Optimal approximation of multivariate periodic functions

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Abstract. Recently there has been an increasing interest in approximation of multivariate functions. This is due to the fact that many problems from areas like finance and chemistry are modeled in function spaces on high-dimensional domains, e.g. the d-dimensional cube or the d-dimensional torus.

Usually one is not only interested in approximating a single function, but whole classes of functions. Typically such classes are unit balls of a function space F, e.g. a Sobolev or Besov space. The error is often measured in the L_2-norm, sometimes also in the uniform norm. In these cases, the quality of approximations can be expressed in terms of approximation numbers a_n of the embedding $F \hookrightarrow L_2$ resp. $F \hookrightarrow L_\infty$.

For classical Sobolev spaces, the exact asymptotic behaviour of these approximation numbers has been known for a long time. However, almost all estimates which are available in the literature involve unspecified constants. It remains unclear how these constants depend on the parameters of the space, in particular on the dimension d of the underlying domain, on the smoothness order, and on the chosen norm.

Often the asymptotic rate can be seen only after "waiting exponentially long", say until $n = 2^d$ or even $n = d^d$. For large d, this is beyond the capacity of computers. Therefore, from the numerical point of view, it is useless to know only the asymptotic order; one needs additional information on the d-dependence of the "hidden" constants. An interesting feature is that the behaviour of the approximation numbers a_n in the preasymptotic range, i.e. for small n, could be completely different than for $n \to \infty$.

In the lectures I will first provide the necessary background on approximation numbers of operators in Hilbert and Banach spaces, and then I will discuss the above-mentioned approximation problems for functions (on the d-dimensional torus) in isotropic Sobolev spaces and in Sobolev spaces of dominating mixed smoothness. Moreover, I will describe a quite general approach that shows how one can derive estimates on L_∞-approximation from L_2-estimates.

The methods are functional-analytic, mostly in the context of Hilbert spaces, combined with combinatorial estimates. The new results that I will present are contained in a number of recent joint papers (2014-2016) with Winfried Sickel (Jena), Tino Ullrich (Bonn) and Fernando Cobos (Madrid).