Low ranks in computational Fourier analysis

Part I: Introduction

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Outline

Part I: Introduction

Part II: Fast matrix vector multiplication

Part III: Efficient Reconstruction

Outline of part I

Fourier transforms

Fourier transform $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$

$$\hat{f}(v) := (\mathcal{F}f)(v) := \int_{\mathbb{R}^d} f(x) \mathrm{e}^{-2\pi \mathrm{i} v x} \, \mathrm{d}x$$

Inversion

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(v) \mathrm{e}^{2\pi \mathrm{i} v x} \, \mathrm{d}v$$

Plancherel

$$\|f\|_2 = \|\hat{f}\|_2$$

Fourier transforms

semidiscrete Fourier transform $\mathcal{F} : L^2([0,1)^d) \to \ell^2(\mathbb{Z}^d)$

$$c_k(f) := \int_{[0,1)^d} f(x) \mathrm{e}^{-2\pi \mathrm{i} kx} \,\mathrm{d}x$$

Inversion (Fourier series)

$$f(x) = \sum_{k \in \mathbb{Z}^d} c_k(f) \mathrm{e}^{2\pi \mathrm{i} kx}$$

Parseval

$$\|f\|_2 = \|c(f)\|_2$$

Fourier transforms

discrete Fourier transform $F : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$

$$\hat{f}_k := \sum_{j=0}^{n} f_j \mathrm{e}^{-2\pi \mathrm{i} kj/(n+1)}$$

Inversion

$$f_j = (n+1)^{-1} \sum_{k=0}^{n} \hat{f}_k \mathrm{e}^{2\pi \mathrm{i} kj/(n+1)}$$

Parseval

$$\|f\|_2 = (n+1)^{-1} \|\hat{f}\|_2$$

Fourier transforms

spatial dimension $d \geq 1$, $[n] := \{0, \ldots, n\}^d$, $N = (n+1)^d$

discrete Fourier transform $F : \mathbb{C}^N \to \mathbb{C}^N$

$$\hat{f}_k := \sum_{j \in [n]} f_j \mathrm{e}^{-2\pi \mathrm{i} kj/(n+1)}$$

Inversion

$$f_j = N^{-1} \sum_{k \in [n]} \hat{f}_k \mathrm{e}^{2\pi \mathrm{i} kj/(n+1)}$$

Parseval

$$\|f\|_2 = N^{-1} \|\hat{f}\|_2$$

Fourier transforms

Equidistant sampling and periodization

Poisson summation formula, $|f(x)|, |\hat{f}(x)| \leq (1 + |x|_2)^{-d-\varepsilon}$

$$\sum_{k \in \mathbb{Z}^d} \hat{f}(k) = \sum_{j \in \mathbb{Z}^d} f(j)$$

$$\sum_{k \in \mathbb{Z}^d} \hat{f}(k) \mathrm{e}^{2\pi \mathrm{i} k x} = \sum_{j \in \mathbb{Z}^d} f(x + j)$$

Alias lemma, $c(f) \in \ell^1(\mathbb{Z}^d)$

$$\hat{f}_0 = \sum_{j \in [n]} f(j/(n+1)) = N^d \sum_{r \in \mathbb{Z}^d} c_{r(n+1)}(f)$$

Fourier transforms

Fourier matrix, $w_{n+1} = \mathrm{e}^{\pm 2\pi \mathrm{i}/(n+1)}$

$$F = \left(w_{n+1}^{kj} \right)_{j,k \in [n]} \in \mathbb{C}^{N \times N}$$

Two objectives

1. fast algorithm to compute
$$f = F\hat{f}$$
2. fast algorithm to solve
$$F\hat{f} = f$$

Fourier transforms

fast Fourier transform, blackboard

- $d = 1$, N even, $W_N = \operatorname{diag}(w_N^j)_{j=0,\ldots,N/2-1}$

$$F_N = P_N \begin{pmatrix} F_{N/2} & 0 \\ 0 & F_{N/2} \end{pmatrix} \begin{pmatrix} I_{N/2} & I_{N/2} \\ W_N & -W_N \end{pmatrix}$$

- $F_{N_1,N_2} = F_{N_1} \otimes F_{N_2}$ for $d = 2$
- in total $\mathcal{O}(N \log N)$

Parseval, F is up to a constant unitary since

$$\sum_{j=0}^{n} w_{n+1}^{kj} = \begin{cases} n+1 & \text{if } k = 0, \\ \frac{1-w_{n+1}^{k(n+1)}}{1-w_{n+1}^{k}} = 0 & \text{otherwise.} \end{cases}$$

Nonequispaced discrete Fourier transform

nonequispaced Fourier matrix, $z_j \in \mathbb{T}^d$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$

$$A = \left(z_j^k\right)_{j=1,\dots,M,k\in[n]} \in \mathbb{C}^{M\times N}$$

Two objectives

1. fast algorithm to compute

$$f = A\hat{f}$$

2. fast algorithm to solve

$$A\hat{f} = f$$

Nonequispaced discrete Fourier transform

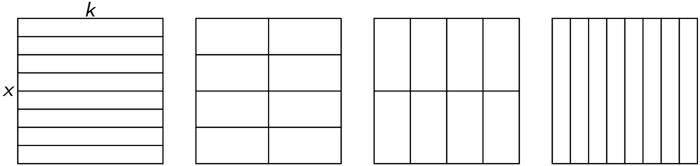
nonequispaced fast Fourier transforms, multiplication with A

- Taylor expansion, polynomial interpolation, space and frequency localized window functions (NFFT, Dutt, Rokhlin; Beylkin; Potts, Steidl, Tasche; ...; Greengard, Lee; Keiner, Knopp, Pippig, Volkmer, Potts, K., ...)

$$A \approx BFD$$

- butterfly schemes and local low rank, $p \geq \lceil \max(2e\pi, |\log_2 \varepsilon|) \rceil$ (Michielsen, Boag; Chew, Song; **Ying**; O'Neil, Woolfe, Rokhlin; Candes, Demanet; Tygert)

$$\left| \mathrm{e}^{2\pi \mathrm{i} kx} - \sum_{s=0}^{p-1} \frac{(2\pi \mathrm{i})^s}{s!} k^s x^s \right| \leq \varepsilon, \qquad |kx| \leq \frac{1}{2}$$



Nonequispaced discrete Fourier transform

inverse problem, $y \in \mathbb{C}^M$ given

- $M \geq N$, least squares

$$\|A\hat{f} - y\|_2 \to \min$$

- $M \leq N$, optimal interpolation

$$\|\hat{f}\|_2 \to \min \quad \text{s.t.} \quad A\hat{f} = y$$

- $M \leq N$, sparse interpolation (a.k.a. compressed sensing)

$$\|\hat{f}\|_0 \to \min \quad \text{s.t.} \quad A\hat{f} = y$$

Nonequispaced discrete Fourier transform

solution strategies, $y \in \mathbb{C}^M$ given

- $M \geq N$, normal equations

$$A^* A \hat{f} = A^* y$$

- $M \leq N$, normal equation of second kind

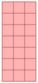
$$A A^* f = y, \quad \hat{f} = A^* f$$

- $M \leq N$, basis pursuit

$$\|\hat{f}\|_1 \to \min \quad \text{s.t.} \quad A\hat{f} = y$$

Nonequispaced discrete Fourier transform

$M \geq N$ (Marcinkiewicz-Zygmund, Duffin-Schäfer, Feichtinger-Gröchenig-Strohmer, Filbir-Mhaskar-Prestin, Narcowich-Ward, Adcock-Gataric-Hansen)

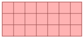


$$\max_{z \in \mathbb{T}^d} \min_{j=1,\dots,M} |z - z_j| =: \delta \leq c_d/n \Rightarrow \operatorname{rank} A = N$$

$M \leq N$ (Candes-Romberg-Tao, Rudelson-Vershynin, Rauhut)

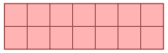
$$\text{i.u.d. } z_j \in \mathbb{T}^d, \; M \geq C\, S \log^4 N \overset{w.h.p.}{\Rightarrow} \operatorname{spark} A \geq S$$

$M \leq N$ (Wiener, Ingham, Kahane, Bazán, Potts-K., Candes, F.-Granda, Potts-Tasche, Liao, Moitra)



$$\min_{j \neq \ell} |z_j - z_\ell| =: q > 1/n \Rightarrow \operatorname{rank} A = M$$

$M \leq N$ (..., Peter-Römer-v.d.Ohe-K.)



$$z_j \in \mathbb{C}^d, \; n \geq M \Rightarrow \operatorname{rank} A = M$$

$$\text{i.u.d. } z_j \in \mathbb{C}^d, \; N \geq M \overset{a.s.}{\Rightarrow} \operatorname{rank} A = M$$

Nonequispaced discrete Fourier transform

compressed sensing vs. Prony's method

compressed sensing, $\hat{f}$ sparse, samples f taken at random on $[0,1)^d \simeq \mathbb{T}^d$

$$A\hat{f} = f$$

Prony's method (a.k.a. superresolution), f sparse (finitely supported complex measure), first moments $\hat{f}$

$$A^*(z)f = \hat{f}$$