = \sqrt{2\delta}\,\text{sinc}(2\delta\pi x)$ with $\varepsilon = 10^{-3}$, $N = 128$, $m = 2, 3, \dots, 10$, as well as $\tau \in \{1/20,\, 1/10,\, 1/4,\, 1/3,\, 9/20\}$, $\delta = \tau N$, and $\lambda \in \{0, 0.5, 1, 2\}$, respectively.

5 B–spline regularized Shannon sampling formula

Now we consider the modified B–spline window function

$$\varphi_{\text{B}}(x) := \frac{1}{M_{2s}(0)}\, M_{2s}\!\left(\frac{Lxs}{m}\right) \tag{5.1}$$

with $s,\, m \in \mathbb{N} \setminus \{1\}$ and $L = N\,(1+\lambda)$, $\lambda \ge 0$, where M_{2s} denotes the centered cardinal B–spline of even order $2s$. Note that (5.1) is supported on $\big[-\frac{m}{L},\, \frac{m}{L}\big]$. According to (3.13) we form the B*–spline regularized* sinc *function*

$$\psi_{\text{B}}(x) := \text{sinc}(L\pi x)\, \varphi_{\text{B}}(x)\,, \quad x \in \mathbb{R}\,. \tag{5.2}$$

Lemma 5.1. *The Fourier transform of the* B*–spline regularized* sinc *function* (5.2) *reads as follows*

$$\hat{\psi}_{\mathrm{B}}(v) = \frac{m}{sL^2\, M_{2s}(0)} \int_{v-L/2}^{v+L/2} \Big(\mathrm{sinc}\, \frac{\pi u m}{sL}\Big)^{2s}\, \mathrm{d}u\,. \tag{5.3}$$

The function (5.3) *is even, smooth, and positive on* $\mathbb{R}$*, where*

$$\hat{\psi}_{\mathrm{B}}(0) = \max_{v\in\mathbb{R}} \hat{\psi}_{\mathrm{B}}(v) < \frac{1}{L}\,.$$

Proof. We apply the equation (2.10) for $f(x) = \mathrm{sinc}(L\pi x)$ and the modified B–spline window function $g(x) = \varphi_{\mathrm{B}}(x)$. These functions possess the Fourier transforms

$$\begin{aligned} \hat{f}(v) &= \frac{1}{L}\, \mathbf{1}_{[-L/2,\, L/2]}(v)\,, \\ \hat{g}(v) &= \frac{m}{sL\, M_{2s}(0)}\, \Big(\mathrm{sinc}\, \frac{\pi v m}{sL}\Big)^{2s}\,, \quad v \in \mathbb{R}\,, \end{aligned} \tag{5.4}$$

(see [9, 10]). Thus, we obtain

$$\begin{aligned} (\hat{f} * \hat{g})(v) &= \int_{\mathbb{R}} \hat{f}(v-u)\, \hat{g}(u)\, \mathrm{d}u = \frac{1}{L} \int_{v-L/2}^{v+L/2} \hat{g}(u)\, \mathrm{d}u \\ &= \frac{m}{sL^2\, M_{2s}(0)} \int_{v-L/2}^{v+L/2} \Big(\mathrm{sinc}\, \frac{\pi u m}{sL}\Big)^{2s}\, \mathrm{d}u\,. \end{aligned}$$

Since it holds

$$(f\,g)\hat{}(v) = \int_{\mathbb{R}} \mathrm{sinc}(L\pi\, x)\, g(x)\, \mathrm{e}^{-2\pi \mathrm{i} v x}\, \mathrm{d}x = \hat{\psi}_{\mathrm{B}}(v)\,,$$

equation (2.10) yields the assertion (5.3).

Obviously, the function (5.3) is even, smooth and positive on whole $\mathbb{R}$. By inverse Fourier transform it holds

$$1 = \varphi_{\mathrm{B}}(0) = \int_{\mathbb{R}} \hat{\varphi}_{\mathrm{B}}(v)\, \mathrm{d}v = \frac{m}{sL\, M_{2s}(0)} \int_{\mathbb{R}} \Big(\mathrm{sinc}\, \frac{\pi v m}{sL}\Big)^{2s}\, \mathrm{d}v$$

such that

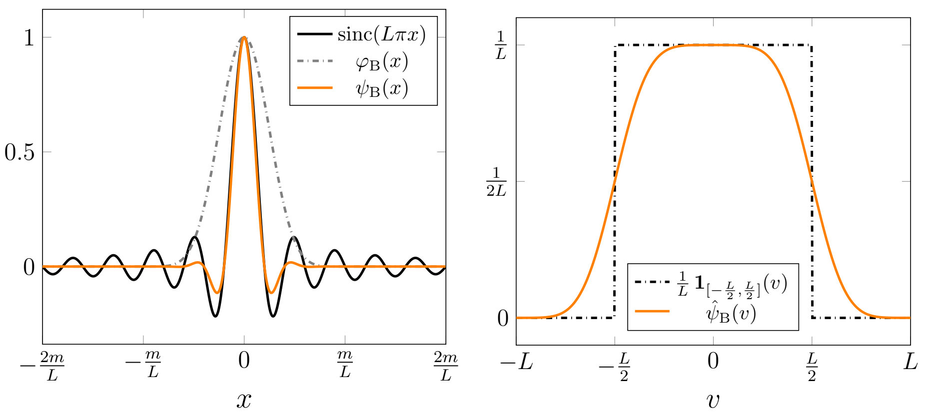
$$\int_{\mathbb{R}} \Big(\mathrm{sinc}\, \frac{\pi v m}{sL}\Big)^{2s}\, \mathrm{d}v = \frac{sL}{m}\, M_{2s}(0)\,. \tag{5.5}$$

Then from (5.3) and (5.5) it follows that

$$\hat{\psi}_{\mathrm{B}}(0) = \frac{m}{sL^2\, M_{2s}(0)} \int_{-L/2}^{L/2} \Big(\mathrm{sinc}\, \frac{\pi u m}{sL}\Big)^{2s}\, \mathrm{d}u < \frac{1}{L}\,.$$

This completes the proof. ■

A visualization of the B–spline regularized sinc function (5.2) and its Fourier transform (5.3) can be found in Figure 5.1.



(a) ψ_{B} from (5.2) (b) $\hat{\psi}_{\mathrm{B}}$ from (5.3)

Figure 5.1: The B–spline regularized sinc function ψ_{B} as well as its Fourier transform $\hat{\psi}_{\mathrm{B}}$ with $m = 5$ and $s = 3$.

Lemma 5.2. *The* B*–spline regularized* sinc *function* (5.2) *is essentially bandlimited on the larger interval* $\big[-\frac{L}{2}\,(1+\varepsilon),\,\frac{L}{2}\,(1+\varepsilon)\big]$ *with* $\varepsilon > \frac{2s}{m\pi}$*, i. e., for all* $v \in \mathbb{R}\setminus\big[-\frac{L}{2}\,(1+\varepsilon),\,\frac{L}{2}\,(1+\varepsilon)\big]$ *it holds*

$$0 < \hat{\psi}_{\mathrm{B}}(v) < \frac{1}{(2s-1)\,\pi\,L\,M_{2s}(0)}\,\Big(\frac{2s}{\varepsilon\,m\,\pi}\Big)^{2s-1}\,.$$

Proof. Since (5.3) is even, we consider $\hat{\psi}_{\mathrm{B}}(v)$ only for $v > \frac{L}{2}\,(1+\varepsilon)$. Then by (5.3) we estimate

$$\begin{aligned} 0 < \hat{\psi}_{\mathrm{B}}(v) &= \frac{m}{sL^2\,M_{2s}(0)} \int_{v-L/2}^{v+L/2} \Big(\mathrm{sinc}\,\frac{\pi u m}{sL}\Big)^{2s}\,\mathrm{d}u \\ &< \frac{m}{sL^2\,M_{2s}(0)} \int_{L\varepsilon/2}^{\infty} \Big(\mathrm{sinc}\,\frac{\pi u m}{sL}\Big)^{2s}\,\mathrm{d}u < \frac{s^{2s-1}\,L^{2s-2}}{m^{2s-1}\,\pi^{2s}\,M_{2s}(0)} \int_{L\varepsilon/2}^{\infty} u^{-2s}\,\mathrm{d}u \\ &= \frac{1}{(2s-1)\,\pi\,L\,M_{2s}(0)}\,\Big(\frac{2s}{\varepsilon\,m\,\pi}\Big)^{2s-1}\,. \end{aligned}$$

This completes the proof. ■

Lemma 5.3. *For the value* $M_{2s}(0)$*,* $s \in \mathbb{N}$*, it holds the formula*

$$M_{2s}(0) = \frac{1}{(2s-1)!}\sum_{j=0}^{s-1}(-1)^j\binom{2s}{j}\,(s-j)^{2s-1}\,. \tag{5.6}$$

The sequence $\big(\sqrt{2s}\,M_{2s}(0)\big)_{s=2}^{\infty}$ *increases monotonously and has the limit*

$$\lim_{s\to\infty}\sqrt{2s}\,M_{2s}(0) = \sqrt{\frac{6}{\pi}} = 1.381976\ldots \tag{5.7}$$

such that for $s \in \mathbb{N}\setminus\{1\}$ *it holds*

$$\frac{4}{3} \le \sqrt{2s}\,M_{2s}(0) < \sqrt{\frac{6}{\pi}}\,. \tag{5.8}$$

Proof. By inverse Fourier transform of $\hat{\varphi}_{\mathrm{B}}$ it holds

$$\varphi_{\mathrm{B}}(x) = \int_{\mathbb{R}} \hat{\varphi}_{\mathrm{B}}(v)\, \mathrm{e}^{2\pi \mathrm{i} v x}\, \mathrm{d}v\,, \quad x \in \mathbb{R}\,.$$

Hence, for $x = 0$ it follows that

$$M_{2s}(0) = \int_{\mathbb{R}} \big(\mathrm{sinc}(\pi v)\big)^{2s}\, \mathrm{d}v = \frac{2}{\pi} \int_0^\infty \big(\mathrm{sinc}\, w\big)^{2s}\, \mathrm{d}w\,.$$

The above integral can be determined in explicit form (see [8, p. 20, 5.12] or [6]) as

$$\int_0^\infty \big(\mathrm{sinc}\, w\big)^{2s}\, \mathrm{d}w = \frac{\pi}{2\,(2s-1)!} \sum_{j=0}^{s-1} (-1)^j \binom{2s}{j} (s-j)^{2s-1}$$

such that (5.6) is shown. Especially, it holds $M_2(0) = 1$, $M_4(0) = \frac{2}{3}$, $M_6(0) = \frac{11}{20}$, $M_8(0) = \frac{151}{315}$, $M_{10}(0) = \frac{15619}{36288}$, and $M_{12}(0) = \frac{655177}{1663200}$. A table with the decimal values of $M_{2s}(0)$ for $m = 15, \ldots, 50$, can be found in [6]. For example, it holds $M_{100}(0) \approx 0.137990$.

By [19], there exists the pointwise limit

$$\lim_{s\to\infty} \sqrt{\frac{s}{6}}\, M_{2s}\left(\sqrt{\frac{s}{6}}\, x\right) = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-x^2/2}$$

such that for $x = 0$ we obtain (5.7). By numerical computations we can see that the sequence $\big(\sqrt{2s}\, M_{2s}(0)\big)_{s=2}^{50}$ increases monotonously (see Figure 5.2). For large s we can use the

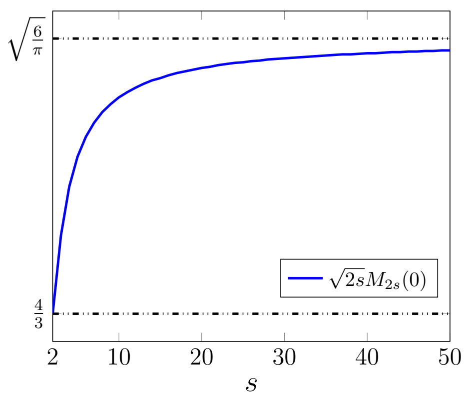


Figure 5.2: The sequence $\big(\sqrt{2s}\, M_{2s}(0)\big)_{s=2}^{50}$.

asymptotic expansion (see [6])

$$\sqrt{2s}\, M_{2s}(0) \approx \sqrt{\frac{6}{\pi}} \left[1 - \frac{3}{40\,s} - \frac{13}{4480\,s^2} + \frac{27}{25600\,s^3} + \frac{52791}{63078400\,s^4} + \frac{482427}{2129920000\,s^5}\right]$$

such that the whole sequence $\big(\sqrt{2s}\, M_{2s}(0)\big)_{s=2}^{\infty}$ increases monotonously. Hence, it holds (5.8).
■

Remark 5.4. The value $M_{2s}(0)$ is closely related to the Eulerian number in combinatorics. For any $n,\, k \in \mathbb{N}$ with $k \le n$, the *Eulerian number* $E(n, k-1)$ denotes the number of permutations of 1 to n in which exactly $k-1$ elements are greater than the previous element. Then it holds

$$E(n,k-1) = \sum_{j=0}^{k-1} (-1)^j \binom{n+1}{j} (k-j)^n\,.$$

Thus, from (5.6) it follows that

$$M_{2s}(0) = \frac{1}{(2s-1)!}\, E(2s-1, s-1)\,.$$

□

Now we show that for the B–spline regularized sinc function (5.2) the uniform approximation error (3.15) decays exponentially with respect to m.

Theorem 5.5. *Let $f \in \mathcal{B}_\delta(\mathbb{R})$ with $\delta = \tau N$, $\tau \in (0,\, 1/2)$, $N \in \mathbb{N}$, $L = N\,(1+\lambda)$, $\lambda \ge 0$, and $m \in \mathbb{N} \setminus \{1\}$ be given. Assume that*

$$\frac{\tau}{1+\lambda} < \frac{1}{2} - \frac{1}{\pi}\,. \tag{5.9}$$

Then the regularized Shannon sampling formula (3.2) *with the* B*–spline window function* (5.1) *and $s = \lceil \frac{m+1}{2} \rceil$ satisfies the error estimate*

$$\|f - R_{\mathrm{B},m} f\|_{C_0(\mathbb{R})} \le \frac{3\,\sqrt{\delta s}}{(2s-1)\,\pi}\, \mathrm{e}^{-m\,(\ln(\pi m\,(1+\lambda-2\tau)) - \ln(2s(1+\lambda)))}\, \|f\|_{L^2(\mathbb{R})}\,. \tag{5.10}$$

*P*roof. By Theorem 3.2 we only have to estimate the regularization error constant (3.17), since it holds $\varphi_{\mathrm{B}}(x) = \varphi_{\mathrm{B}}(x)\, \mathbf{1}_{[-m/L,\, m/L]}(x)$ for all $x \in \mathbb{R}$ and therefore the truncation error constant (3.18) vanishes for the B–spline window function (5.1) by Remark 3.3.

By (5.4) we recognize that the auxiliary function (3.23) is given by

$$\eta(v) = \mathbf{1}_{[-\delta,\,\delta]}(v) - \frac{m}{sL\, M_{2s}(0)} \int_{v-L/2}^{v+L/2} \Big(\mathrm{sinc}\, \frac{\pi u m}{sL}\Big)^{2s}\, \mathrm{d}u\,, \quad v \in \mathbb{R}\,.$$

For $v \in [-\delta,\, \delta]$, the function η can be determined by (5.5) in the following form

$$\begin{aligned}
\eta(v) \;&=\; \frac{m}{sL\, M_{2s}(0)} \bigg[\int_{\mathbb{R}} \Big(\mathrm{sinc}\, \frac{\pi u m}{sL}\Big)^{2s} \mathrm{d}u - \int_{v-L/2}^{v+L/2} \Big(\mathrm{sinc}\, \frac{\pi u m}{sL}\Big)^{2s} \mathrm{d}u\bigg] \\
&=\; \frac{m}{sL\, M_{2s}(0)} \bigg[\int_{-\infty}^{v-L/2} \Big(\mathrm{sinc}\, \frac{\pi u m}{sL}\Big)^{2s} \mathrm{d}u + \int_{v+L/2}^{\infty} \Big(\mathrm{sinc}\, \frac{\pi u m}{sL}\Big)^{2s} \mathrm{d}u\bigg] \\
&=\; \frac{m}{sL\, M_{2s}(0)} \bigg[\int_{L/2-v}^{\infty} \Big(\mathrm{sinc}\, \frac{\pi u m}{sL}\Big)^{2s} \mathrm{d}u + \int_{v+L/2}^{\infty} \Big(\mathrm{sinc}\, \frac{\pi u m}{sL}\Big)^{2s} \mathrm{d}u\bigg]\,.
\end{aligned}$$

Applying the simple estimates

$$\int_{L/2-v}^{\infty} \Big(\mathrm{sinc}\, \frac{\pi u m}{sL}\Big)^{2s} \mathrm{d}u \;\le\; \frac{s^{2s}\, L^{2s}}{m^{2s}\, \pi^{2s}} \int_{L/2-v}^{\infty} u^{-2s}\, \mathrm{d}u = \frac{s^{2s}\, L^{2s}}{(2s-1)\, m^{2s}\, \pi^{2s}\, (L/2-v)^{2s-1}}\,,$$

$$\int_{v+L/2}^{\infty} \left(\operatorname{sinc} \frac{\pi u m}{sL}\right)^{2s} \mathrm{d}u \;\le\; \frac{s^{2s}\, L^{2s}}{m^{2s}\, \pi^{2s}} \int_{v+L/2}^{\infty} u^{-2s}\, \mathrm{d}u = \frac{s^{2s}\, L^{2s}}{(2s-1)\, m^{2s}\, \pi^{2s}\, (v+L/2)^{2s-1}}\,,$$

the function η can be estimated for $v \in [-\delta, \delta]$ by

$$\eta(v) \le \frac{s^{2s-1}\, L^{2s-1}}{(2s-1)\, m^{2s-1}\, \pi^{2s}\, M_{2s}(0)} \left[\frac{1}{(L/2-v)^{2s-1}} + \frac{1}{(L/2+v)^{2s-1}}\right].$$

By $v \in [-\delta,\, \delta]$ with $0 < \delta < N/2 \le L/2$, it holds $L/2 - v,\, L/2 + v \in [L/2-\delta,\, L/2+\delta]$. Since the function x^{1-2s} decreases for $x > 0$, we conclude that

$$\max_{v\in[-\delta,\delta]} |\eta(v)| \le \frac{2\, s^{2s-1}\, L^{2s-1}}{(2s-1)\, m^{2s-1}\, \pi^{2s}\, M_{2s}(0)\, (L/2-\delta)^{2s-1}}\,.$$

Hence, by (3.17), (3.23) and (5.8) we receive

$$\begin{aligned} E_1(m,\delta,L) &\le \frac{2\sqrt{2\delta}}{(2s-1)\,\pi\, M_{2s}(0)} \left(\frac{2sL}{\pi m L - 2\pi m \delta}\right)^{2s-1} \\ &\le \frac{3\sqrt{\delta s}}{(2s-1)\,\pi} \left(\frac{2sL}{\pi m L - 2\pi m \delta}\right)^{2s-1}. \end{aligned} \tag{5.11}$$

For achieving convergence we have to satisfy

$$\frac{2sL}{\pi m L - 2\pi m\delta} = \frac{2s(1+\lambda)}{\pi m\,(1+\lambda-2\tau)} =: c < 1\,.$$

This condition holds if (5.9) is fulfilled. By means of logarithmic laws we recognize that $c^{2s-1} = \mathrm{e}^{\ln(c^{2s-1})} = \mathrm{e}^{(2s-1)\,\ln c}$. Thus, the condition $c < 1$ yields $\ln c < 0$ and therefore an exponential decay of (5.11) with respect to $(2s-1)$. Since the aim is achieving an exponential decay with a rate of at least m, the condition $2s-1 \ge m$ can now be used to pick a suitable parameter $s \in \mathbb{N}$ in the form $s = \lceil \frac{m+1}{2} \rceil$. Then

$$\begin{aligned} c^{2s-1} &= \mathrm{e}^{(2s-1)\,\ln c} = \mathrm{e}^{(2s-1)\,(\ln(2s(1+\lambda)) - \ln(\pi m\,(1+\lambda-2\tau)))} \\ &= \mathrm{e}^{-(2s-1)\,(\ln(\pi m\,(1+\lambda-2\tau)) - \ln(2s(1+\lambda)))} \le \mathrm{e}^{-m\,(\ln(\pi m\,(1+\lambda-2\tau)) - \ln(2s(1+\lambda)))} \end{aligned}$$

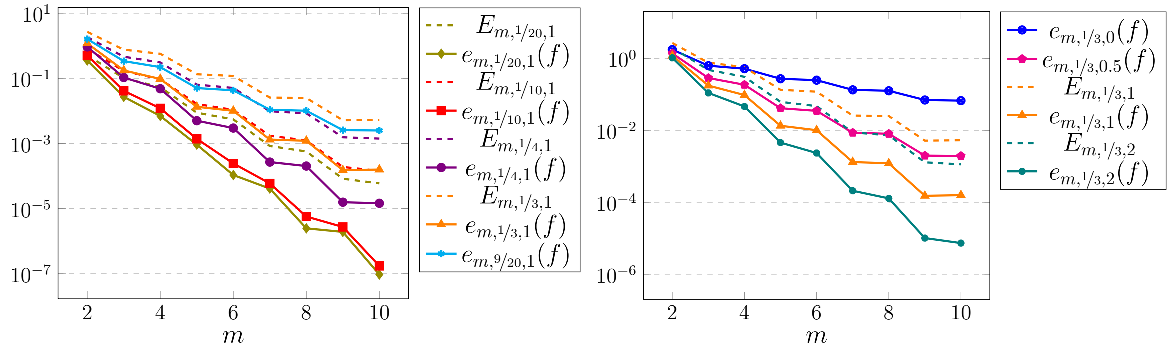
yields the assertion. We remark that it holds $\pi m\,(1+\lambda-2\tau) > 2s(1+\lambda)$ since $c < 1$. ■

Example 5.6. Analogous to Example 4.4, we now aim to visualize the error bound from Theorem 5.5, i. e., for $\varphi = \varphi_{\mathrm{B}}$ we show that for the approximation error (4.9) it holds by (5.10) that $e_{m,\tau,\lambda}(f) \le E_{m,\tau,\lambda}\, \|f\|_{L^2(\mathbb{R})}$, where

$$E_1(m,\delta,L) \le E_{m,\tau,\lambda} := \frac{3\sqrt{\delta s}}{(2s-1)\,\pi}\, \mathrm{e}^{-m\,(\ln(\pi m\,(1+\lambda-2\tau)) - \ln(2s(1+\lambda)))} \tag{5.12}$$

with $s = \lceil \frac{m+1}{2} \rceil$. Additionally, we now have to observe the condition (5.9). For the first experiment in Example 4.4 with $\lambda = 1$ this leads to $\tau < 1 - \frac{2}{\pi} \approx 0.3634$, while in the second experiment we fixed $\tau = \frac{1}{3}$ and therefore have to satisfy $\lambda > \frac{2\pi}{3\pi-6} - 1 \approx 0.8346$. Thus, only in these settings the requirements of Theorem 5.5 are fulfilled, and therefore only those error bounds are plotted in Figure 5.3 while the approximation error (4.9) is computed for all constellations of parameters as given in Example 4.4. We recognize that we have almost the same behavior as in Figure 4.2, which means that there is hardly any improvement using the B–spline window function in comparison to the well-studied Gaussian window function. □

Now we show that for the regularized Shannon sampling formula with the B–spline window function (5.1) the uniform perturbation error (3.25) only grows as $\mathcal{O}(\sqrt{m})$.



(a) $\lambda = 1$ and various $\tau < \frac{1}{2}$ (b) $\tau = \frac{1}{3}$ and various $\lambda \geq 0$

Figure 5.3: Maximum approximation error (4.9) and error constant (5.12) using φ_{B} in (5.1) and $s = \left\lceil \frac{m+1}{2} \right\rceil$ for the function $f(x) = \sqrt{2\delta}\,\mathrm{sinc}(2\delta\pi x)$ with $N = 128$, $m = 2, 3, \ldots, 10$, as well as $\tau \in \{1/20,\ 1/10,\ 1/4,\ 1/3,\ 9/20\}$, $\delta = \tau N$, and $\lambda \in \{0, 0.5, 1, 2\}$, respectively.

Theorem 5.7. *Let $f \in \mathcal{B}_\delta(\mathbb{R})$ with $\delta = \tau N$, $\tau \in (0,\, 1/2)$, $N \in \mathbb{N}$, $L = N(1+\lambda)$ with $\lambda \geq 0$ and $m \in \mathbb{N} \setminus \{1\}$ be given. Let $s \in \mathbb{N}$ be defined by $s = \left\lceil \frac{m+1}{2} \right\rceil$. Further let $R_{\mathrm{B},m}\tilde{f}$ be as in* (3.24) *with the noisy samples $\tilde{f}_\ell = f\left(\frac{\ell}{L}\right) + \varepsilon_\ell$, where $|\varepsilon_\ell| \leq \varepsilon$ for all $\ell \in \mathbb{Z}$.*
Then the regularized Shannon sampling formula (3.2) *with the* B*–spline window function* (5.1) *and $s = \left\lceil \frac{m+1}{2} \right\rceil$ is numerically robust and satisfies*

$$\|R_{\mathrm{B},m}\tilde{f} - R_{\mathrm{B},m}f\|_{C_0(\mathbb{R})} \leq \varepsilon \left(2 + \frac{3}{2}\sqrt{m}\right). \tag{5.13}$$

Proof. By Theorem 3.4 we only have to compute $\hat{\varphi}_{\mathrm{B}}(0)$ for the B–spline window function (5.1). By (5.4) we recognize that

$$\hat{\varphi}_{\mathrm{B}}(0) = \frac{m}{sL\, M_{2s}(0)}\,.$$

From (5.8) it follows that

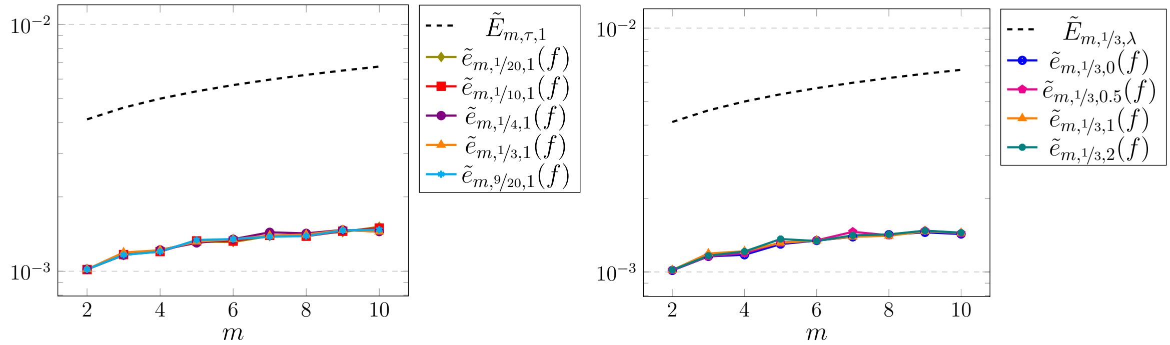
$$\frac{1}{M_{2s}(0)} \leq \frac{3\sqrt{2}}{4}\,\sqrt{s}\,.$$

Due to $s = \left\lceil \frac{m+1}{2} \right\rceil$ it holds $\sqrt{s} \geq \frac{\sqrt{m}}{\sqrt{2}}$ such that (3.25) yields the assertion. ■

Example 5.8. Now we aim to visualize the error bound from Theorem 5.7. Similar to Example 4.6, we show that for the perturbation error (4.12) with $\varphi = \varphi_{\mathrm{B}}$ it holds by (5.13) that $\tilde{e}_{m,\tau,\lambda}(f) \leq \tilde{E}_{m,\tau,\lambda}$, where

$$\tilde{E}_{m,\tau,\lambda} := \varepsilon \left(2 + \frac{3}{2}\sqrt{m}\right). \tag{5.14}$$

We conduct the same experiments as in Example 4.6 using a maximum perturbation of $\varepsilon = 10^{-3}$ as well as uniformly distributed random numbers ε_ℓ in $(-\varepsilon, \varepsilon)$. Due to the randomness we perform the experiments 100 times and then take the maximum error over all runs. The corresponding outcomes are depicted in Figure 5.4. □



(a) $\lambda = 1$ and various $\tau < \frac{1}{2}$ (b) $\tau = \frac{1}{3}$ and various $\lambda \geq 0$

Figure 5.4: Maximum perturbation error (4.12) over 100 runs and error constant (5.14) using φ_{B} in (5.1) and $s = \lceil \frac{m+1}{2} \rceil$ for the function $f(x) = \sqrt{2\delta}\,\mathrm{sinc}(2\delta\pi x)$ with $\varepsilon = 10^{-3}$, $N = 128$, $m = 2, 3, \ldots, 10$, as well as $\tau \in \{1/20,\, 1/10,\, 1/4,\, 1/3,\, 9/20\}$, $\delta = \tau N$, and $\lambda \in \{0, 0.5, 1, 2\}$, respectively.

6 sinh-type regularized Shannon sampling formula

We consider the sinh-type window function

$$\varphi_{\sinh}(x) := \begin{cases} \frac{1}{\sinh \beta}\, \sinh\big(\beta\, \sqrt{1 - (Lx/m)^2}\big) & : \quad x \in [-m/L,\, m/L]\,, \\ 0 & : \quad x \in \mathbb{R} \setminus [-m/L,\, m/L]\,, \end{cases} \tag{6.1}$$

with the parameter $\beta := \frac{s\pi\,(1+2\lambda)}{1+\lambda}$ for $s > 0$, $m \in \mathbb{N} \setminus \{1\}$, and $L = N\,(1+\lambda)$, $\lambda \geq 0$. Later we will see that the values $s = \frac{m\,(1+\lambda\pm 2\tau)}{1+2\lambda}$ are of special interest. Then we form the sinh-*type regularized* sinc *function*

$$\psi_{\sinh}(x) := \mathrm{sinc}(L\pi x)\, \varphi_{\sinh}(x)\,, \quad x \in \mathbb{R}\,. \tag{6.2}$$

Lemma 6.1. *The Fourier transform of the* sinh*-type regularized* sinc *function* (6.2) *reads as follows*

$$\hat{\psi}_{\sinh}(v) = \frac{m\beta}{2L^2\, \sinh \beta} \int_{v-L/2}^{v+L/2} \frac{J_1\big(2\pi\, \sqrt{m^2u^2/L^2 - s^2(1+2\lambda)^2/(2+2\lambda)^2}\big)}{\sqrt{m^2u^2/L^2 - s^2(1+2\lambda)^2/(2+2\lambda)^2}}\, \mathrm{d}u\,. \tag{6.3}$$

Note that the integrand of (6.3) *is real-valued, since* $J_1(\mathrm{i}\,z) = \mathrm{i}\,I_1(z)$ *for* $z \in \mathbb{C}$. *Here* J_α *denotes the Bessel function of first kind and* I_α *is the modified Bessel function of first kind.*

Proof. We apply the equation (2.10) for $f(x) = \mathrm{sinc}(L\pi x)$ and the sinh-type window function $g(x) = \varphi_{\sinh}(x)$. These functions possess the Fourier transforms

$$\begin{aligned} \hat{f}(v) &= \frac{1}{L}\, \mathbf{1}_{[-L/2,\, L/2]}(v)\,, \\ \hat{g}(v) &= \frac{2}{\sinh \beta} \int_0^{m/L} \sinh\big(\beta\, \sqrt{1 - (Lx/m)^2}\big)\, \cos(2\pi x v)\, \mathrm{d}x \\ &= \frac{\pi m \beta}{L\, \sinh \beta} \cdot \begin{cases} (w^2 - \beta^2)^{-1/2}\, J_1\big(\sqrt{w^2 - \beta^2}\big) & w \in \mathbb{R} \setminus \{-\beta,\, \beta\}\,, \\ 1/2 & w = \pm\beta \end{cases} \end{aligned} \tag{6.4}$$

(see [8, p. 38, 7.58] or [11]), where $w := 2\pi m v/L$ denotes a scaled frequency. Thus, we obtain

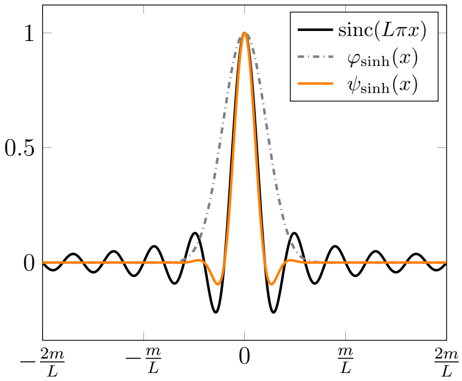
$$\begin{aligned}(\hat f * \hat g)(v) &= \int_{\mathbb R} \hat f(v-u)\,\hat g(u)\,\mathrm du = \frac{1}{L}\int_{v-L/2}^{v+L/2} \hat g(u)\,\mathrm du \\ &= \frac{m\beta}{2L^2\,\sinh\beta}\int_{v-L/2}^{v+L/2} \frac{J_1\big(2\pi\,\sqrt{m^2u^2/L^2 - s^2(1+2\lambda)^2/(2+2\lambda)^2}\big)}{\sqrt{m^2u^2/L^2 - s^2(1+2\lambda)^2/(2+2\lambda)^2}}\,\mathrm du\,.\end{aligned}$$

On the other hand, it holds

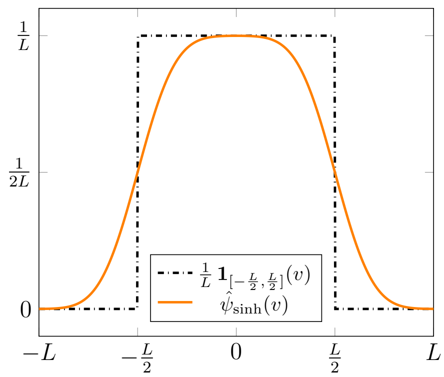
$$(f\,g)\check{}(v) = \int_{\mathbb R} \mathrm{sinc}(L\pi\,x)\,g(x)\,\mathrm e^{-2\pi \mathrm i\,vx}\,\mathrm dx = \hat\psi_{\mathrm{sinh}}(v)\,.$$

From (2.10) it follows the assertion (6.3). ■

A visualization of the sinh-type regularized sinc function (6.2) and its Fourier transform (6.3) can be found in Figure 6.1.



(a) ψ_{sinh} from (6.2)



(b) $\hat\psi_{\mathrm{sinh}}$ from (6.3)

Figure 6.1: The sinh-type regularized sinc function ψ_{sinh} as well as its Fourier transform $\hat\psi_{\mathrm{sinh}}$ with $m = 5$ and $\beta = \frac{55}{8}\,\pi$.

Lemma 6.2. *Assume that* $\tau \in (0, \frac12)$, $N \in \mathbb N$, $L = N(1+\lambda)$ *with* $\lambda \ge 0$, $s > 0$, *and* $m \in \mathbb N \setminus \{1\}$. *Let* $\varepsilon \ge \frac{4s}{m}$ *be given.*
Then the sinh*-type regularized* sinc *function* (6.2) *with* $\beta = \frac{s\pi\,(1+2\lambda)}{1+\lambda}$ *is essentially bandlimited, i. e., for all* $v \in \mathbb R \setminus \big[-\frac{L}{2}\,(1+\varepsilon),\, \frac{L}{2}\,(1+\varepsilon)\big]$ *it holds*

$$|\hat\psi_{\mathrm{sinh}}(v)| < \frac{5\,\sqrt{2s\beta}}{4L\,\sqrt{m\varepsilon}\,\sinh\beta}\,.$$

Proof. Since (6.3) is even, we consider $\hat\psi_{\mathrm{sinh}}(v)$ only for $v > \frac{L}{2}\,(1+\varepsilon)$. Then for all $u \in \big[v - \frac{L}{2},\, v + \frac{L}{2}\big]$ it holds

$$\frac{m^2u^2}{L^2} - \frac{s^2\,(1+2\lambda)^2}{(2+2\lambda)^2} > 0$$

such that from (6.3) it follows that

$$|\hat\psi_{\mathrm{sinh}}(v)| \le \frac{m\beta}{2L^2\,\sinh\beta}\int_{v-L/2}^{v+L/2} \frac{\big|J_1\big(2\pi\,\sqrt{m^2u^2/L^2 - s^2(1+2\lambda)^2/(2+2\lambda)^2}\big)\big|}{\sqrt{m^2u^2/L^2 - s^2(1+2\lambda)^2/(2+2\lambda)^2}}\,\mathrm du\,.$$

Since it holds $|J_1(x)| < \frac{1}{\sqrt{x}}$ for all $x > 0$, we obtain

$$\begin{aligned} |\hat{\psi}_{\mathrm{sinh}}(v)| &\le \frac{m\beta}{2L^2\,\sqrt{2\pi}\,\sinh\beta} \int_{v-L/2}^{v+L/2} \bigg(\frac{m^2\,u^2}{L^2} - \frac{s^2\,(1+2\lambda)^2}{(2+2\lambda)^2}\bigg)^{-3/4} \,\mathrm{d}u \\ &< \frac{m\beta}{2L^2\,\sqrt{2\pi}\,\sinh\beta} \int_{\varepsilon L/2}^{\infty} \bigg(\frac{m^2\,u^2}{L^2} - \frac{s^2\,(1+2\lambda)^2}{(2+2\lambda)^2}\bigg)^{-3/4} \,\mathrm{d}u\,. \end{aligned}$$

Substituting $u = \frac{sL\,(1+2\lambda)}{m(2+2\lambda)}\,w$, we conclude

$$|\hat{\psi}_{\mathrm{sinh}}(v)| < \frac{\sqrt{\beta}}{2L\,\sinh\beta} \int_{\varepsilon m\,(1+\lambda)/(s+2s\lambda)}^{\infty} (w^2-1)^{-3/4}\,\mathrm{d}w \le \frac{\sqrt{\beta}}{2L\,\sinh\beta} \int_{\varepsilon m/(2s)}^{\infty} (w^2-1)^{-3/4}\,\mathrm{d}w\,,$$

because $\frac{\varepsilon m\,(1+\lambda)}{s+2s\lambda} \ge \frac{\varepsilon m}{2s}$ for $\lambda \ge 0$. Since $\varepsilon \ge \frac{4s}{m}$ by assumption and

$$\max_{w\ge 2} \frac{w^{3/2}}{(w^2-1)^{3/4}} < \frac{5}{4}\,,$$

we obtain

$$|\hat{\psi}_{\mathrm{sinh}}(v)| < \frac{5\,\sqrt{\beta}}{8L\,\sinh\beta} \int_{\varepsilon m/(2s)}^{\infty} w^{-3/2}\,\mathrm{d}w = \frac{5\,\sqrt{2s\beta}}{4L\,\sqrt{m\varepsilon}\,\sinh\beta}\,.$$

This completes the proof. ■

Now we show that for the sinh-type regularized sinc function (6.2) the uniform approximation error (3.15) decays exponentially with respect to m.

Theorem 6.3. *Let $f \in \mathcal{B}_\delta(\mathbb{R})$ with $\delta = \tau N$, $\tau \in \big(0,\,\frac{1}{2}\big)$, $N \in \mathbb{N}$, $L = N\,(1+\lambda)$, $\lambda \ge 0$, and $m \in \mathbb{N}\setminus\{1\}$ be given.*
Then the regularized Shannon sampling formula (3.2) *with the* sinh-*type window function* (6.1) *and $\beta = \frac{\pi m\,(1+\lambda+2\tau)}{1+\lambda}$ satisfies the error estimate*

$$\|f - R_{\mathrm{sinh},m}f\|_{C_0(\mathbb{R})} \le \bigg(\frac{\sqrt{\beta}\,\pi\delta}{(1-2\,\mathrm{e}^{-\beta})(1-w_0^2)^{1/4}}\;\mathrm{e}^{-\beta\left(1-\sqrt{1-w_0^2}\right)} + \frac{2\,\sqrt{2\delta}}{1-\mathrm{e}^{-2\beta}}\,\mathrm{e}^{-\beta}\bigg)\,\|f\|_{L^2(\mathbb{R})}\,,$$

where $w_0 = \frac{1+\lambda-2\tau}{1+\lambda+2\tau} \in (0,\,1)$. Further, the regularized Shannon sampling formula (3.2) *with the* sinh-*type window function* (6.1) *and $\beta = \frac{\pi m\,(1+\lambda-2\tau)}{1+\lambda}$ fulfills*

$$\|f - R_{\mathrm{sinh},m}f\|_{C_0(\mathbb{R})} \le 3\,\sqrt{2\delta}\,\mathrm{e}^{-\beta}\,\|f\|_{L^2(\mathbb{R})}\,.$$

Proof. By Theorem 3.2 we only have to estimate the regularization error constant (3.17), since it holds $\varphi_{\mathrm{sinh}}(x) = \varphi_{\mathrm{sinh}}(x)\,\mathbf{1}_{[-m/L,\,m/L]}(x)$ for all $x \in \mathbb{R}$ and therefore the truncation error constant (3.18) vanishes for the sinh-type window function (6.1) by Remark 3.3.

By (6.4) we recognize that the auxiliary function (3.23) is given by

$$\eta(v) = \mathbf{1}_{[-\delta,\,\delta]}(v) - \frac{m\beta}{2L\,\sinh\beta} \int_{v-L/2}^{v+L/2} \frac{J_1\big(2\pi\,\sqrt{m^2u^2/L^2 - s^2(1+2\lambda)^2/(2+2\lambda)^2}\big)}{\sqrt{m^2u^2/L^2 - s^2(1+2\lambda)^2/(2+2\lambda)^2}}\,\mathrm{d}u\,,\ v\in\mathbb{R}\,.$$

Substituting $u = \frac{sL\,(1+2\lambda)}{m(2+2\lambda)}\, w$, we obtain for $v \in [-\delta,\, \delta]$ that

$$\eta(v) = 1 - \frac{\beta}{2\,\sinh\beta} \int_{-w_1(-v)}^{w_1(v)} \frac{J_1\big(\beta\sqrt{w^2-1}\big)}{\sqrt{w^2-1}}\,\mathrm{d}w \tag{6.5}$$

with

$$w_1(v) := \frac{m(v+L/2)(2+2\lambda)}{sL\,(1+2\lambda)} > 0\,, \quad v \in [-\delta,\, \delta]\,. \tag{6.6}$$

Since the integrand of (6.5) behaves differently for $w \in [-1,1]$ and $w \in \mathbb{R} \setminus (-1,1)$ we have to distinguish between the cases $w_1(v) \le 1$ and $w_1(v) \ge 1$ for all $v \in [-\delta,\, \delta]$. By definition $w_1(v)$ is linear and monotonously increasing. Thus, we have $\min\{w_1(v) : v \in [-\delta,\, \delta]\} = w_1(-\delta)$ and $\max\{w_1(v) : v \in [-\delta,\, \delta]\} = w_1(\delta)$. In the following we can choose an optimal parameter $s = s(m,\tau,\lambda) > 0$ such that either $w_1(\delta) \le 1$ or $w_1(-\delta) \ge 1$ is fulfilled.

Case 1 ($w_1(\delta) \le 1$): Note that by [4, 6.681–3] and [1, 10.2.13] as well as $J_1(\mathrm{i}\,z) = \mathrm{i}\,I_1(z)$ for $z \in \mathbb{C}$ it holds

$$\begin{aligned} \int_{-1}^{1} \frac{J_1\big(\beta\sqrt{w^2-1}\big)}{\sqrt{w^2-1}}\,\mathrm{d}w &= \int_{-1}^{1} \frac{I_1\big(\beta\sqrt{1-w^2}\big)}{\sqrt{1-w^2}}\,\mathrm{d}w = \int_{-\pi/2}^{\pi/2} I_1(\beta\,\cos s)\,\mathrm{d}s \\ &= \pi\left(I_{1/2}\Big(\frac{\beta}{2}\Big)\right)^2 = \frac{4}{\beta}\left(\sinh\frac{\beta}{2}\right)^2 . \end{aligned} \tag{6.7}$$

Then from (6.5) and (6.7) it follows that

$$\begin{aligned} \eta(v) &= \frac{\beta}{4\,\big(\sinh\frac{\beta}{2}\big)^2} \int_{-1}^{1} \frac{I_1\big(\beta\sqrt{1-w^2}\big)}{\sqrt{1-w^2}}\,\mathrm{d}w - \frac{\beta}{2\,\sinh\beta}\int_{-w_1(-v)}^{w_1(v)} \frac{I_1\big(\beta\sqrt{1-w^2}\big)}{\sqrt{1-w^2}}\,\mathrm{d}w \\ &= \eta_1(v) + \eta_2(v) \end{aligned} \tag{6.8}$$

with

$$\begin{aligned} \eta_1(v) &:= \left(\frac{\beta}{4\,\big(\sinh\frac{\beta}{2}\big)^2} - \frac{\beta}{2\,\sinh\beta}\right)\int_{-w_1(-v)}^{w_1(v)} \frac{I_1\big(\beta\sqrt{1-w^2}\big)}{\sqrt{1-w^2}}\,\mathrm{d}w\,, \\ \eta_2(v) &:= \frac{\beta}{4\,\big(\sinh\frac{\beta}{2}\big)^2}\left(\int_{-1}^{1} - \int_{-w_1(-v)}^{w_1(v)}\right)\frac{I_1\big(\beta\sqrt{1-w^2}\big)}{\sqrt{1-w^2}}\,\mathrm{d}w\,. \end{aligned}$$

By $2\,\big(\sinh\frac{\beta}{2}\big)^2 < \sinh\beta$ we have

$$\frac{\beta}{4\,\big(\sinh\frac{\beta}{2}\big)^2} - \frac{\beta}{2\,\sinh\beta} > 0\,. \tag{6.9}$$

Since the integrand of (6.8) is positive, it is easy to find an upper bound of $\eta_1(v)$ for all $v \in [-\delta,\, \delta]$, because by (6.7) it holds

$$0 \le \eta_1(v) \;\le\; \left(\frac{\beta}{4\,\big(\sinh\frac{\beta}{2}\big)^2} - \frac{\beta}{2\,\sinh\beta}\right)\int_{-1}^{1} \frac{I_1\big(\beta\sqrt{1-w^2}\big)}{\sqrt{1-w^2}}\,\mathrm{d}w$$

$$= \; 1 - \frac{2\left(\sinh\frac{\beta}{2}\right)^2}{\sinh\beta} = \frac{2 - 2\,\mathrm{e}^{-\beta}}{\mathrm{e}^{\beta} - \mathrm{e}^{-\beta}} < \frac{2}{1 - \mathrm{e}^{-2\beta}}\,\mathrm{e}^{-\beta}\,.$$

Further, for arbitrary $v \in [-\delta,\, \delta]$ we obtain

$$\begin{aligned} 0 \le \eta_2(v) \;&=\; \frac{\beta}{4\left(\sinh\frac{\beta}{2}\right)^2}\left(\int_{-1}^{-w_1(-v)} + \int_{w_1(v)}^{1}\right)\frac{I_1\big(\beta\sqrt{1-w^2}\big)}{\sqrt{1-w^2}}\,\mathrm{d}w \\ &=\; \frac{\beta}{4\left(\sinh\frac{\beta}{2}\right)^2}\left(\int_{w_1(-v)}^{1} + \int_{w_1(v)}^{1}\right)\frac{I_1\big(\beta\sqrt{1-w^2}\big)}{\sqrt{1-w^2}}\,\mathrm{d}w \\ &\le\; \frac{\beta}{2\left(\sinh\frac{\beta}{2}\right)^2}\int_{w_0}^{1}\frac{I_1\big(\beta\sqrt{1-w^2}\big)}{\sqrt{1-w^2}}\,\mathrm{d}w\,, \end{aligned} \tag{6.10}$$

since the integrand is positive and $w_0 \coloneqq w_1(-\delta) = \min\{w_1(v) \,:\, v \in [-\delta,\, \delta]\}$. Substituting $w = \sin t$ in (6.10), for all $v \in [-\delta,\, \delta]$ we can estimate

$$\eta_2(v) \le \frac{\beta}{2\left(\sinh\frac{\beta}{2}\right)^2}\int_{\arcsin w_0}^{\pi/2} I_1(\beta\,\cos t)\,\mathrm{d}t$$

with $\arcsin w_0 \in \left(0,\, \frac{\pi}{2}\right)$. The above integral can now be approximated by the rectangular rule (see Figure 6.2) such that

$$\eta_2(v) \le \frac{\beta}{2\left(\sinh\frac{\beta}{2}\right)^2}\left(\frac{\pi}{2} - \arcsin w_0\right) I_1\left(\beta\sqrt{1-w_0^2}\right).$$

Further it holds $4\,(\sinh\frac{\beta}{2})^2 = \mathrm{e}^{\beta} - 2 + \mathrm{e}^{-\beta} > \mathrm{e}^{\beta} - 2$. Since by [10, Lemma 7] we have $\sqrt{2\pi x}\,\mathrm{e}^{-x}\,I_1(x) < 1$, it holds

$$I_1\left(\beta\sqrt{1-w_0^2}\right) < \frac{1}{\sqrt{2\pi\beta}}\left(1 - w_0^2\right)^{-1/4}\mathrm{e}^{\beta\sqrt{1-w_0^2}}\,,$$

and therefore we obtain

$$\eta_2(v) \le \frac{\sqrt{\beta}\,(\pi - 2\arcsin w_0)}{\sqrt{2\pi}\,(1-w_0^2)^{1/4}\,(1 - 2\,\mathrm{e}^{-\beta})}\,\mathrm{e}^{-\beta\left(1-\sqrt{1-w_0^2}\right)}\,.$$

Additionally using (3.17) and (3.23) as well as $\arcsin w_0 \in \left(0,\, \frac{\pi}{2}\right)$ this yields

$$E_1(m,\delta,L) \le \frac{\sqrt{\beta\,\pi\delta}}{(1 - 2\,\mathrm{e}^{-\beta})(1-w_0^2)^{1/4}}\,\mathrm{e}^{-\beta\left(1-\sqrt{1-w_0^2}\right)} + \frac{2\sqrt{2\delta}}{1-\mathrm{e}^{-2\beta}}\,\mathrm{e}^{-\beta}\,. \tag{6.11}$$

What remains is the choice of the optimal parameter $s > 0$, where we have to fulfill $w_1(\delta) \le 1$. To obtain the smallest error bound we are looking for an $s > 0$ that minimizes the error term $\max_{v\in[-\delta,\,\delta]} |\eta(v)|$. By (6.8) and (6.9) we maximize the second integral in (6.8). Since the integrand of (6.8) is positive, the integration limit $w_1(v)$ should be as large as possible for all $v \in [-\delta,\, \delta]$ and therefore $w_1(\delta) = 1$. Rearranging this by (6.6) in terms of s we see immediately that

$$s = \frac{m\,(1 + \lambda + 2\tau)}{1 + 2\lambda}$$

and hence

$$\beta = \frac{\pi m\,(1 + \lambda + 2\tau)}{1 + \lambda}\,, \quad w_0 = w_1(-\delta) = \frac{1 + \lambda - 2\tau}{1 + \lambda + 2\tau} \in (0,\, 1)$$

such that β depends linearly on m by definition.

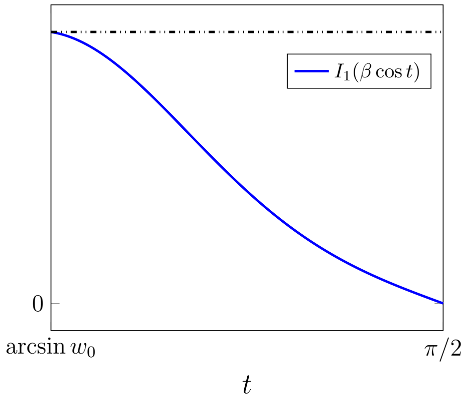


Figure 6.2: The integrand $I_1(\beta\ \cos t)$ on the interval $[\text{arcsin}\ w_0,\ \pi/2]$.

Case 2 ($w_1(-\delta) \geq 1$): From (6.5) it follows that

$$\eta(v) = \eta_3(v) - \eta_4(v)\,, \quad v \in [-\delta,\ \delta]\,,$$

with

$$\begin{aligned} \eta_3(v) &:= 1 - \frac{\beta}{2\ \sinh\beta} \int_{-1}^{1} \frac{I_1\big(\beta\sqrt{1-w^2}\big)}{\sqrt{1-w^2}}\,\mathrm{d}w\,, \\ \eta_4(v) &:= \frac{\beta}{2\ \sinh\beta}\left(\int_{-w_1(-v)}^{-1} + \int_{1}^{w_1(v)}\right) \frac{J_1\big(\beta\sqrt{w^2-1}\big)}{\sqrt{w^2-1}}\,\mathrm{d}w\,. \end{aligned}$$

By (6.7) we obtain

$$\eta_3(v) = 1 - \frac{2\big(\sinh\frac{\beta}{2}\big)^2}{\sinh\ \beta} = \frac{2\,\mathrm{e}^{-\beta}}{1+\mathrm{e}^{-\beta}} > 0\,.$$

Further it holds

$$\eta_4(v) = \frac{\beta}{2\ \sinh\beta}\left(\int_{1}^{w_1(-v)} + \int_{1}^{w_1(v)}\right) \frac{J_1\big(\beta\sqrt{w^2-1}\big)}{\sqrt{w^2-1}}\,\mathrm{d}w\,.$$

Substituting $w = \cosh t$ in above integrals, we have

$$\eta_4(v) = \frac{\beta}{2\ \sinh\beta}\left(\int_{0}^{\mathrm{arcosh}(w_1(-v))} + \int_{0}^{\mathrm{arcosh}(w_1(v))}\right) J_1(\beta\ \sinh t)\,\mathrm{d}t\,.$$

In order to estimate these integrals properly we now have a closer look at the integrand. As known, the Bessel function J_1 oscillates on $[0,\ \infty)$ and has the non-negative simple zeros $j_{1,n}$, $n \in \mathbb{N}_0$, with $j_{1,0} = 0$. The zeros $j_{1,n}$, $n = 1, \ldots, 40$, are tabulated in [20, p. 748]. On each interval $\big[\mathrm{arsinh}\,\frac{j_{1,2n}}{\beta},\ \mathrm{arsinh}\,\frac{j_{1,2n+2}}{\beta}\big]$, $n \in \mathbb{N}_0$, the integrand $J_1(\beta\ \sinh t)$ is firstly non-negative and then non-positive, see Figure 6.3. Due to this properties and the fact that the amplitude is decreasing when $x \to \infty$, the integrals are positive on each interval $\big[\mathrm{arsinh}\,\frac{j_{1,2n}}{\beta},\ \mathrm{arsinh}\,\frac{j_{1,2n+2}}{\beta}\big]$, $n \in \mathbb{N}_0$. Note that by [4, 6.645–1] it holds

$$\int_0^\infty J_1(\beta\sinh t)\,\mathrm{d}t = I_{1/2}\Big(\frac{\beta}{2}\Big)\,K_{1/2}\Big(\frac{\beta}{2}\Big) = \frac{2}{\sqrt{\pi\beta}}\ \sinh\frac{\beta}{2}\cdot\sqrt{\frac{\pi}{\beta}}\,\mathrm{e}^{-\beta/2} = \frac{1-\mathrm{e}^{-\beta}}{\beta}\,,$$

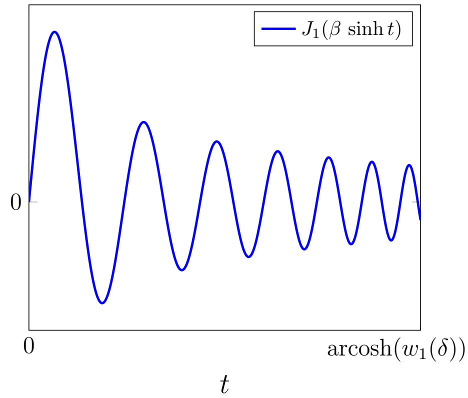


Figure 6.3: The integrand $J_1(\beta \sinh t)$ on the interval $[0, \operatorname{arcosh}(w_1(\delta))]$.

where K_α denotes the modified Bessel function of second kind and $I_{1/2}$, $K_{1/2}$ denote modified Bessel functions of half order (see [1, 10.2.13, 10.2.14, and 10.2.17]). In addition, numerical experiments have shown that for all $T \geq 0$ it holds

$$0 \leq \int_0^T J_1(\beta \sinh t)\, \mathrm{d}t \leq \frac{3\,(1 - \mathrm{e}^{-\beta})}{2\,\beta}\,.$$

Therefore, we obtain

$$0 \leq \eta_4(v) \leq \frac{\beta}{2\,\sinh \beta} \cdot \frac{3\,(1 - \mathrm{e}^{\beta})}{\beta} = \frac{3\,\mathrm{e}^{-\beta}}{1 + \mathrm{e}^{-\beta}} < 3\,\mathrm{e}^{-\beta}$$

and hence

$$\max_{v \in [-\delta,\,\delta]} |\eta(v)| = \max_{v \in [-\delta,\,\delta]} |\eta_3(v) - \eta_4(v)| < 3\,\mathrm{e}^{-\beta}\,. \tag{6.12}$$

Thus, by (3.17) and (3.23) we conclude that

$$E_1(m, \delta, L) \leq 3\,\sqrt{2\delta}\,\mathrm{e}^{-\beta}\,. \tag{6.13}$$

What remains is the choice of the optimal parameter $s > 0$, where we have to fulfill $w_1(-\delta) = c$ with $c \geq 1$. Rearranging this by (6.6) in terms of s we see that

$$s = s(c) = \frac{m\,(1 + \lambda - 2\tau)}{c\,(1 + 2\lambda)}\,, \quad \beta = \beta(c) = \frac{\pi m\,(1 + \lambda - 2\tau)}{c\,(1 + \lambda)}.$$

To obtain the smallest error bound we are looking for a constant $c \geq 1$ that minimizes the error term $\max_{v \in [-\delta,\,\delta]} |\eta(v)|$. By (6.12) we minimize the upper bound $3\,\mathrm{e}^{-\beta(c)}$. Since $3\,\mathrm{e}^{-\beta(c)}$ is monotonously increasing for $c \geq 1$ we recognize that the minimum value is $c = 1$. Hence, the suggested parameters are

$$s = \frac{m\,(1 + \lambda - 2\tau)}{1 + 2\lambda}\,, \quad \beta = \frac{\pi m\,(1 + \lambda - 2\tau)}{1 + \lambda}$$

such that β depends linearly on m by definition. This completes the proof. ■

Now we compare the actual decay rates of the error constants (6.11) with $\beta = \frac{\pi m\,(1+\lambda+2\tau)}{1+\lambda}$ and (6.13) with $\beta = \frac{\pi m\,(1+\lambda-2\tau)}{1+\lambda}$. It can be seen that the decay rate of (6.11) reads as

$$\frac{\pi m\,(1+\lambda+2\tau)}{1+\lambda}\,\Big(1-\sqrt{1-w_0^2}\,\Big)$$

with $w_0 = \frac{1+\lambda-2\tau}{1+\lambda+2\tau}$. On the other hand, the decay rate of (6.13) is $\frac{\pi m\,(1+\lambda-2\tau)}{1+\lambda}$. Since $1+\lambda > 2\tau$ for all $\lambda \geq 0$ and $\tau \in \big(0,\,\frac{1}{2}\big)$, simple calculation shows that

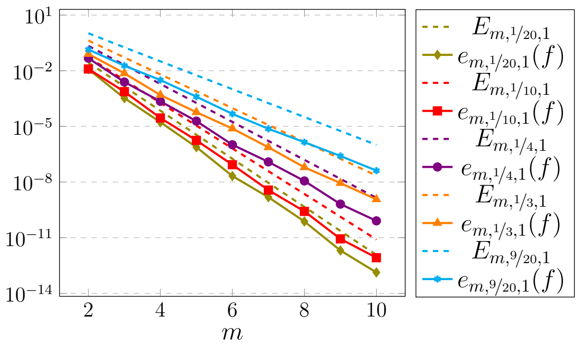
$$\frac{\pi m\,(1+\lambda-2\tau)}{1+\lambda} > \frac{\pi m\,(1+\lambda+2\tau)}{1+\lambda}\,\Big(1-\sqrt{1-w_0^2}\,\Big)\,.$$

Hence, the error constant (6.13) decays faster than the error constant (6.11). Therefore, we always use the regularized Shannon sampling formula (3.2) with the sinh-type window function (6.1) with $\beta = \frac{\pi m\,(1+\lambda-2\tau)}{1+\lambda}$.

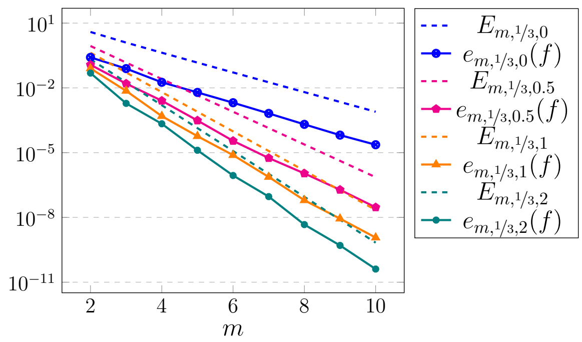
Example 6.4. Analogous to Example 4.4, we now aim to visualize the error bound from Theorem 6.3, i. e., for $\varphi = \varphi_{\mathrm{sinh}}$ we show that for the approximation error (4.9) it holds by (6.13) that $e_{m,\tau,\lambda}(f) \leq E_{m,\tau,\lambda}\,\|f\|_{L^2(\mathbb{R})}$, where

$$E_1(m,\delta,L) \leq E_{m,\tau,\lambda} := 3\,\sqrt{2\delta}\,\mathrm{e}^{-\beta}\,, \tag{6.14}$$

with $\beta = \frac{\pi m\,(1+\lambda-2\tau)}{1+\lambda}$. The associated results are displayed in Figure 6.4. In both parts we see a substantial improvement in the results compared to both Figure 4.2 and 5.3. We also remark that for larger choices of N, the line plots in Figure 6.4 would only be shifted slightly upwards, such that for all N we receive almost the same error results. This is to say, we can see that the sinh-type window function is by far the best choice as a regularization function for regularized Shannon sampling sums. □



(a) $\lambda = 1$ and various $\tau < \frac{1}{2}$



(b) $\tau = \frac{1}{3}$ and various $\lambda \geq 0$

Figure 6.4: Maximum approximation error (4.9) and error constant (6.14) using φ_{sinh} and $\beta = \frac{\pi m\,(1+\lambda-2\tau)}{1+\lambda}$ in (6.1) for the function $f(x) = \sqrt{2\delta}\,\mathrm{sinc}(2\delta\pi x)$ with $N = 128$, $m = 2, 3, \ldots, 10$, as well as $\tau \in \{1/20,\, 1/10,\, 1/4,\, 1/3,\, 9/20\}$, $\delta = \tau N$, and $\lambda \in \{0, 0.5, 1, 2\}$, respectively.

Now we show that for the regularized Shannon sampling formula with the Gaussian window function (6.1) the uniform perturbation error (3.25) only grows as $\mathcal{O}(\sqrt{m})$.

Theorem 6.5. *Let $f \in \mathcal{B}_\delta(\mathbb{R})$ with $\delta = \tau N$, $\tau \in (0, 1/2)$, $N \in \mathbb{N}$, $L = N(1+\lambda)$ with $\lambda \ge 0$ and $m \in \mathbb{N} \setminus \{1\}$ be given. Further let $R_{\mathrm{sinh},m}\tilde{f}$ be as in* (3.24) *with the noisy samples $\tilde{f}_\ell = f\big(\frac{\ell}{L}\big) + \varepsilon_\ell$, where $|\varepsilon_\ell| \le \varepsilon$ for all $\ell \in \mathbb{Z}$.*
Then the regularized Shannon sampling formula (3.2) *with the* sinh*-type window function* (6.1) *and $\beta = \frac{\pi m\,(1+\lambda+2\tau)}{1+\lambda}$ is numerically robust and satisfies*

$$\|R_{\mathrm{sinh},m}\tilde{f} - R_{\mathrm{sinh},m}f\|_{C_0(\mathbb{R})} \le \varepsilon \left(2 + \sqrt{\frac{2+2\lambda}{1+\lambda-2\tau}}\,\frac{1}{1-\mathrm{e}^{-2\beta}}\,\sqrt{m}\right). \tag{6.15}$$

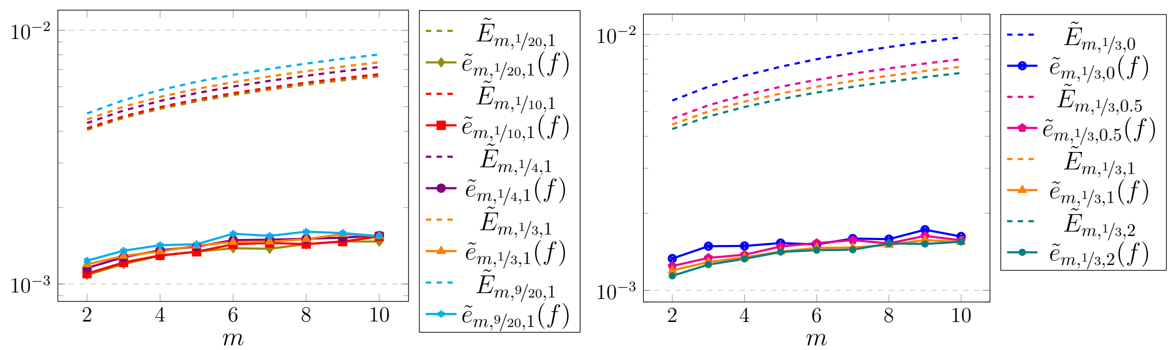
Proof. By Theorem 3.4 we only have to compute $\hat{\varphi}_{\mathrm{sinh}}(0)$ for the sinh-type window function (6.1). By (6.4) we recognize that

$$\hat{\varphi}_{\mathrm{sinh}}(0) = \frac{\pi m \beta}{L\,\sinh\beta} \cdot \frac{I_1(\beta)}{\beta} = \frac{\pi m\, I_1(\beta)}{L\,\sinh\beta}.$$

By [10, Lemma 7] it holds $\sqrt{2\pi\beta}\,\mathrm{e}^{-\beta}\, I_1(\beta) < 1$. Thus, we have

$$\frac{\pi m\, I_1(\beta)}{\sinh\beta} \le \frac{\pi m\,\mathrm{e}^{\beta}}{\sqrt{2\pi\beta}\,\sinh\beta} = \frac{\sqrt{2\pi}\,m}{\sqrt{\beta}\,(1-\mathrm{e}^{-2\beta})}.$$

If we now use $\beta = \frac{\pi m\,(1+\lambda-2\tau)}{1+\lambda}$, then (3.25) yields the assertion. ∎



(a) $\lambda = 1$ and various $\tau < \frac{1}{2}$ (b) $\tau = \frac{1}{3}$ and various $\lambda \ge 0$

Figure 6.5: Maximum perturbation error (4.12) over 100 runs and error constant (6.16) using φ_{sinh} in (6.1) and $\beta = \frac{\pi m\,(1+\lambda-2\tau)}{1+\lambda}$ for the function $f(x) = \sqrt{2\delta}\,\mathrm{sinc}(2\delta\pi x)$ with $\varepsilon = 10^{-3}$, $N = 128$, $m = 2, 3, \dots, 10$, as well as $\tau \in \{1/20, 1/10, 1/4, 1/3, 9/20\}$, $\delta = \tau N$, and $\lambda \in \{0, 0.5, 1, 2\}$, respectively.

Example 6.6. Now we aim to visualize the error bound from Theorem 6.5. Similar to Example 4.6, we show that for the perturbation error (4.12) with $\varphi = \varphi_{\mathrm{sinh}}$ it holds by (6.15) that $\tilde{e}_{m,\tau,\lambda}(f) \le \tilde{E}_{m,\tau,\lambda}$ where

$$\tilde{E}_{m,\tau,\lambda} := \varepsilon \left(2 + \sqrt{\frac{2+2\lambda}{1+\lambda-2\tau}}\,\frac{1}{1-\mathrm{e}^{-2\beta}}\,\sqrt{m}\right). \tag{6.16}$$

We conduct the same experiments as in Example 4.6 using a maximum perturbation of $\varepsilon = 10^{-3}$ as well as uniformly distributed random numbers ε_ℓ in $(-\varepsilon, \varepsilon)$. Due to the randomness we perform the experiments 100 times and then take the maximum error over all runs. The corresponding outcomes are depicted in Figure 6.5. □

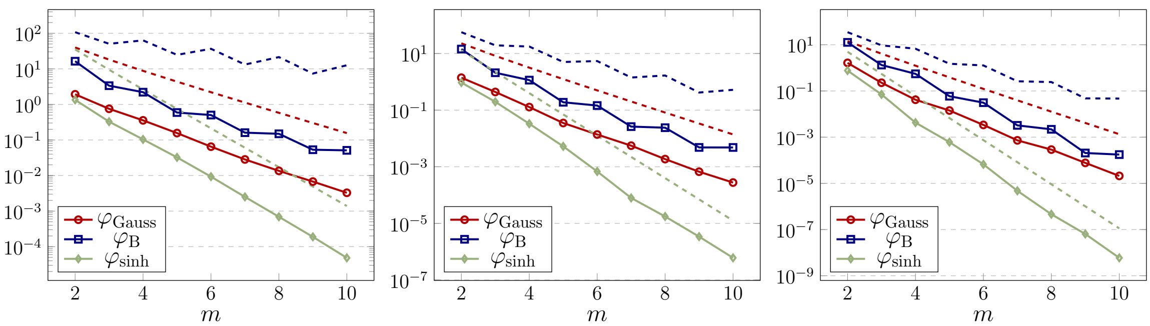
7 Conclusion

To overcome the drawbacks of classical Shannon sampling series – which are poor convergence and non-robustness in the presence of noise – in this paper we considered regularized Shannon sampling formulas with localized sampling. To this end, we considered bandlimited functions $f \in \mathcal{B}_\delta(\mathbb{R})$ and introduced a set $\Phi_{m,L}$ of window functions. Despite the original result, where $\varphi \in \Phi_{m,L}$ is chosen as the rectangular window function, and the well–studied approach of using the Gaussian window function, we proposed new window functions with compact support $[-m/L,\, m/L]$, namely the B–spline and sinh-type window function, which are well-studied in the context of the nonequispaced fast Fourier transform (NFFT).

In Section 3 we considered an arbitrary window function $\varphi \in \Phi_{m,L}$ and presented a unified approach to error estimates of the uniform approximation error for regularized Shannon sampling formulas in Theorem 3.2, as well as a unified approach to the numerical robustness of regularized Shannon sampling formulas in Theorem 3.4.

Then, in the next sections, we concretized the results for special window functions. More precisely, it was shown that the uniform approximation error decays exponentially with respect to the truncation parameter m, if $\varphi \in \Phi_{m,L}$ is the Gaussian, B–spline, or sinh-type window function. Moreover, we have shown that the regularized Shannon sampling formulas are numerically robust for noisy samples, i. e., if $\varphi \in \Phi_{m,L}$ is the Gaussian, B–spline, or sinh-type window function, then the uniform perturbation error only grows as $m^{1/2}$. While the Gaussian window function from Section 4 has already been studied in numerous papers such as [12, 13, 14, 15, 5], we remarked that Theorem 4.3 improves a corresponding result in [5], since we improved the exponential decay rate from $(m-1)$ to m.

Throughout this paper, several numerical experiments illustrated the corresponding theoretical results. Finally, comparing the proposed window functions as done in Figure 7.1, the superiority of the new proposed sinh-type window function can easily be seen, since even small choices of the truncation parameter $m \le 10$ are sufficient for achieving high precision. Due to the usage of localized sampling the evaluation of $R_{\varphi,m}f$ on an interval $[0, 1/L]$ requires only $2m$ samples and therefore has a computational cost of $\mathcal{O}(2m)$ flops. Thus, a reduction of the truncation parameter m is desirable to obtain an efficient method.



(a) $\lambda = 0.5$ (b) $\lambda = 1$ (c) $\lambda = 2$

Figure 7.1: Maximum approximation error (4.9) and error constant (3.16) using $\varphi \in \{\varphi_{\text{Gauss}}, \varphi_{\text{B}}, \varphi_{\text{sinh}}\}$ for the function $f(x) = \delta\,\text{sinc}^2(\delta\pi x)$ with $N = 256$, $\tau = 0.45$, $\delta = \tau N$, as well as $m = 2, 3, \dots, 10$, and $\lambda \in \{0.5, 1, 2\}$.

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