Surjectivity of linear partial differential operators on spaces of scalar valued and vector valued distributions

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1. Introduction

In many mathematical models linear partial differential operators show up, e.g.

$$\Delta = \Delta_x = \sum_{j=1}^{d} \partial_j^2 \qquad \text{(Laplace operator)},$$
$$\partial_t - \Delta_x \qquad \text{(Heat operator)},$$
$$\partial_t^2 - \Delta_x \qquad \text{(Wave operator)},$$
$$-i\partial_t - \Delta_x \qquad \text{(time dependent free Schrödinger operator)},$$
$$\tfrac{1}{2}(\partial_1 + i\partial_2) \qquad \text{(Cauchy Riemann operator)}.$$

For general $P \in \mathbb{C}[X_1, \ldots, X_d]$ set

$$P(D) := P(-i\partial_1, \ldots, -i\partial_d).$$

E.g. $\Delta = P_L(D)$ for $P_L(\xi) = -\sum_{j=1}^d \xi_j^2$

$\partial_t - \Delta_x = P_H(D)$ for $P_H(\xi_1, \ldots, \xi_d) = i\xi_1 + \sum_{j=2}^d \xi_j^2$

$\partial_t^2 - \Delta_x = P_W(D)$ for $P_W(\xi_1, \ldots, \xi_d) = -\xi_1^2 + \sum_{j=2}^d \xi_j^2$

$-i\partial_t - \Delta_x = P_S(D)$ for $P_S(\xi_1, \ldots, \xi_d) = \xi_1 + \sum_{j=2}^d \xi_j^2$

For $X \subseteq \mathbb{R}^d$ open and f given, solve $P(D)u = f$ in X.

Possible for every f from a fixed space of functions? "Solution" in which sense; classical, distributional?

Let $P \in \mathbb{C}[X_1, \dots, X_d] \backslash \{0\}$ and let $X \subseteq \mathbb{R}^d$ be open.

i) When is $P(D) : C^\infty(X) \to C^\infty(X)$ surjective?

ii) When is $C^\infty(X) \subseteq P(D)(\mathscr{D}'(X))$?

iii) When is $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ surjective?

Answers will depend on combined properties of P and X.

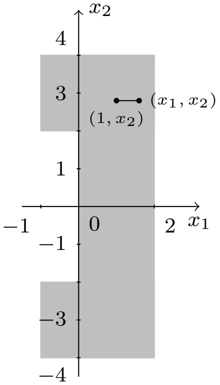
Example:

$$X = \Big((0,2)\times(-4,4)\Big) \cup \Big((-1,1)\times(-4,-2)\Big) \cup \Big((-1,1)\times(2,4)\Big)$$

$P_1(\xi_1,\xi_2) = i\xi_1 \ \Rightarrow P_1(D) = \partial_1$;

given $f \in C^\infty(X) \ \Rightarrow$
$u(x_1,x_2) := \int_1^{x_1} f(t,x_2)\,dt \ \in C^\infty(X)$
satisfies $\partial_1 u = f$

$\Rightarrow P_1(D) : C^\infty(X) \to C^\infty(X)$ surjective



Example:

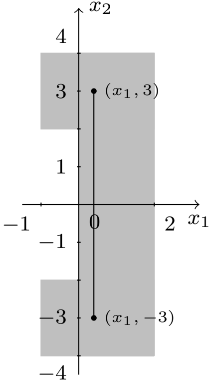
$$X = \Big((0,2)\times(-4,4)\Big) \cup \Big((-1,1)\times(-4,-2)\Big) \cup \Big((-1,1)\times(2,4)\Big)$$

$P_2(\xi_1,\xi_2) = i\xi_2 \;\Rightarrow P_2(D) = \partial_2$;

choose $\eta \in C^\infty(\mathbb{R})$ with $\eta(t) = 0$ for $t \notin [-1,1]$ and $\int_{-1}^{1} \eta(t)\,dt > 0$; set

$$f(x_1,x_2) = \begin{cases} \frac{\eta(x_2)}{x_1}, & \text{if } x_1 > 0 \\ 0, & \text{if } x_1 \le 0 \end{cases}$$

$\Rightarrow f \in C^\infty(X)$; suppose
$\exists\, u \in C^1(X) :\ \partial_2 u = f$;
for $x_1 \in (0,2)$ we then have
$u(x_1,3) - u(x_1,-3) = \int_{-3}^{3} \partial_2 u(x_1,t)dt$
$= \frac{1}{x_1}\int_{-1}^{1} \eta(t)\,dt \rightarrow_{x_1\to 0} \infty$



$\Rightarrow P_2(D) : C^1(X) \rightarrow C^\infty(X)$ not surjective

Example:

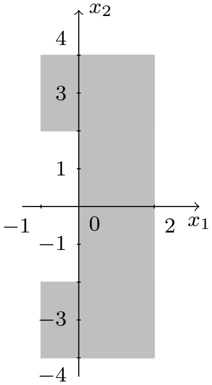
$$X = \Big((0,2)\times(-4,4)\Big)\cup\Big((-1,1)\times(-4,-2)\Big)\cup\Big((-1,1)\times(2,4)\Big)$$

For $P_1(\xi_1,\xi_2) = i\xi_1$ resp. $P_2(\xi_1,\xi_2) = i\xi_2$ is

$P_1(D) : C^\infty(X) \to C^\infty(X)$ surjective,

$P_2(D) : C^1(X) \to C^\infty(X)$ not surjective.

Is it possible to "see" this without calculation? What about $P_2(D)$ if we allow for more general solutions of $P_2(D)u = f, f \in C^\infty(X)$, than $u \in C^1(X)$?



2. Distributions and differential operators

$X \subseteq \mathbb{R}^d$ open, $K \Subset X (:\Leftrightarrow K \subseteq X$ compact), $l \in \mathbb{N}_0$

$$\|\cdot\|_{l,K} : C^\infty(X) \to [0,\infty), f \mapsto \sup_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq l} \sup_{x \in K} |\partial^\alpha f(x)|$$

defines a seminorm on $C^\infty(X)$.
$(f_n)_{n\in\mathbb{N}} \in C^\infty(X)^{\mathbb{N}}$ converges to $f \in C^\infty(X)$ $:\Leftrightarrow$

$$\forall\, K \Subset X, l \in \mathbb{N}_0 : \lim_{n\to\infty} \|f_n - f\|_{l,K} = 0$$

This convergence can be described by a metric on $C^\infty(X)$ which is complete; we denote by $\mathscr{E}(X)$ the space $C^\infty(X)$ equipped with this notion of convergence.

For $M \subseteq \mathbb{R}^d$ we set $\mathscr{D}(M) := \{\varphi \in C^\infty(\mathbb{R}^d);\ \operatorname{supp}\varphi \subseteq M \text{ compact}\}$, where $\operatorname{supp}\varphi = \overline{\{x \in \mathbb{R}^d;\ \varphi(x) \neq 0\}}$; $\mathscr{D}(M)$ is a subspace of $C^\infty(\mathbb{R}^d)$.

$(\varphi_n)_{n\in\mathbb{N}} \in \mathscr{D}(M)^{\mathbb{N}}$ converges to $\varphi \in \mathscr{D}(M)$:$\Leftrightarrow$

- $\lim_{n\to\infty} \varphi_n = \varphi$ in $\mathscr{E}(\mathbb{R}^d)$,
- $\exists\, K \Subset M$: $\cup_{n\in\mathbb{N}} \operatorname{supp}\varphi_n \cup \operatorname{supp}\varphi \subseteq K$

For every non-compact M, this convergence cannot be described by a metric on $\mathscr{D}(M)$ but by a (locally convex) topology which is complete; from now on we always equip $\mathscr{D}(M)$ with the above notion of convergence.

For open $X \subseteq \mathbb{R}^d$ the "inclusion" $i : \mathscr{D}(X) \hookrightarrow \mathscr{E}(X), \varphi \mapsto \varphi_{|X}$ is continuous, has dense range; thus, every continuous $u : \mathscr{E}(X) \to \mathbb{C}$ induces continuous $u : \mathscr{D}(X) \to \mathbb{C}$, and u uniquely determined by $u_{|\mathscr{D}(X)}$.

For $X \subseteq \mathbb{R}^d$ open we define

$$\mathscr{D}'(X) := \{u : \mathscr{D}(X) \to \mathbb{C};\ u \text{ linear, continuous}\}$$

$$\mathscr{E}'(X) := \{u : \mathscr{E}(X) \to \mathbb{C};\ u \text{ linear, continuous}\}$$

$\mathscr{D}'(X), \mathscr{E}'(X)$ are vector spaces, $u \in \mathscr{D}'(X)$ is called a distribution on X

By the previous slide:

$$\mathscr{E}'(X) \to \mathscr{D}'(X), u \mapsto u_{|\mathscr{D}(X)}$$

is well-defined, obviously linear, and one-to-one.

2.1 Proposition

a) For linear $u : \mathscr{E}(X) \to \mathbb{C}$ tfae:
   i) $u \in \mathscr{E}'(X)$,
   ii) $\exists\, K \Subset X, l \in \mathbb{N}_0, C > 0 \,\forall\, f \in \mathscr{E}(X) : \; |u(f)| \leq C\|f\|_{l,K}$.

b) For linear $u : \mathscr{D}(X) \to \mathbb{C}$ tfae:
   i) $u \in \mathscr{D}'(X)$,
   ii) $\forall\, K \Subset X \,\exists\, l \in \mathbb{N}_0, C > 0 \,\forall\, \varphi \in \mathscr{D}(K) : \; |u(\varphi)| \leq C\|\varphi\|_{l,K}$.

Notation: $\langle u, \varphi \rangle := u(\varphi)$

If in b) ii) $l \in \mathbb{N}_0$ may be chosen independently of $K \Subset X$ then u is of finite order and

ord$(u) := \min\{l \in \mathbb{N}_0; \forall\, K \Subset X \,\exists\, C > 0 \,\forall\, \varphi \in \mathscr{D}(K) : \; |u(\varphi)| \leq C\|\varphi\|_{l,K}\}$

is called order of u; $\mathscr{D}'_F(X) := \{u \in \mathscr{D}'(X); \text{ord}(u) < \infty\}$ is a subspace of $\mathscr{D}'(X)$ with $\mathscr{E}'(X) \subsetneq \mathscr{D}'_F(X)$.

Examples:

i) For $f \in L^1_{\text{loc}}(X)$

$$u_f : \mathscr{D}(X) \to \mathbb{C}, \varphi \mapsto \int_X f(x)\varphi(x)dx$$

is a well-defined linear mapping, $\forall\, K \Subset X, \varphi \in \mathscr{D}(K)$:

$$|\langle u_f, \varphi\rangle| \leq \int_K |f(x)\varphi(x)|dx \leq \int_K |f(x)|dx\, \|\varphi\|_{0,K},$$

$\Rightarrow u_f \in \mathscr{D}'(X)$, $\text{ord}(u_f) = 0$.
Recall the "Fundamental lemma of calculus of variations":

$$\forall\, f \in L^1_{\text{loc}}(X) :\ (\forall\, \varphi \in \mathscr{D}(X) :\ \int_X f(x)\varphi(x)dx = 0 \Rightarrow f = 0)$$

$\Rightarrow$ the linear mapping $L^1_{\text{loc}}(X) \to \mathscr{D}'(X), f \mapsto u_f$ is one-to-one
$\Rightarrow$ we can/will write f instead of the distribution u_f, i.e.
$\langle f, \varphi\rangle = \int_X f(x)\varphi(x)\, dx$

Examples continued:

ii) For every regular, resp. complex, measure μ on the Borel-σ-algebra over X

$$u_\mu : \mathscr{D}(X) \to \mathbb{C}, \varphi \mapsto \int_X \varphi(x) d\mu(x)$$

is a well-defined linear mapping, $\forall\, K \Subset X, \varphi \in \mathscr{D}(K)$:

$$|\langle u_\mu, \varphi\rangle| \leq |\mu|(K)\|\varphi\|_{0,K}$$

$\Rightarrow u_\mu \in \mathscr{D}'(X)$, $\operatorname{ord}(u_\mu) = 0$.
By the Riesz-Markov Theorem, $\mu \mapsto u_\mu$ is one-to-one, so we write μ instead of u_μ.

Concrete example: $\mu = \delta_x, x \in X$

Examples continued:

iii) σ surface measure on S^{d-1}, $f \in L^1(\sigma)$ with $\int_{S^{d-1}} f(\omega)\, d\sigma(\omega) = 0$. For $\varphi \in \mathscr{D}(\mathbb{R}^d)$ we have:

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \int_{\varepsilon \le |x|} \frac{\varphi(x)}{|x|^d} f(\frac{x}{|x|}) dx &= \lim_{\varepsilon \downarrow 0} \int_{\varepsilon \le |x|} \frac{\varphi(x) - \varphi(0)}{|x|^d} f(\frac{x}{|x|}) dx \\ &= \int \frac{\varphi(x) - \varphi(0)}{|x|^d} f(\frac{x}{|x|}) dx, \end{aligned}$$

where the last integral exists due to $|\varphi(x) - \varphi(0)| \le \|\nabla \varphi\|_\infty |x|$ (polar coordinates, Lebesgue's Theorem, ...)

By the same argument: $\forall\, \varphi \in \mathscr{D}(K)$ where $K \subseteq B[0, R]$:

$$|\lim_{\varepsilon \downarrow 0} \int_{\varepsilon \le |x|} \frac{\varphi(x)}{|x|^d} f(\frac{x}{|x|}) dx| \le R \int_{S^{d-1}} |f(\omega)| d\sigma(\omega) \|\varphi\|_{1,K}$$

$\Rightarrow \langle vp\big(|x|^{-d} f(\frac{x}{|x|})\big), \varphi\rangle := \lim_{\varepsilon \downarrow 0} \int_{\varepsilon \le |x|} \frac{\varphi(x)}{|x|^d} f(\frac{x}{|x|}) dx$ defines a distribution on $\mathbb{R}^d$ of order 1; these are kernels of classical singular integral operators, e.g. Hilbert transform on $\mathbb{R}$ ($f(\omega) = \text{sign}(\omega)$), Riesz operators ($f(\omega) = \omega_j, 1, \ldots, d$).

$X \subseteq \mathbb{R}^d$ open, $M \subseteq X \Rightarrow \mathscr{D}(M) \subseteq \mathscr{D}(X)$ subspace

For $u \in \mathscr{D}'(X)$ we set $u_{|M} := u_{|\mathscr{D}(M)}$ the restriction of u to M

$u \in \mathscr{D}'(X)$ vanishes in M $:\Leftrightarrow u_{|M} = 0$, i.e. $\forall\, \varphi \in \mathscr{D}(M) : \langle u, \varphi \rangle = 0$

$$\operatorname{supp} u := \{x \in X;\ \nexists\, V \subseteq X \text{ open},\ x \in V : u_{|V} = 0\}$$

is called support of u. For $f \in C(X)$ it holds

$$\operatorname{supp} u_f = \overline{\{x \in X;\ f(x) \neq 0\}}^X$$

For $u \in \mathscr{D}'(X)$ we have

- $\operatorname{supp} u$ is a closed subset of X (by definition)
- $X \backslash \operatorname{supp} u$ is the largest open subset of X where u vanishes, i.e.

$$\forall\, \varphi \in \mathscr{D}(X) : (\operatorname{supp} \varphi \cap \operatorname{supp} u = \emptyset \Rightarrow \langle u, \varphi \rangle = 0)$$

2.2 Theorem

For $X \subseteq \mathbb{R}^d$ open we have $\mathscr{E}'(X) = \{u \in \mathscr{D}'(\mathbb{R}^d);\ \operatorname{supp} u \subseteq X \text{ compact}\}$.

For $h \in \mathscr{E}(X)$ and $1 \leq j \leq d$ the operators

$$m_h : \mathscr{D}(X) \to \mathscr{D}(X), \varphi \mapsto h\varphi \text{ and } \partial_j : \mathscr{D}(X) \to \mathscr{D}(X), \varphi \mapsto \partial_j \varphi$$

are well-defined, linear, and continuous.

For arbitrary $\varphi \in \mathscr{D}(X)$ we have

$$\forall\, f \in L^1_{\mathsf{loc}}(X) :\ \langle hf, \varphi \rangle = \int_X h(x)f(x)\varphi(x)dx = \langle f, m_h(\varphi) \rangle$$

and if $f \in C^1(X)(\subseteq L^1_{\mathsf{loc}}(X))$ integration by parts gives

$$\langle \partial_j f, \varphi \rangle = \int_X \partial_j f(x)\varphi(x)dx = -\int_X f(x)\partial_j\varphi(x)dx = -\langle f, \partial_j \varphi \rangle.$$

For arbitrary $u \in \mathscr{D}'(X)$ we <u>define</u> $\langle hu, \varphi \rangle := \langle u, m_h(\varphi) \rangle$ and $\langle \partial_j u, \varphi \rangle := -\langle u, \partial_j \varphi \rangle \ \Rightarrow\ hu, \partial_j u \in \mathscr{D}'(X)$ and $u \mapsto hu, u \mapsto \partial_j u$ are linear.

For $P \in \mathbb{C}[X_1, \ldots, X_d]$ it follows $P(D)u \in \mathscr{D}'(X)$ and

$$\langle P(D)u, \varphi \rangle = \langle u, \check{P}(D)\varphi \rangle, \text{ where } \check{P}(\xi) = P(-\xi).$$

2.3 Proposition

For $h \in \mathscr{E}(X)$ and $P \in \mathbb{C}[X_1, \ldots, X_d]$ the following hold.

i) $\forall\, u \in \mathscr{D}'(X)$: supp $(hu) \subseteq$ supp $h \cap$ supp u and ord $(hu) \leq$ ord u.

ii) $\forall\, u \in \mathscr{D}'(X)$: supp $P(D)u \subseteq$ supp u and if P of degree m then ord $\big(P(D)u\big) \leq$ ord $u + m$.

iii) $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X), u \mapsto P(D)$ is a linear mapping with $P(D)(\mathscr{E}'(X)) \subseteq \mathscr{E}'(X)$ and $P(D)(\mathscr{D}'_F(X)) \subseteq \mathscr{D}'_F(X)$.

Examples:

i) For the Heaviside function $Y = \mathbb{1}_{(0,\infty)}$ we have for $\varphi \in \mathscr{D}(\mathbb{R})$

$$\langle Y', \varphi \rangle = -\langle Y, \varphi' \rangle = -\int_0^\infty \varphi'(x)dx = \varphi(0) = \langle \delta_0, \varphi \rangle$$

ii) $X \subset \mathbb{R}^d$ be open with C^1-boundary. For $\varphi \in \mathscr{D}'(\mathbb{R}^d)$:

$$\langle \partial_j \mathbb{1}_X, \varphi \rangle = -\int_X \partial_j \varphi(x)dx = -\int_{\partial X} \nu_j(\omega)\varphi(\omega)d\sigma(\omega) = \langle -\nu_j \sigma, \varphi \rangle,$$

with $\nu(\omega) = (\nu_1(\omega), \ldots, \nu_d(\omega))$ denoting the outer unit normal in $\omega \in \partial X$ and σ the surface measure on ∂X.

For $m \in \mathbb{N}_0$ we define the local Sobolev space of order m over X as

$$H^m_{\mathrm{loc}}(X) = \{f \in L^2_{\mathrm{loc}}(X); \forall\, |\alpha| \leq m : \partial^\alpha f \in L^2_{\mathrm{loc}}(X)\}$$

which is a subspace of $\mathscr{D}'_F(X)$.

$\rightsquigarrow$ differential equations for distributions or in any subspace E of $\mathscr{D}'(X)$ like, e.g. $\mathscr{E}(X), H^m_{\text{loc}}(X)$, $L^1_{\text{loc}}(X), \mathscr{D}'_F(X)$:
given arbitrary $f \in E$ is there $u \in \mathscr{D}'(X)$ (resp. $u \in E$) with $P(D)u = f$, i.e.

$$\forall\, \varphi \in \mathscr{D}(X) : \langle f, \varphi\rangle = \langle P(D)u, \varphi\rangle \big(= \langle u, P(-D)\varphi\rangle\big)?$$

2.4 Theorem (Malgrange, 1955, see ALPDO II, Section 10.6)

For open $X \subseteq \mathbb{R}^d$ and $P \in \mathbb{C}[X_1, \ldots, X_d]$ tfae:

i) $P(D) : \mathscr{E}(X) \to \mathscr{E}(X)$ is surjective.

ii) $\forall\, f \in \mathscr{E}(X)\, \exists\, u \in \mathscr{D}'(X) : P(D)u = f$.

iii) $P(D) : \mathscr{D}'_F(X) \to \mathscr{D}'_F(X)$ is surjective.

iv) $\forall\, f \in H^m_{\text{loc}}(X)\, \exists\, u \in H^m_{\text{loc}}(X) : P(D)u = f$.

v) $\forall\, u \in \mathscr{E}'(X) : \text{dist}(\text{supp}\, P(-D)u, X^c) = \text{dist}(\text{supp}\, u, X^c)$.

In v) "$\forall\, u \in \mathscr{E}'(X)$" can be replaced by "$\forall\, u \in \mathscr{D}(X)$".

Given $P \in \mathbb{C}[X_1, \ldots, X_d] \backslash \{0\}$. X is called P-convex for supports iff

$$\forall\, u \in \mathscr{E}'(X) : \operatorname{dist}(\operatorname{supp} P(-D)u, X^c) = \operatorname{dist}(\operatorname{supp} u, X^c).$$

Recall: $\operatorname{supp} P(-D)u \subseteq \operatorname{supp} u$, thus we always have

$$\forall\, u \in \mathscr{E}'(X) : \operatorname{dist}(\operatorname{supp} P(-D)u, X^c) \geq \operatorname{dist}(\operatorname{supp} u, X^c).$$

Consequence of "Theorem of Supports":

$$\forall\, u \in \mathscr{E}'(\mathbb{R}^d) : \operatorname{conv}(\operatorname{supp} u) = \operatorname{conv}(\operatorname{supp} P(-D)u),$$

which implies: every convex open set $X \subseteq \mathbb{R}^d$ is P-convex for supports.

If $(X_\iota)_{\iota \in I}$ is a family of open sets which are P-convex for supports then $\operatorname{int}\big(\bigcap_{\iota \in I} X_\iota\big)$ is P-convex for supports, too.

Geometrical conditions for/characterisation of P-convexity for supports?

Problem: <u>not</u> a local property!

Every open $X \subseteq \mathbb{R}^d$ is P-convex for supports iff P is elliptic, i.e. if $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$ then

$$\forall\, \xi \in \mathbb{R}^d \backslash \{0\}; 0 \neq P_m(\xi) := \sum_{|\alpha|=m} a_\alpha \xi^\alpha \,(\text{principal part of } P)$$

If P acts along a subspace of $\mathbb{R}^d$ and is elliptic there, then P-convexity for supports is completely characterized (Nakane, 1979).

For polynomials with principal part $P_2(\xi) = \xi_d^2 - \sum_{j=1}^{d-1} \xi_j^2$ P-convexity for supports is completely characterized (Persson, 1981).

For P of real principal type there are characterizations if

- X is bounded and ∂X is analytic (Tintarev, 1988)
- $X \subseteq \mathbb{R}^3$ (Tintarev, 1992)

For $d = 2$ P-convexity for supports is completely characterized (Hörmander, 1971).

When is $P(D)(\mathscr{D}'(X)) = \mathscr{D}'(X)$? Unfortunately, P-convexity for supports of X is not enough!

Idea (Hörmander): Because $P(D)(\mathscr{E}(X)) \subseteq \mathscr{E}(X)$, iff

- $\mathscr{E}(X) \subseteq P(D)(\mathscr{D}'(X))$ ($\Leftrightarrow$ X P-convex for supports)
- $P(D)$ surjective on $\mathscr{D}'(X)/\mathscr{E}(X)$

For open $V \subseteq X \subseteq \mathbb{R}^d$ and $u \in \mathscr{D}'(X)$, we say that u is smooth in V :$\Leftrightarrow$ $u_{|V} \in \mathscr{E}(V)$, i.e.

$$\exists\, f \in \mathscr{E}(V)\,\forall\,\varphi \in \mathscr{D}(V) : \langle u, \varphi\rangle = \int_V f(x)\varphi(x)dx.$$

$$\text{sing supp}\, u := \{x \in X;\, \nexists\, V \subseteq X \text{ open}, x \in V : u \text{ smooth in } V\}$$

is called singular support of u.

For $u \in \mathscr{D}'(X), h \in \mathscr{E}(X)$, and $P \neq 0$ we have

- $\text{sing supp}\, u$ is a closed subset of X (by definition)
- $X \backslash \text{sing supp}\, u$ is the largest open subset of X where u is smooth
- $\text{sing supp}\, u \subseteq \text{supp}\, u$ and $\text{sing supp}\,(hu) \subseteq \text{supp}\, h \cap \text{sing supp}\, u$
- $\text{sing supp}\, P(D)u \subseteq \text{sing supp}\, u$

2.5 Theorem (Hörmander, 1962, see ALPDO Section 10.7)

For open $X \subseteq \mathbb{R}^d$ we have $\mathscr{D}'(X)/\mathscr{E}(X) = P(D)(\mathscr{D}'(X)/\mathscr{E}(X))$ iff X *P-convex for singular supports*, i.e.

$$\forall\, u \in \mathscr{E}'(X) : \operatorname{dist}(\operatorname{sing\,supp} P(-D)u, X^c) = \operatorname{dist}(\operatorname{sing\,supp} u, X^c).$$

Because $\operatorname{sing\,supp} P(-D)u \subseteq \operatorname{sing\,supp} u$ we always have $\operatorname{dist}(\operatorname{sing\,supp} P(-D)u, X^c) \geq \operatorname{dist}(\operatorname{sing\,supp} u, X^c)$.

Consequence of "Theorem of Singular Supports":

$$\forall\, u \in \mathscr{E}'(\mathbb{R}^d), P \neq 0 : \operatorname{conv}(\operatorname{sing\,supp} u) = \operatorname{conv}(\operatorname{sing\,supp} P(-D)u),$$

which implies: every convex open set $X \subseteq \mathbb{R}^d$ is P-convex for singular supports.

If $(X_\iota)_{\iota\in I}$ is a family of open sets which are P-convex for singular supports then $\operatorname{int}\big(\bigcap_{\iota\in I} X_\iota\big)$ is P-convex for singular supports, too.

X *strongly P-convex* $:\Leftrightarrow$ X P-convex for supports and singular supports

Geometric conditions for/characterisation of P-convexity for singular supports?

Problem: not a local property!

Every open $X \subseteq \mathbb{R}^d$ is P-convex for singular supports iff P is hypoelliptic, i.e.

$$\forall\, X \subseteq \mathbb{R}^d \text{ open}, u \in \mathscr{D}'(X) :\ \text{sing supp}\, P(D)u = \text{sing supp}\, u$$

(e.g. elliptic and parabolic operators are hypoelliptic)
Algebraic characterisation of hypoellipticity of P (Hörmander, 1955):

$$\forall\, \alpha \neq 0 : \lim_{\xi \in \mathbb{R}^d, |\xi| \to \infty} \frac{P^{(\alpha)}(\xi)}{P(\xi)} = 0,$$

thus P hypoelliptic $\Leftrightarrow$ $\check{P}$ hypoelliptic

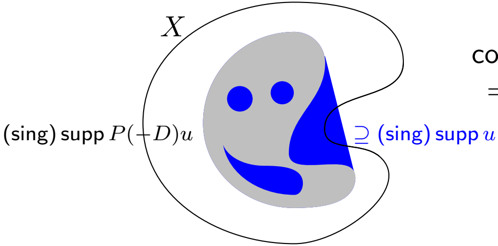
For $d = 2$ P-convexity for singular supports is completely characterized (K., '10).

3. Conditions for P-convexity for (singular) supports

X P-convex for (singular) supports $\Leftrightarrow$

$$\forall\, u \in \mathscr{E}'(X): \ \mathrm{dist}((\mathrm{sing})\,\mathrm{supp}\, P(-D)u, X^c) = \mathrm{dist}((\mathrm{sing})\,\mathrm{supp}\, u, X^c)$$

What can we say about the location of (sing) supp u if we know (sing) supp $P(-D)u$?

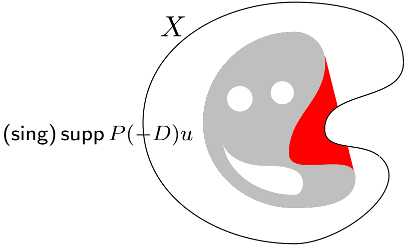


$$\mathrm{conv}\big((\mathrm{sing})\,\mathrm{supp}\, P(-D)u\big) = \mathrm{conv}\big((\mathrm{sing})\,\mathrm{supp}\, u\big)$$

X P-convex for (singular) supports $\Leftrightarrow$

$$\forall\, u \in \mathscr{E}'(X):\ \text{dist}((\text{sing})\,\text{supp}\, P(-D)u, X^c) = \text{dist}((\text{sing})\,\text{supp}\, u, X^c)$$

What can we say about the location of (sing) supp u if we know (sing) supp $P(-D)u$?

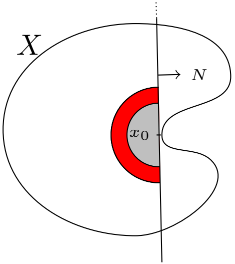


$$\text{conv}\big((\text{sing})\,\text{supp}\, P(-D)u\big) = \text{conv}\big((\text{sing})\,\text{supp}\, u\big)$$

A hyperplane $H = \{x \in \mathbb{R}^d; \langle x, N\rangle = \gamma\}$ ($N \in S^{d-1}, \gamma \in \mathbb{R}$) is called characteristic for P if $P_m(N) = 0$ (P_m principal part of P).

3.1 Theorem (Hörmander, 1955, see ALPDO I, Theorem 8.6.7)

Let $H = \{x \in \mathbb{R}^d; \langle N, x\rangle = \gamma\}$ be a characteristic hyperplane for P. Then there is $f \in \mathscr{E}(\mathbb{R}^d)$ with supp $f = \{x \in \mathbb{R}^d; \langle x, N\rangle \leq \gamma\}$ and $P(-D)f = 0$.



f as above for $\gamma = \langle N, x_0\rangle, \chi \in \mathscr{D}(\mathbb{R}^d)$ with supp $\chi = B(x_0, 2\varepsilon), \chi = 1$ in $B(x_0, \varepsilon), u := \chi f$

supp $u = B(x_0, 2\varepsilon) \cap \{x; \langle x, N\rangle \leq \gamma\}$

supp $P(-D)u \subseteq (\text{supp}\, u) \backslash B(x_0, \varepsilon)$

$\Rightarrow$ X not P-convex for supports

A hyperplane $H=\{x\in\mathbb{R}^d;\langle x,N\rangle=\gamma\}$ $(N\in S^{d-1},\gamma\in\mathbb{R})$ is called characteristic for P if $P_m(N)=0$ (P_m principal part of P).

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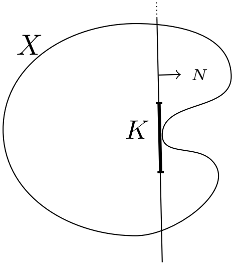


$g:X\to\mathbb{R}$ satisfies the minimum principle in a closed subset C of $\mathbb{R}^d$ if for every compact set $K\subseteq C\cap X$ we have $\inf_{x\in K}g(x)=\inf_{\partial_C K}g(x)$. We set $d_X:X\to\mathbb{R},x\mapsto\text{dist}(x,X^c)$, the boundary distance of X.

3.2 Corollary (Hörmander, 1971, see ALPDO II, Theorem 10.8.1)

If X is P-convex for supports then d_X satisfies the minimum principle in every characteristic hyperplane for P.

For $d = 2$ this necessary condition is also sufficient:

3.3 Theorem (Hörmander, 1971, see ALPDO II, Theorem 10.8.3)

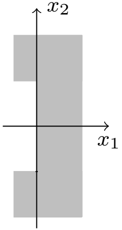
Let $X \subseteq \mathbb{R}^2$ be open and connected, $P \in \mathbb{C}[X_1, X_2]$. Tfae:

i) X is P-convex for supports.

ii) d_X satisfies the minimum principle in every characteristic hyperplane for P.

$P_1(\xi_1, \xi_2) = i\xi_1 \Rightarrow P_1(D) = \partial_1$
characteristic hyperplanes are parallels to x_1-axis

$P_2(\xi_1, \xi_2) = i\xi_2 \Rightarrow P_2(D) = \partial_2$
characteristic hyperplanes are parallels to x_2-axis



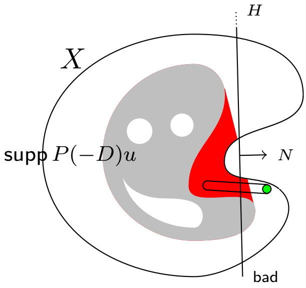
We now come to sufficient conditions for P-convexity for supports for arbitrary d. A starting point is a unique continuation result due to Hörmander:

3.4 Theorem (Hörmander, 1955, see ALPDO I, Theorem 8.6.8)

Let $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ be open and convex. Tfae:

i) $\forall\, v \in \mathscr{D}'(X_2), P(-D)v = 0 : \big(v_{|X_1} = 0 \Rightarrow v = 0\big)$

ii) Every characteristic hyperplane for P which intersects X_2 already intersects X_1.



$v := u_{|X_2}$ satisfies $P(-D)v = 0$ and $v_{|X_1} = 0$

$H = \{x \in \mathbb{R}^d; \langle x, N\rangle = \alpha\}$ with $P_m(N) = 0$

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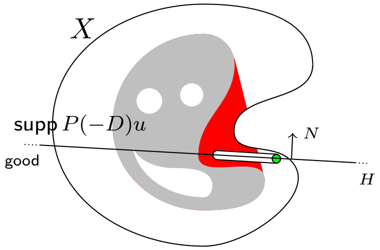


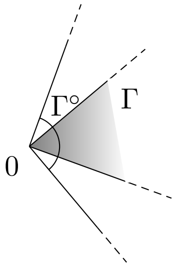
$v := u_{|X_2}$ satisfies $P(-D)v = 0$ and $v_{|X_1} = 0$

$H = \{x \in \mathbb{R}^d; \langle x, N\rangle = \alpha\}$ with $P_m(N) = 0$

Let $\emptyset \neq \Gamma \subset \mathbb{R}^d$ be an open convex cone and

$$\Gamma^\circ := \{\xi \in \mathbb{R}^d; \forall\, x \in \Gamma : \langle x, \xi \rangle \geq 0\}$$

its dual cone.



Γ° is a closed, proper, convex cone

Conversely: Every closed proper convex cone C is the dual cone of a unique open convex cone

From now on always $\emptyset \neq \Gamma \neq \mathbb{R}^d \Rightarrow 0 \notin \Gamma$ and $\Gamma^\circ \notin \{\mathbb{R}^d, \{0\}\}$

3.5 Theorem (Exterior Cone Condition I - K., '12)

Let $P \in \mathbb{C}[X_1, \ldots, X_d]$ with principal part P_m.

i) X is P-convex for supports if for every $x \in \partial X$ there is an open convex cone $\Gamma \subset \mathbb{R}^d$ such that

$$(x + \Gamma^\circ) \cap X = \emptyset \text{ and } P_m(\xi) \neq 0 \, \forall \, \xi \in \Gamma.$$

ii) If Γ is an open convex cone and $X := \mathbb{R}^d \backslash \Gamma^\circ$ then X is P-convex for supports iff $P_m(\xi) \neq 0$ for every $\xi \in \Gamma$.

As another sufficient condition for P-convexity for supports we have:

3.6 Theorem (K., '14)

Let $\{0\} \neq W \subseteq \mathbb{R}^d$ be a subspace such that d_X satisfies the minimum principle in every affine subspace parallel to W.
If $\{\xi \in \mathbb{R}^d; P_m(\xi) = 0\} \subseteq W^\perp$ then X is P-convex for supports.

The above condition easily implies that for every elliptic P each open $X \subseteq \mathbb{R}^d$ is P-convex for supports (take $W = \mathbb{R}^d$).

3.7 Corollary (K., '14)

If $\{\xi \in \mathbb{R}^d; P_m(\xi) = 0\}$ is a one-dimensional subspace then X is P-convex for supports iff d_X satisfies the minimum principle in in every characteristic hyperplane for P.

Applicable to the free Schrödinger operator $-i\partial_t - \Delta_x$ and parabolic operators, i.e. $P(\xi) = Q(\xi_1, \ldots, \xi_{d-1}) + i\xi_d$ with elliptic $Q \in \mathbb{C}[X_1, \ldots, X_{d-1}]$, e.g. $\partial_t - \Delta_x$.

We now consider P-convexity for singular supports of X, i.e. conditions for

$$\forall\, \mathscr{E}'(X): \ \mathrm{dist}(\mathrm{sing\,supp}\, P(-D)u, X^c) = \mathrm{dist}(\mathrm{sing\,supp}\, u, X^c)$$

("$\geq$" always holds).

Some preparations have to be made: for $\zeta \in \mathbb{C}^d$ we define

$$e_\zeta : \mathbb{R}^d \to \mathbb{C}, x \mapsto e^{-i\langle x,\zeta\rangle} \text{ (where } \langle x,\zeta\rangle = \sum_{j=1}^{d} x_j\zeta_j)$$

and for $u \in \mathscr{E}'(\mathbb{R}^d)$

$$\mathscr{F}(u) := \hat{u} : \mathbb{C}^d \to \mathbb{C}, \zeta \mapsto u(e_\zeta)$$

the Fourier-Laplace transform of u which is a entire analytic function.

3.8 Theorem (Paley-Wiener-Schwartz, 1952, see ALPDO I, Theorem 7.3.1)

$\hat{u}$ is an entire analytic function for each $u \in \mathscr{E}'(\mathbb{R}^d)$.

i) If $u \in \mathscr{E}'(\mathbb{R}^d)$ satisfies $\operatorname{supp} u \subseteq B[0,R]$ then

$$\exists\, N \in \mathbb{N}_0, C > 0 \,\forall\, \zeta \in \mathbb{C}^d : \, |\hat{u}(\zeta)| \leq C(1+|\zeta|)^N e^{R|\operatorname{Im}\zeta|}$$

(one can choose $N = \operatorname{ord}(u)$). Conversely, every entire analytic function satisfying an estimate like the above is the Fourier-Laplace transform of a distribution with support in $B[0,R]$.

ii) If $u \in \mathscr{D}(\mathbb{R}^d)$ satisfies $\operatorname{supp} u \subseteq B[0,R]$ then

$$\forall\, N \in \mathbb{N}_0 \,\exists\, C > 0 \,\forall\, \zeta \in \mathbb{C}^d : \, |\hat{u}(\zeta)| \leq C(1+|\zeta|)^{-N} e^{R|\operatorname{Im}\zeta|}.$$

Conversely, every entire analytic function satisfying estimates like the above is the Fourier-Laplace transform of a test function with support in $B[0,R]$.

Fix $u \in \mathscr{E}'(X)(\subseteq \mathscr{E}'(\mathbb{R}^d))$. For every $\varphi \in \mathscr{D}(X \backslash \operatorname{sing\,supp} P(-D)u)$, $\eta \in \mathbb{R}^d$:

$$\langle e_\eta P(-D)u, \varphi \rangle \to_{|\eta| \to \infty} 0.$$

Thus, in $\mathscr{D}'(X \backslash \operatorname{sing\,supp} P(-D)u)$,

$$0 = \lim_{|\eta| \to \infty} \frac{\check{P}_\eta(D)}{\tilde{\check{P}}(\eta, 1)} \big(\tilde{\check{P}}(\eta, 1) e_\eta u\big).$$

$\forall\, (\eta_k)_{k \in \mathbb{N}}, \lim_{k \to \infty} |\eta_k| = \infty \,\exists\, (\eta_{k_l})_{l \in \mathbb{N}} : \exists \lim_{l \to \infty} \tilde{\check{P}}(\eta_{k_l}, 1) e_{\eta_{k_l}} u$ in $\mathscr{D}'(\mathbb{R}^d)$ (limit $= 0$ in $\mathbb{R}^d \backslash \operatorname{sing\,supp} u$)

$\forall\, (\eta_k)_{k \in \mathbb{N}}, \lim_{k \to \infty} |\eta_k| = \infty \,\exists\, (\eta_{k_l})_{l \in \mathbb{N}} : \exists \lim_{l \to \infty} \frac{\check{P}_{\eta_{k_l}}(\xi)}{\tilde{\check{P}}(\eta_{k_l}, 1)} =: Q(\xi)$ in $\mathbb{C}[X_1, \ldots, X_d]$, Q invariant under some non-trivial subspace $V \subseteq \mathbb{R}^d$, i.e.

$$\forall\, x \in V, \xi \in \mathbb{R}^d : Q(\xi + x) = Q(\xi)$$

so - if Q does not have a constant term - every $w \in \mathscr{E}'(\mathbb{R}^d)$ depending only on variables from $V^\perp$ satisfies $Q(D)w = 0$

$\rightsquigarrow$ plausibility/conjecture: to every such $V \,\exists\, w \in \mathscr{E}'(\mathbb{R}^d)$: $P(-D)w \in \mathscr{E}(\mathbb{R}^d)$ and $\operatorname{sing\,supp} w = V^\perp \cap \operatorname{supp} w$

How to recognize these V?

Q non-constant $\Rightarrow \infty = \lim_{t\to\infty} \tilde{Q}(0,t) (= \lim_{t\to\infty} \sup_{|\xi|\le t} |Q(\xi)|)$ while

$\tilde{Q}_V(0,t) := \sup_{x\in V, |x|\le t} |Q(x+0)| = |Q(0)|$ by definition of V

For suitable $(\eta_n)_{n\in\mathbb{N}}$ tending to infinity:

$$\begin{aligned} 0 &= \inf_{t\ge 1} \frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)} = \inf_{t\ge 1} \lim_{n\to\infty} \frac{\tilde{\check{\tilde{P}}}_V(\eta_n,t)}{\tilde{\check{P}}(\eta_n,t)} \\ &\ge \inf_{t\ge 1} \liminf_{\eta\to\infty} \frac{\tilde{\check{\tilde{P}}}_V(\eta,t)}{\tilde{\check{P}}(\eta,t)}, \end{aligned}$$

where $\tilde{\check{\tilde{P}}}_V(\eta,t) = \sup_{\xi\in V, |\xi|\le t} |\check{P}(\xi+\eta)|$

Hörmander: For $V \subseteq \mathbb{R}^d$ subspace define

$$\sigma_P(V) = \inf_{t\ge 1} \liminf_{\eta\to\infty} \frac{\tilde{\check{\tilde{P}}}_V(\eta,t)}{\tilde{\check{P}}(\eta,t)}.$$

Abbreviation: $\forall\, y \in \mathbb{R}^d: \; \sigma_P(y) = \sigma_P(\text{span}\{y\})$

3.9 Theorem (Hörmander, 1972, see ALPDO II, Theorem 11.3.1)

Let $V \subseteq \mathbb{R}^d$ be a subspace with $\sigma_P(V) = 0$. Then there is $u \in \mathscr{D}'(\mathbb{R}^d)$ with $P(-D)u = 0$ and $\operatorname{sing\,supp} u = V^\perp$.

Like Theorem 3.1 is used to prove Corollary 3.2 the above theorem gives a necessary condition for P-convexity for singular supports:

3.10 Corollary (Hörmander, 1972, see ALPDO II, Corollary 11.3.2)

Let $V \subseteq \mathbb{R}^d$ be a subspace with $\sigma_P(V) = 0$. If X is P-convex for singular supports then d_X satisfies the minimum principle in every affine subspace parallel to $V^\perp$.

This necessary condition is also sufficient for $d = 2$:

3.11 Theorem (K., '11)

Let $X \subseteq \mathbb{R}^2$ be open and connected, $P \in \mathbb{C}[X_1, X_2]$. Tfae:

i) X is P-convex for singular supports.

ii) d_X satisfies the minimum principle in every hyperplane $H = \{x \in \mathbb{R}^2; \langle x, N\rangle = \gamma\}$ with $\sigma_P(N) = 0$.

σ_P can also be used to give sufficient conditions for P-convexity for singular supports for arbitrary d.

3.12 Theorem (Exterior Cone Condition II - K., '12)

Let $P \in \mathbb{C}[X_1, \dots, X_d]$.

i) X is P-convex for singular supports if for every $x \in \partial X$ there is an open convex cone $\Gamma \subset \mathbb{R}^d$ such that

$$(x + \Gamma^\circ) \cap X = \emptyset \text{ and } \sigma_P(\xi) \neq 0 \, \forall \, \xi \in \Gamma.$$

ii) If Γ is an open convex cone and $X := \mathbb{R}^d \backslash \Gamma^\circ$ then X is P-convex for singular supports iff $\sigma_P(\xi) \neq 0$ for every $\xi \in \Gamma$.

4. Interlude: Some Functional Analysis

General references: IFA and AFO

E be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

a) A family of seminorms $\mathscr{P}$ is called directed if

$$\forall\, p, q \in \mathscr{P}\, \exists\, r \in \mathscr{P} :\ p \leq r \text{ and } q \leq r.$$

b) A locally convex space (lcs for short) is a pair $(E, \mathscr{P})$ consisting of a vector space E over $\mathbb{K}$ and a directed family of seminorms $\mathscr{P}$.

c) A lcs $(E, \mathscr{P})$ is called separated if

$$\forall\, x \in E \backslash \{0\}\, \exists\, p \in \mathscr{P} : p(x) > 0.$$

$(E,\mathscr{P})$ lcs, $U\subseteq E$ is called open (in $(E,\mathscr{P})$) $:\Leftrightarrow$

$$\forall\, x\in U\,\exists\, p\in\mathscr{P},\varepsilon>0:\; B_p(x,\varepsilon)\subseteq U,$$

where $B_p(x,\varepsilon):=\{y\in E;\, p(x-y)<\varepsilon\}$

Since $\mathscr{P}$ is a directed family of seminorms

$$\{U\subseteq E;\, U \text{ open in } (E,\mathscr{P})\}$$

is stable under finite intersections (and obviously under arbitrary unions) and thus a topology on E ($B_p(x,\varepsilon)$ convex $\rightsquigarrow$ "locally convex") which is Hausdorff iff $(E,\mathscr{P})$ is separated,

$$E\times E\to E,(x,y)\mapsto x+y \text{ and } \mathbb{K}\times E\to E,(\lambda,x)\mapsto\lambda x$$

are both continuous

Examples:

a) Every normed space is a separated lcs.

b) For $X \subseteq \mathbb{R}^d$ open $\mathscr{P}_{\infty,c} := \{\|\cdot\|_{l,K};\, l \in \mathbb{N}_0, K \Subset X\}$ is a directed family of seminorms on $C^\infty(X)$. (Recall that

$$\|f\|_{l,K} = \sup_{\alpha \in \mathbb{N}_0^d, |\alpha| \le l} \sup_{x \in K} |\partial^\alpha f(x)|).$$

This (separated) lcs is denoted by $\mathscr{E}(X)$.

c) $X \subseteq \mathbb{R}^d$ open, $K \Subset X$, $f \in C(X)$ we set $\|f\|_K := \sup_{x \in \mathbb{K}} |f(x)|$. Then $\mathscr{P}_c := \{\|\cdot\|_K;\, K \Subset X\}$ is a directed family of seminorms making $C(X)$ a (separated) lcs.

$(E, \mathscr{P})$ be a lcs $\mathscr{P}_0 \subseteq \mathscr{P}$ is called fundamental system of seminorms iff

$$\forall\, q \in \mathscr{P}\, \exists\, p \in \mathscr{P}_0, C > 0\, \forall\, x \in E:\ q(x) \leq Cp(x)$$

$(E, \mathscr{P})$ is called Fréchet space $:\Leftrightarrow$ $(E, \mathscr{P})$ is separated, there is a countable fundamental sequence of seminorms, and $(E, \mathscr{P})$ is (sequentially) complete, i.e. every Cauchy sequence converges

Examples:

a) Every Banach space is a Fréchet space.

b) $(E, \mathscr{P})$ Fréchet space, $F \subseteq E$ closed subspace $\Rightarrow$ $(F, \mathscr{P})$ Fréchet space.

c) $(K_n)_{n \in \mathbb{N}_0}$ compact exhaustion of $X \subseteq \mathbb{R}^d$ open $\Rightarrow \{\| \cdot \|_{n,K_n}; n \in \mathbb{N}_0\}$ is a countable fundamental system of seminorms for $\mathscr{E}(X)$ and $\{\| \cdot \|_{n,K_n}; n \in \mathbb{N}_0\}$ for $(C(X), \mathscr{P}_c)$. Both lcs are Fréchet spaces.

A linear $T : E_1 \to E_2$ between lcs $(E_1, \mathscr{P}_1)$ and $(E_2, \mathscr{P}_2)$ is continuous iff

$$\forall\, q \in \mathscr{P}_2 \,\exists\, p \in \mathscr{P}_1, C > 0 \,\forall\, x \in E_1 :\; q(Tx) \leq Cp(x).$$

$L(E_1, E_2) := \{T : E_1 \to E_2;\;$ linear and continuous$\}$.

Dual space of the lcs $(E, \mathscr{P})$

$$E' := (E, \mathscr{P})' := \{u : E \to \mathbb{K};\, u \text{ linear, continuous}\}$$

$u : E \to \mathbb{K}$ linear belongs to E' iff

$$\exists\, p \in \mathscr{P}, C > 0 \,\forall\, x \in E :\; |u(x)| \leq Cp(x).$$

We want to make $(E, \mathscr{P})'$ into a lcs. $B \subseteq E$ is called bounded iff

$$\forall\, p \in \mathscr{P} : \sup_{x \in B} p(x) < \infty.$$

For bounded B, $p_B : E' \to \mathbb{R}, u \mapsto \sup_{x \in B} |u(x)|$ is a well-defined seminorm and

$$b(E', E) := \{p_B;\, B \subseteq E \text{ bounded}\}$$

is a directed family of seminorms on E'.

The lcs $(E', b(E', E))$ is called strong dual of E.

For a normed space $(E, \|\cdot\|)$ a fundamental system of seminorms for $b(E', E)$ is $\{\|\cdot\|_{\text{op}}\}$ with $\|u\|_{\text{op}} = \sup_{\|x\| \leq 1} |u(x)|$.

5. Vector valued distributions and differential operators

Although we do not give a directed family of seminorms for $\mathscr{D}(X)$ explicitly, there is a unique way to turn $\mathscr{D}(X)$ into a (reasonable) separated, complete lcs. For a lcs $(E, \mathscr{P})$ a linear $T : \mathscr{D}(X) \to E$ is continuous iff

$$(*)\,\forall\, q \in \mathscr{P}\,\forall\, K \Subset X\,\exists\, l \in \mathbb{N}_0, C > 0\,\forall\, \varphi \in \mathscr{D}(K) :\; q(T\varphi) \le C\|\varphi\|_{l,K}.$$

$Y \subseteq \mathbb{R}^n$ open, $T : Y \to \mathscr{D}'(X), y \mapsto T_y$ continuous $:\Leftrightarrow$

$$\forall\, \varphi \in \mathscr{D}(X) :\; \lambda(T)(\varphi) : Y \to \mathbb{C}, y \mapsto \langle T_y, \varphi\rangle$$

is continuous. With $(*)$ and 2.1 b): $\lambda(T) \in L\big(\mathscr{D}(X), (C(Y), \mathscr{P}_c)\big)$.

λ is an isomorphism between $\{T : Y \to \mathscr{D}'(X); T \text{ continuous}\}$ and $L\big(\mathscr{D}(X), (C(Y), \mathscr{P}_c)\big)$.

Moreover, for continuous $T : Y \to \mathscr{D}'(X)$ we also have that

$$P(D)T : Y \to \mathscr{D}'(X), y \mapsto P(D)T_y$$

is continuous with $\lambda(P(D)T)(\varphi) = \lambda(T)(P(-D)\varphi)$.

For general lcs E instead of $C(Y)$ we define $\mathscr{D}'(X,E) := L(\mathscr{D}(X),E)$ E-valued distributions over $X \subseteq \mathbb{R}^d$ and

$$P(D) : \mathscr{D}'(X,E) \to \mathscr{D}'(X,E), \big(P(D)T\big)(\varphi) := T(P(-D)\varphi).$$

For E a space of functions the problem of surjectivity of $P(D)$ on $\mathscr{D}'(X,E)$ translates to the corresponding problem of parameter dependence: for each f_y in $\mathscr{D}'(X)$ depending on the parameter y as the functions in E, is there a solution u_y of $P(D)u_y = f_y$ depending in the same way on y (e.g. $E \in \{C(Y), C^\infty(Y), \ldots\}$)?

We also consider the question of surjectivity of $P(D)$ on $C^\infty(X,E)$.

We restrict ourselves to E being a Fréchet space or the strong dual of a Fréchet space.

A Fréchet space E has property (DN) ($E \in (DN)$) iff there is a fundamental system of seminorms $\{p_k;\, k \in \mathbb{N}\}$ with

$$\forall\, k \geq 2\, \forall\, x \in E:\; p_k(x)^2 \leq p_{k-1}(x) p_{k+1}(x).$$

p_1 is then a norm on E (so-called dominating norm)

Banach spaces have (DN)

The space of rapidly decreasing sequences

$$s := \{x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}};\, \forall\, k \in \mathbb{N}:\; p_k(x)^2 := \sum_{n=1}^{\infty} |x_n|^2 n^{2k} < \infty\}$$

with the sequence of seminorms $(p_k)_{k\in\mathbb{N}}$ is a Fréchet space with (DN) (by Hölder).

Spaces linearly homeomorphic to s: $C_p^\infty(\mathbb{R}^d), H(\mathbb{C})$, $C^\infty(\overline{X})$ ($X \subseteq \mathbb{R}^d$ open, bounded, C^1-boundary), $\mathscr{D}(K)$ ($K \Subset \mathbb{R}^d$), $\mathscr{S}(\mathbb{R}^d)$

5.1 Theorem

Let $X \subseteq \mathbb{R}^d, P \in \mathbb{C}[X_1, \ldots, X_d], P^+(\xi_1, \ldots, \xi_{d+1}) := P(\xi_1, \ldots, \xi_d)$

i) (Grothendieck, 1955) X be P-convex for supports and E be a Fréchet space. Then $P(D) : C^\infty(X, E) \to C^\infty(X, E)$ is surjective.

ii) (Vogt, 1983) P be elliptic and $E = F'$ the strong dual of a Fréchet space F. Then $P(D) : C^\infty(X, E) \to C^\infty(X, E)$ is surjective iff $F \in (DN)$.

iii) (Vogt, 1983 + Bonet, Domański '06) P be hypoelliptic, X P-convex for supports, and $E = F'$ the strong dual of a Fréchet space $F \in (DN)$. Then $P(D) : C^\infty(X, E) \to C^\infty(X, E)$ is surjective if $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is surjective. This condition is also necessary for $F \cong s$.

iv) (Bonet, Domański, '06) X be strongly P-convex and $E = F'$ be the strong dual of a Fréchet space $F \cong$ closed subspace of s. Then $P(D) : \mathscr{D}'(X, E) \to \mathscr{D}'(X, E)$ is surjective if this is true for $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$.

Given $P \in \mathbb{C}[X_1, \ldots, X_d]$ and $X \subseteq \mathbb{R}^d$ open such that

$$P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$$

is surjective. When is

$$P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$$

surjective, too, where $P^+(\xi_1, \ldots, \xi_{d+1}) = P(\xi_1, \ldots, \xi_d)$?

Equivalent formulation: X strongly P-convex $\overset{?}{\Rightarrow}$ $X \times \mathbb{R}$ strongly P^+-convex

If X is convex then $X \times \mathbb{R}$ is convex, so then "yes".

If P is elliptic, then "yes" due to Vogt (see Theorem 5.1 ii), iii)).

X P-convex for supports $\Rightarrow P^+(D): C^\infty(X\times\mathbb{R})\to C^\infty(X\times\mathbb{R})$ surjective due to Grothendieck (compare Theorem 5.1 i)), i.e. $X\times\mathbb{R}$ P^+-convex for supports

Thus, the question is:
X strongly P-convex $\overset{?}{\Rightarrow}$ $X\times\mathbb{R}$ P^+-convex for singular supports

Conditions for P^+-convexity for singular supports from section 3 involve σ_{P^+}. However, σ_{P^+} is not appropriate to evaluate conditions for P^+-convexity for singular supports of $X\times\mathbb{R}$ in terms of P and X. To achieve this, we define for a subspace $V\subseteq\mathbb{R}^d$

$$\sigma_P^0(V) := \inf_{t\geq 1}\inf_{\eta\in\mathbb{R}^d}\frac{\tilde{\check{P}}_V(\eta,t)}{\tilde{\check{P}}(\eta,t)},$$

recall that $\tilde{\check{P}}_V(\eta,t)=\sup_{\xi\in V,|\xi|\leq t}|\check{P}(\xi+\eta)|$ and $\tilde{\check{P}}(\eta,t)=\tilde{\check{P}}_{\mathbb{R}^d}(\eta,t)$.
Again we abbreviate

$$\forall\, y\in\mathbb{R}^d\backslash\{0\}:\ \sigma_P^0(y):=\sigma_P^0(\text{span}\{y\}).$$

5.2 Theorem (Exterior Cone Condition III - K., '12)

If Γ is an open convex cone and $X := \mathbb{R}^d \backslash \Gamma^\circ$ then $X \times \mathbb{R}$ is P^+-convex for singular supports iff $\sigma_P^0(\xi) \neq 0$ for every $\xi \in \Gamma$.

5.3 Lemma (K., '12)

Let P have principal part P_m and let $y \in \mathbb{R}^d \backslash \{0\}$.

i) $\sigma_P^0(y) \leq \sigma_P(y)$ and $\forall\, k \in \mathbb{N}: \sigma_{P^k}^0(y) = (\sigma_P^0(y))^k$.

ii) $\sigma_P^0(y) \leq \sigma_{P_m}^0(y)$.

Let $d \geq 3, A(\xi) = \xi_1^2 - \xi_2^2 - \ldots - \xi_d^2 \Rightarrow A(e_d) \neq 0, \sigma_A(e_d) = 0$ (Here, $d \geq 3$ is needed!)

$\overset{i)}{\Rightarrow} \forall\, k \in \mathbb{N}: \sigma_{A^k}^0(e_d) = 0$

$\overset{ii)}{\Rightarrow}$ Each P with principal part $P_m = A^k$ satisfies $\sigma_P^0(e_d) = 0$ and $P_m(e_d) \neq 0$.

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Let $d \geq 3, A(\xi) = \xi_1^2 - \xi_2^2 - \ldots - \xi_d^2$
$\Rightarrow$ Each P with principal part $P_m = A^k$ satisfies $\sigma_P^0(e_d) = 0$ and $P_m(e_d) \neq 0$.
$\Rightarrow\ \exists\, \Gamma \subset \mathbb{R}^d$ open proper convex cone, $e_d \in \Gamma\, \forall\, x \in \Gamma : P_m(x) \neq 0$

$X := \mathbb{R}^d \backslash \Gamma^\circ$ is P-convex for supports (by 3.5 ii) and $X \times \mathbb{R}$ is not P^+-convex for singular supports for every such P.

With $R(\xi) = (\xi_1^2 + \ldots + \xi_d^2)^3$ set $P(\xi) := A^4(\xi) + R(\xi)$. Then P is hypoelliptic so that X is P-convex for singular support. Thus:

5.4 Theorem (K., '12)

For $d \geq 3$ there are hypoelliptic P and open $X \subseteq \mathbb{R}^d$ such that $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ is surjective but $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is not surjective. In particular, $P(D)$ is surjective on $C^\infty(X)$ but not on $C^\infty(X, \mathscr{S}'(\mathbb{R}^n))$.

$d \geq 3$ is essential here:

5.5 Theorem (K., '12)

For $P \in \mathbb{C}[X_1, X_2]$ and $X \subseteq \mathbb{R}^2$ tfae:

i) $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ is surjective.

ii) $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is surjective.

Positive results for arbitrary dimension:

5.6 Theorem (K., '14)

Let $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ be surjective. Then $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is surjective in the following cases.

i) P is parabolic, e.g. the heat operator $P(D) = \partial_t - \Delta_x$.

ii) P acts along a subspace W and is elliptic as a polynomial on W, e.g. $P(D) = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ on $\mathbb{R}^3$.

iii) P factorises into linear factors, i.e. $P(\xi) = \alpha \prod_{j=1}^{k}(\langle \xi, a_j \rangle - \beta_j)$, $\alpha, \beta_j \in \mathbb{C}, a_j \in \mathbb{C}^d$.

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