Summer School on Applied Analysis
Chemnitz, 21-25 September 2015

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Operator Factorization and Boundary Value Problems

Part 3: Sommerfeld problems in $\mathbb{R}^n$

Abstract

Motivated by the classical Sommerfeld diffraction problem we consider interface problems in weak formulation for the n-dimensional Helmholtz equation in $\Omega = \mathbb{R}^n_+ \cup \mathbb{R}^n_-$ (due to $x_n > 0$ or $x_n < 0$, respectively), where the interface $\Gamma = \partial\Omega$ is identified with $\mathbb{R}^{n-1}$ and divided into two parts, Σ and Σ', with different transmission conditions of first and second kind. These two parts are half-spaces of $\mathbb{R}^{n-1}$ (half-planes for $n = 3$) and more general sets in the first part of the paper. The aim of this work is to construct explicitly resolvent operators acting from the interface data into the energy space $H^1(\Omega)$. The approach is based upon a factorization conception for Wiener-Hopf operators (according to the interface equations), the so-called Wiener-Hopf factorization through an intermediate space, that includes Simonenko's well-known "generalized factorization of matrix functions in L^p spaces" and avoids an interpretation of the factors as unbounded operators. In a natural way, we meet non-isotropic Sobolev spaces which reflect the wedge asymptotic of diffracted waves.

Formulation of the problems

We consider interface problems (IFPs) for the Helmholtz equation (HE) in $\Omega = \mathbb{R}^n_+ \cup \mathbb{R}^n_-$ where

$$\mathbb{R}^n_\pm = \{x = (x', x_n) = (x_1, ..., x_{n-1}, x_n) : \pm x_n > 0\}$$

and the interface $\Gamma = \mathbb{R}^{n-1} \times \{0\}$ is divided into two parts which are identified with open subsets $\Sigma \subset \mathbb{R}^{n-1}$ and $\Sigma' = \mathbb{R}^{n-1} \setminus \overline{\Sigma}$ assuming: (i) that $\operatorname{mes}(\overline{\Sigma} \cap \overline{\Sigma'}) = 0$ and (ii) that Σ has a so-called extension property for $s = \pm 1/2$. The present IFPs are briefly written as

$$\begin{aligned} &(\Delta + k^2)u = 0 \quad in \quad \Omega \\ a_0 u_0^+ + b_0 u_0^- &= g_0 \quad , \quad a_1 u_1^+ + b_1 u_1^- = g_1 \quad on \quad \Sigma \\ a_0' u_0^+ + b_0' u_0^- &= g_0' \quad , \quad a_1' u_1^+ + b_1' u_1^- = g_1' \quad on \quad \Sigma'. \end{aligned} \tag{1}$$

Herein $u_0^\pm$ denote the traces on the upper/lower bank $\Gamma^\pm$ of Γ and $u_1^\pm$ the normal derivatives. Further $k \in \mathbb{C}$, $\Im m k > 0$ and $a_0, ..., b_1' \in \mathbb{C}$. For constructive results (explicit solution) the *screen* Σ is moreover assumed to be a half-space, say $\Sigma = \mathbb{R}^{n-1}_+$ according to $x_{n-1} > 0$. Sommerfeld problems are easily identified as a special subclass of IFPs.

Spaces

We assume that the notion and basic properties of Sobolev spaces (here synonymously for fractional Sobolev spaces or Bessel potential spaces) are known using the common notation of $H^s = H^s(\mathbb{R}^n)$ and $H^s(\Omega)$ for the restricted function(al)s, $s \in \mathbb{R}$, further we write H^s_Σ for those defined on $\mathbb{R}^{n-1}$ but supported on $\overline{\Sigma}$, and $\widetilde{H}^s(\Sigma)$ for those defined on Σ, which admit an extension by zero within $H^s(\mathbb{R}^{n-1})$, see [CDS14,ENS11,Esk81,HW08] for details.

Representation formulas

The weak solutions $u \in H^1(\Omega)$ of the HE can be written in the form (see [CDS14,ENS11,MS87], e.g.):

$$u = \mathcal{K}_{D,\Omega}(u_0^+, u_0^-) = \begin{cases} \mathcal{K}_{D,\Omega^+}\ u_0^+ & \text{in} \quad \Omega^+ \\ \mathcal{K}_{D,\Omega^-}\ u_0^- & \text{in} \quad \Omega^- \end{cases} \tag{2}$$

$$\mathcal{K}_{D,\Omega^+} u_0^+(x) = \mathcal{F}^{-1}_{\xi' \mapsto x'} e^{-t(\xi')x_n} \widehat{u_0^+}(\xi') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i\xi' x' - t(\xi')x_n} \widehat{u_0^+}(\xi') d\xi'$$

$$\mathcal{K}_{D,\Omega^-} u_0^-(x) = \mathcal{F}^{-1}_{\xi' \mapsto x'} e^{t(\xi')x_n} \widehat{u_0^-}(\xi') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i\xi' x' + t(\xi')x_n} \widehat{u_0^-}(\xi') d\xi'$$

where $u_0^\pm \in H^{1/2}(\mathbb{R}^{n-1})$, writing $x' = (x_1, ..., x_{n-1})$, $\xi' = (\xi_1, ..., \xi_{n-1}) \in \mathbb{R}^{n-1}$, $d\xi' = d\xi_1 ... d\xi_{n-1}$, $\xi' x' = \xi_1 x_1 + ... + \xi_{n-1} x_{n-1}$ and $t(\xi') = (\xi_1^2 + ... + \xi_{n-1}^2 - k^2)^{1/2}$ (with vertical branch cut from k to $-k$ via ∞ such that t is continuous on $\mathbb{R}^{n-1}$ and $t(\xi') \approx |\xi'|$ as $\xi' \to \infty$ in $\mathbb{R}^{n-1}$). The space of weak solutions of the HE, i.e., of functions $u \in H^1(\Omega)$ representable in this form, is denoted by $\mathcal{H}^1(\Omega)$. It forms a Hilbert (sub-)space with the common inner product induced by $H^1(\Omega)$.

Reduction to the semi-homogeneous problem

For clarity it is convenient to confine ourselves on semi-homogeneous problems where g_0' and g_1' vanish. It has been shown in [S12] how the "equivalent reduction" to semi-homogeneous problems can be arranged in a very strong sense, in terms of operator relations. With this simplification we define the operator associated to the semi-homogeneous IFP by

$$B_0 \quad : \quad \mathcal{H}_0^1(\Omega) \to H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma) \quad , \quad u \mapsto (g_0, g_1) \tag{3}$$

that is explicitly given by formula (1) with $g_0' = 0$ and $g_1' = 0$. For short, $\mathcal{H}_0^1(\Omega)$ denotes the subspace of $\mathcal{H}^1(\Omega)$ with vanishing boundary data g_0', g_1' on Σ' (see third line of (1)). It is evident that the (semi-homogeneous) IFP is then well-posed if and only if B_0 is boundedly invertible. In this case, B_0^{-1} represents the resolvent operator.

Screens with extension property

In general we admit screens $\Sigma \subset \mathbb{R}^m$ $(m = n - 1)$ which have the following *extension property*: For $s = \pm 1/2$ there exist bounded linear operators $E^s_\Sigma : H^s(\Sigma) \to H^s(\mathbb{R}^m)$ which are left invertible by restriction, denoted by $r_\Sigma = r^s_\Sigma$ (the fact that r_Σ depends on s as an operator is suppressed), in brief $r_\Sigma E^s_\Sigma = I_{H^s(\Sigma)}$ (cf. [HW08,Wlo87]). For $s \in \mathbb{R}^2$, particularly for $s = (s_1, s_2) = (1/2, -1/2)$, we briefly write $E^s_\Sigma = \operatorname{diag}(E^{s_1}_\Sigma, E^{s_2}_\Sigma)$, provided Σ has the extension property for s_1 and s_2.

Normal type IFPs and their characteristic number

We say that the IFP (and B_0, as well,) is of *normal type* [S89] if

$$a_0 b_1 + b_0 a_1 \neq 0 \quad , \quad a'_0 b'_1 + b'_0 a'_1 \neq 0. \tag{4}$$

These conditions guarantee that certain Fourier symbols do not degenerate (and corresponding pseudo-differential operators are normally solvable). Further let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{a'_0 b'_1 + b'_0 a'_1} \begin{pmatrix} a_0 b'_1 + b_0 a'_1 & -a_0 b'_0 + b_0 a'_0 \\ -a_1 b'_1 + b_1 a'_1 & a_1 b'_0 + b_1 a'_0 \end{pmatrix} \tag{5}$$

and $bc \neq 0$ (see remarks in Section 5 about "decomposing systems"). Then we call

$$\lambda = ad/bc \tag{6}$$

the *characteristic number* of the IFP. The following matrix function plays a central role:

$$\sigma_\lambda = \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix} . \tag{7}$$

Main Theorem

Let Σ be an open half-space in $\mathbb{R}^{n-1}$ $(n \geq 2)$ and B_0 defined by (3) be of normal type. Let $k \in \mathbb{C}, \Im m\, k > 0, abcd \neq 0$ be satisfied and $\lambda^{-1} \in \mathbb{C} \setminus [0,1]$. Then the IFP is well-posed and the resolvent operator can be calculated by means of a canonical Wiener-Hopf factorization of $A_\lambda = \mathcal{F}^{-1}\sigma_\lambda \cdot \mathcal{F}$ through an unisotropic Sobolev space that will be given later explicitly.

The existence of such a factorization is also necessary for the IFP to be well-posed.

Interface operators (IFOs)

We continue considering the semi-homogeneous problem. Thus it is convenient to formulate the interface equations in terms of new unknowns v_Σ, w_Σ which vanish on Σ' instead of the traces u_0^+, u_0^-. This leads us to the following data management (operator theoretical formulation), analogously to the case $n = 2$ [S89]. Let

$$B_\pm \,:\, H^{1/2}(\mathbb{R}^{n-1})^2 \to H^{1/2}(\mathbb{R}^{n-1}) \times H^{-1/2}(\mathbb{R}^{n-1})$$

$$B_+ = \mathcal{F}^{-1} \begin{pmatrix} a_0 & b_0 \\ -a_1 t & b_1 t \end{pmatrix} \cdot \mathcal{F} = \mathcal{F}^{-1} \sigma_{B_+} \mathcal{F}^{-1} \tag{8}$$

$$B_- = \mathcal{F}^{-1} \begin{pmatrix} a_0' & b_0' \\ -a_1' t & b_1' t \end{pmatrix} \cdot \mathcal{F} = \mathcal{F}^{-1} \sigma_{B_-} \mathcal{F}^{-1}$$

which are boundedly invertible linear operators if (4) is satisfied. In this case we define the IFO (associated to the IFP) as

$$T = r_\Sigma\ \mathcal{F}^{-1} \sigma \cdot \mathcal{F} \,:\, H_\Sigma^{1/2} \times H_\Sigma^{-1/2} \to H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma) \tag{9}$$

$$\sigma = \sigma_{B_+} \sigma_{B_-}^{-1}.$$

Theorem 1

Let B_0 be the operator associated to an IFP of normal type (see (3), (4)) and T the corresponding IFO defined by (9). Then these two operators are equivalent.

Proof The representation formula (2) and the definition of $B_\pm$ imply the operator factorization

$$\begin{array}{ccccccc} T & = & B_0 & & \mathcal{K} & & B_-^{-1} \\ & & \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} \longleftarrow & u & \longleftarrow & \begin{pmatrix} u_0^+ \\ u_0^- \end{pmatrix} \longleftarrow & \begin{pmatrix} v_\Sigma \\ w_\Sigma \end{pmatrix} \\ & & H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma) \leftarrow & \mathcal{H}_0^1 & \leftarrow & \dots \qquad \leftarrow & H_\Sigma^{1/2} \times H_\Sigma^{-1/2} \end{array} \tag{10}$$

in which the composition of the last two operators on the right hand side is a linear homeomorphisms according to the normality assumption and the definitions of B_- and $\mathcal{H}_0^1$. □

Corollary: Explicit solution in terms of T^{-1}

The IFP is well-posed if and only if T has a bounded inverse. In this case the resolvent operator is represented by

$$B_0^{-1} = \mathcal{K} B_-^{-1} T^{-1}$$

where $\mathcal{K}$ and B_-^{-1} are given by (2) and (8) (inverting σ_{B_-}), resp.

Theorem 2: Reduction to a normal form

Let T be given as before and $abc \neq 0$ in (5). Then $T \sim T_{\lambda,\Sigma}$ where

$$T_{\lambda,\Sigma} = r_\Sigma \ \mathcal{F}^{-1}\sigma_\lambda \cdot \mathcal{F} \ : \ H_\Sigma^{1/2} \times H_\Sigma^{-1/2} \to H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma) \tag{11}$$

$$\sigma_\lambda = \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix}$$

and λ is given by (6).

Proof Obviously σ can be transformed into σ_λ by multiplication with constant matrices, explicitly by writing:

$$\begin{aligned} \sigma &= \frac{1}{a_0'b_1' + b_0'a_1'} \begin{pmatrix} a_0b_1' + b_0a_1' & (-a_0b_0' + b_0a_0')t^{-1} \\ (-a_1b_1' + b_1a_1')t & a_1b_0' + b_1a_0' \end{pmatrix} \\ &= \begin{pmatrix} a & bt^{-1} \\ ct & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b/a \end{pmatrix} \square \end{aligned} \tag{12}$$

General WHOs and canonical WH-factorization through an intermediate space

Let $A : X \to Y$ be a bounded linear operator acting in Banach spaces and $P_1 \in \mathcal{L}(X), P_2 \in \mathcal{L}(Y)$ projectors, i.e., $P_j^2 = P_j, j = 1, 2$. Then

$$W \quad = \quad P_2 A|_{P_1 X} \quad : \quad P_1 X \to P_2 Y \tag{13}$$

is referred to as a *general Wiener-Hopf operator (WHO)* [DevShi69,S85]. Further

$$A \quad = \quad A_- \, A_+ \quad : \quad Y \leftarrow Z \leftarrow X \tag{14}$$

is said to be a *canonical WH-factorization of A through an intermediate space Z* (in short *FIS*), if Z is a Banach space, as well, $A_+ \in \mathcal{L}(X, Z), A_- \in \mathcal{L}(Z, Y)$ are boundedly invertible and if there is a projector $P \in \mathcal{L}(Z)$ such that

$$A_+ P_1 X \quad = \quad PZ \quad , \quad A_- PZ \quad = \quad P_2 Y \, . \tag{15}$$

See [Cas95,CS95,...,S14].

Theorem 3: Factorization theorem

Let $A : X \to Y$ be a bounded linear operator acting in Banach spaces and $P_1 \in \mathcal{L}(X), P_2 \in \mathcal{L}(Y)$ projectors. Then the WHO (13) is boundedly invertible if and only if A admits a FIS (14). In this case the inverse of W reads

$$W^{-1} \quad = \quad A_+^{-1} \, P \, A_-^{-1}|_{P_2 Y} \quad : \quad P_2 Y \to P_1 X \, . \tag{16}$$

Proof Sufficiency follows from a verification of formula (16), i.e., showing $W^{-1}W = I_X$ and $WW^{-1} = I_Y$ by help of the *factor properties* resulting from (15):

$$\begin{aligned} A_+ P_1 &= P A_+ P_1 \quad , \quad & A_+^{-1} P &= P_1 A_+^{-1} P \\ P_2 A_- &= P_2 A_- P \quad , \quad & P A_-^{-1} &= P A_-^{-1} P_2 \, . \end{aligned} \tag{17}$$

This is just a simple modification of an old idea known from classical, symmetric WHOs [Shi61,Goh...?].

Necessity: If W is invertible, put

$$\begin{aligned}
Z &= P_2Y \times Q_1X \quad , \quad P = \begin{pmatrix} I|_{P_2Y} & 0 \\ 0 & 0 \end{pmatrix} : Z \to Z \\
\iota_1 x &= \begin{pmatrix} P_1 x \\ Q_1 x \end{pmatrix}, x \in X \quad , \quad \iota_2 y = \begin{pmatrix} P_2 y \\ Q_2 y \end{pmatrix}, y \in Y \\
A_+ &= \begin{pmatrix} P_2AP_1 & P_2AQ_1 \\ 0 & Q_1 \end{pmatrix} \iota_1 : X \to Z \qquad (18) \\
A_- &= \iota_2^{-1} \begin{pmatrix} P_2 & 0 \\ Q_1A^{-1}P_2 & Q_1A^{-1}Q_2 \end{pmatrix}^{-1} : Z \to Y \quad \square
\end{aligned}$$

Remark This part of the proof is new, although just a modification of Cebotarev's brilliant idea [Ceb67,S85] to write, in the symmetric case,

$$A = A_- \; A_+ = (...)(PA + Q) .$$

However, there is a (non-trivial) modification for the generalized invertibility of W called WH-factorization through an intermediate space, to be published [S14].

Realization of WHOs in diffraction theory

Let $\Sigma \subset \mathbb{R}^m$ be open and such that Σ and $\Sigma' = \mathbb{R}^m \setminus \overline{\Sigma}$ satisfy the extension property for $s = \pm 1/2$, i.e., there exist bounded linear operators $E^s_\Sigma : H^s(\Sigma) \to H^s(\mathbb{R}^m)$ with $r_\Sigma E^s_\Sigma = I_{H^s(\Sigma)}$, for $s = (1/2, -1/2)$. Then the operator $T = T_{\lambda,\Sigma}$ of (10) (corresponding with $\sigma = \sigma_\lambda$) is equivalent to a (general) WHO, namely

$$T_{\lambda,\Sigma} = r_\Sigma W \quad , \quad W = E^s_\Sigma T_{\lambda,\Sigma} = P_2 \, A_\lambda|_{P_1 X} \tag{19}$$

$$X = Y = H^{1/2} \times H^{-1/2} \quad , \quad A_\lambda = \mathcal{F}^{-1} \sigma_\lambda \cdot \mathcal{F}$$

where P_1 is any projector onto $H^{1/2}_\Sigma \times H^{-1/2}_\Sigma$, P_2 is any other projector along $H^{1/2}_{\Sigma'} \times H^{-1/2}_{\Sigma'}$. In particular it is possible to choose for P_1 the orthogonal projector onto $H^{1/2}_\Sigma \times H^{-1/2}_\Sigma$ and for P_2 the orthogonal projector along $H^{1/2}_{\Sigma'} \times H^{-1/2}_{\Sigma'}$.

This leads to the general question of existence and representation of extension operators in spaces of Bessel potentials, corresponding projectors and the construction of Bessel potential operators, see [DS93].

Example 1

For $s = 0$, $H^s(\Sigma) = L^2(\Sigma)$ and any measurable set $\Sigma \subset \mathbb{R}^m$, the extension by zero represents a bounded linear operator and "produces" an orthogonal projector

$$\ell_0 \in \mathcal{L}(L^2(\Sigma), L^2(\mathbb{R}^m)) \quad , \quad r_\Sigma \ell_0 = I|_{L^2(\Sigma)}$$
$$P = \ell_0 r_\Sigma = P^2 = P^* \in \mathcal{L}(L^2(\mathbb{R}^m)\,.$$

If we replace $s = 0$ by $0 < |s| < 1/2$ and consider $H^s(\Sigma)$, where $\Sigma \subset \mathbb{R}^m$ is open, all this remains true up to the fact that $P = \ell_0 r_\Sigma$ is not orthogonal. However P is still a projector along $H^s(\Sigma')$, $\Sigma' = \mathbb{R}^m \setminus \overline{\Sigma}$.

Example 2

The question of explicit representation of the orthogonal projector onto (or along) $H^s(\Sigma)$, $s \in \mathbb{R}$, is quickly answered for a half-space, $\Sigma = \mathbb{R}^m_+$ say, by using Bessel potential operators Λ_+:

$$P = \Lambda_+^{-s}\ \ell_0 r_\Sigma\ \Lambda_+^s \quad , \quad \Lambda_+ = \mathcal{F}^{-1}\lambda_+ \cdot \mathcal{F} \tag{20}$$
$$\lambda_+(\xi) = \xi_m + i(\xi'^2+1)^{1/2} \quad , \quad \xi = (\xi', \xi_m) \in \mathbb{R}^m ,$$

see [Esk81,CS98], for instance.

One observes that the orthogonal projector (20) represents the projector onto $H^s(\Sigma)$ along $\Lambda^{-2s}H^s(\Sigma')^{-s}$ where Λ is the Bessel potential operator with symbol t in the special case of $k = i$:

$$\Lambda = A_{t_i} = \mathcal{F}^{-1} t_i \cdot \mathcal{F} \quad , \quad t_i(\xi) = (\xi^2+1)^{1/2} , \xi \in \mathbb{R}^m . \tag{21}$$

Note that t_i is often denoted by λ [Esk81] (which stands here already for the characteristic parameter that appeared in (6)).

Example 3

Now, let us work with $t(\xi) = (\xi^2 - k^2)^{1/2}$ instead of $t_i = (\xi^2 + 1)^{1/2}$ where $k \neq i$, moreover $\Re e k > 0, \Im m k > 0$. Replacing $\lambda_+(\xi)$ by the last factor of the factorization

$$(\xi^2 - k^2) \quad = \quad (\xi_m - i(\xi'^2 - k^2)^{1/2})\,(\xi_m + i(\xi'^2 - k^2)^{1/2})$$

we obtain similar results but no orthogonality.

Explicit Wiener-Hopf factorization of A_λ through an intermediate space

The problem is to factorize the operator $A_\lambda = \mathcal{F}^{-1}\sigma_\lambda \cdot \mathcal{F}$ with respect to $X = Y = H^{1/2} \times H^{-1/2}$ and projectors P_1 onto $H^{1/2}_{\Sigma} \times H^{-1/2}_{\Sigma}$, P_2 along $H^{1/2}_{\Sigma'} \times H^{-1/2}_{\Sigma'}$ where $\Sigma = \mathbb{R}^m_+$ and σ_λ is the matrix of (9)

$$\sigma_\lambda = \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix}$$

with $\lambda \in \mathbb{C} \setminus \{0, 1\}$, $t(\xi) = (\xi^2 - k^2)^{1/2}$ and

$$\xi = (\xi', \xi_m) = (\xi_1, ..., \xi_{m-1}, \xi_m) \in \mathbb{R}^m$$

in case of $m \geq 2$ (i.e., $n \geq 3$).

Certainly this will be done by factorization of σ_λ.

The one-dimensional case

For convenience we use the following abbreviations assuming that $k, \lambda \in \mathbb{C}\,,\, \Im m\, k > 0\,,\, 0 \neq \lambda \neq 1$:

$$\begin{aligned}
\gamma_\pm(\xi) &= \frac{\sqrt{k \pm \xi} + i\sqrt{k \mp \xi}}{\sqrt{2k}}\ , \quad \xi \in \mathbb{R} \\
C &= \frac{i}{\pi} \log \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1}\ , \quad \delta = \Re e\, C \qquad (22)\\
c_\pm(\xi) &= \cosh[C \log \gamma_\pm(\xi)] \\
s_\pm(\xi) &= \sinh[C \log \gamma_\pm(\xi)]
\end{aligned}$$

with the usual branch cuts compatible with $t(\xi) = (\xi^2 - k^2)^{1/2}$ (see the introduction). More precisely, $t = t_- t_+\,,\, t_\pm(\xi) = (\xi \pm k)^{1/2} \to +\infty$ as $\xi \to +\infty$ with vertical branch cuts from k to $i\,\infty$ and from $-k$ to $-i\,\infty$, respectively. Further

$$\sqrt{k-\xi} \quad = \quad i(\xi-k)^{1/2} \quad = \quad it_-(\xi)\,.$$

Note that here $t_\pm$ have order 1/2 (as in [S89]) and not order 1.

Theorem 4

For $\Sigma = \mathbb{R}_+, \lambda \in \mathbb{C}\backslash\{0\}$ the operator $T_{\lambda,\Sigma}$ defined by (11) is boundedly invertible if and only if $\lambda^{-1} \in \mathbb{C} \setminus [0,1]$ holds. In this case the inverse $T_{\lambda,\Sigma}^{-1}$ is given by a bounded composition of linear operators:

$$\begin{aligned}
T_{\lambda,\Sigma}^{-1} &= A_{\lambda+}^{-1}\,\ell_0 r_+ \cdot A_{\lambda-}^{-1}\,\ell \qquad (23)\\
A_{\lambda\pm}^{-1} &= \mathcal{F}^{-1}\,\sigma_{\lambda\pm}^{-1} \cdot \mathcal{F}\\
\sigma_{\lambda+} &= (1-\lambda^{-1})^{-1/4} \begin{pmatrix} c_+ & -s_+\sqrt{\lambda}/t \\ -c_+\xi/\sqrt{\lambda} - s_+ t/\sqrt{\lambda} & s_+\xi/t + c_+ \end{pmatrix}\\
\sigma_{\lambda-} &= (1-\lambda^{-1})^{-1/4} \begin{pmatrix} c_- - s_-\xi/t & -s_-\sqrt{\lambda}/t \\ -s_- t/\sqrt{\lambda} + c_-\xi/\sqrt{\lambda} & c_- \end{pmatrix}
\end{aligned}$$

where ℓg denotes any extension of $g \in H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma)$ into $H^{1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$. $A_{\lambda\pm}$ and $A_{\lambda\pm}^{-1}$ are considered as densely defined, unbounded operators whose composition in the first line of (23) has a bounded extension.

Proof of Theorem 4 (sketch)

It is well-known that $T_{\lambda,\Sigma}$ has a bounded inverse if and only if the lifted Fourier symbol

$$\sigma_{\lambda 0} \quad = \quad \begin{pmatrix} t_- & 0 \\ 0 & t_-^{-1} \end{pmatrix} \sigma_\lambda \begin{pmatrix} t_+^{-1} & 0 \\ 0 & t_+ \end{pmatrix} \tag{24}$$

admits a canonical generalized factorization in $L^2(\mathbb{R})^2$, see Example 1 and [S89], Theorem 2.1 for more details.

There it was shown that the Fredholm criterion for WHOs acting in $L^2(\mathbb{R}_+)^2$ with Fourier symbol (24) (belonging to $C^\nu(\ddot{\mathbb{R}})^{2\times 2}$) excludes $\lambda^{-1} \in [0,1]$. For $\lambda^{-1} \in \mathbb{C} \setminus \{0,1\}$ a factorization of $\sigma_{\lambda 0}$ into lower/upper holomorphic function matrices was constructed (see Theorem 3.1 of [S89]) by means of the Daniele-Khrapkov formulas. This was not a generalized factorization in $L^2(\mathbb{R})^2$ for any λ. However, for $\lambda^{-1} \in \mathbb{C} \setminus [0,1]$, it could be transformed into a generalized factorization with the help of the so-called Daniele trick [Dan84], splitting a rational ansatz function in a convenient way, see [S89], Section 4, for details, resulting in the factorization (23). □

Theorem 5

Let $\Sigma = \mathbb{R}_+$, $\lambda^{-1} \in \mathbb{C} \setminus [0,1]$. The factorization of $A_\lambda = A_{\lambda-} A_{\lambda+}$ in (23) represents a FIS where

$$\begin{aligned} \sigma_\lambda &= \sigma_{\lambda-} \quad \sigma_{\lambda+} \\ A_\lambda &= A_{\lambda-} \quad A_{\lambda+} \quad = \quad \mathcal{F}^{-1}\sigma_{\lambda-}\cdot\mathcal{F} \quad \mathcal{F}^{-1}\sigma_{\lambda+}\cdot\mathcal{F} \\ & H^{1/2}\times H^{-1/2} \leftarrow Z \leftarrow H^{1/2}\times H^{-1/2} \\ Z &= H^z(\mathbb{R}) \quad , \quad z=(z_1,z_2)=(\frac{1}{2}(1-\delta),\frac{1}{2}(\delta-1)) \end{aligned} \tag{25}$$

and $\delta = \frac{-1}{\pi}\arg\frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} \in]0,1]$.

Proof of Theorem 5

It is convenient to consider the corresponding factorization of $A_{\lambda 0} = \mathcal{F}^{-1}\sigma_{\lambda 0} \cdot \mathcal{F} \,:\, L^2(\mathbb{R})^2 \to L^2(\mathbb{R})^2$

$$\begin{array}{ccccc} A_{\lambda 0} & = & A_{\lambda 0-} \quad A_{\lambda 0+} & = & \mathcal{F}^{-1}\,\sigma_{\lambda 0-}\mathcal{F} \quad \mathcal{F}^{-1}\sigma_{\lambda 0-}\mathcal{F} \\ & & L^2(\mathbb{R})^2 \leftarrow \; Z \; \leftarrow L^2(\mathbb{R})^2 & & \end{array} \tag{26}$$

as a factorization of a bounded linear operator acting in $L^2(\mathbb{R})^2$ into injective unbounded operators (densely defined in $L^2(\mathbb{R})^2$) such that the composition

$$A_{\lambda 0+}^{-1}\, P\, A_{\lambda 0-}^{-1} \tag{27}$$

where $P = 1_+ \cdot$ acts on a dense subspace, as well, admitting a bounded extension in $L^2(\mathbb{R})^2$.

In this situation, the factorization (25) can be regarded as a *bounded operator factorization through an intermediate space* Z which is simply the image of $A_{\lambda 0+}$ equipped with the induced norm topology $\|.\|_Z = \|A_{\lambda 0+}^{-1}.\|_{L^2(\mathbb{R})^2}$. It turns out that the factorization (25) represents a FIS in the sense of the statement because of the asymptotic behavior of the lifted symbol factors

$$\begin{aligned} \operatorname{ord}\sigma_{\lambda 0-} &= \operatorname{ord}\sigma_{\lambda 0+}^{-1} = \begin{pmatrix} \frac{1}{2}(1-\delta) & \frac{1}{2}(\delta-1) \\ \frac{1}{2}(1-\delta) & \frac{1}{2}(\delta-1) \end{pmatrix} \qquad (28) \\ \operatorname{ord}\sigma_{\lambda 0+} &= \operatorname{ord}\sigma_{\lambda 0-}^{-1} = \begin{pmatrix} \frac{1}{2}(\delta-1) & \frac{1}{2}(\delta-1) \\ \frac{1}{2}(1-\delta) & \frac{1}{2}(1-\delta) \end{pmatrix}, \end{aligned}$$

The crucial point is that $P_+ = \ell_0 r_+ \in \mathcal{L}(Z)$. In other words: The multiplication operator generated by the characteristic function of $\mathbb{R}_+$ acts as a continuous operator in $Z = H^z(\mathbb{R})$, which is true since $|z_j| < 1/2, j = 1, 2$. The range of δ results from the definition, namely $\delta = \frac{-1}{\pi} \arg \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} \in [0, 1]$ and $\delta = 0$ is excluded (corresponding with $\lambda \geq 1$ where the operator is not Fredholm. □

Theorem 6

Let $\lambda \neq 0, \Sigma = \mathbb{R}_+$ and $T_{\lambda,\Sigma}$ be given by (11). The following assertions are equivalent:

1. $T_{\lambda,\Sigma}$ is normally solvable (i.e., it has a closed image),
2. $T_{\lambda,\Sigma}$ is boundedly invertible,
3. $\lambda^{-1} \in \mathbb{C} \setminus [0,1]$.

Proof It remains to consider $\lambda^{-1} \in]0,1]$. If $\lambda = 1$, the matrix σ_λ degenerates which implies that A_λ is not normally solvable and $T_{\lambda,\Sigma}$, as well [MikPro80]. If $\lambda^{-1} \in]0,1[$, we know from [S89] that $T_{\lambda,\Sigma}$ is not normally solvable. In the other cases where $\lambda^{-1} \in \mathbb{C} \setminus [0,1]$ it was proved already in Theorem 3.2 that $T_{\lambda,\Sigma}$ is invertible. □

Theorem 7: The higher-dimensional case

Let $m \in \mathbb{N}, m \geq 2, k, \Im m\, k > 0, \lambda^{-1} \in \mathbb{C} \setminus [0,1], t(\xi) = (\xi^2 - k^2)^{1/2} = (\xi_m^2 + \xi'^2 - k^2)^{1/2}, \xi = (\xi', \xi_m) \in \mathbb{R}^m$, σ_λ be given by (11), and $\sigma_{\lambda\pm}$ by (23) replacing ξ by ξ_m and k by $(k^2 - \xi'^2)^{1/2}$ in (22). Consider

$$
\begin{aligned}
W_{\lambda,\mathbb{R}^m_+} &= P_2 A|_{P_1 X} \qquad (29)\\
X &= H^{1/2}(\mathbb{R}^m) \times H^{-1/2}(\mathbb{R}^m)\\
A &= A_\lambda = \mathcal{F}^{-1}\sigma_\lambda \cdot \mathcal{F}\\
P_j^2 &= P_j \in \mathcal{L}(X)\,, j = 1,2\\
\operatorname{im} P_1 &= H_\Sigma^{1/2} \times H_\Sigma^{-1/2} \quad , \quad \ker P_2 = H_{\Sigma'}^{1/2} \times H_{\Sigma'}^{-1/2}\,.
\end{aligned}
$$

Then, with these substitutions, (25) represents a FIS where the intermediate space is an anisotropic vector Sobolev space

$$
\begin{aligned}
Z &= H^z(\mathbb{R}^m) \times H^{-z}(\mathbb{R}^m) \qquad (30)\\
H^z(\mathbb{R}^m) &= \mathcal{F}(w_z L^2(\mathbb{R}^m)) \quad , \quad w_z(\xi) = (1 + |\xi'|^2)^{z_1/2})\,(1 + \xi_m^2)^{z_2/2})\\
z &= (z_1, z_2) = (\frac{1}{2}(\delta - 1), \frac{1}{2}(1 - \delta))\,.
\end{aligned}
$$

Proof of Theorem 7

The explicit factorization formulas (23) show, after substitution of ξ by ξ_m and k by $(k^2 - \xi'^2)^{1/2}, \xi' \in \mathbb{R}^{m-1}$ (considered as parameter), the desired properties of the factors with respect to analytic extension in $\zeta_m = \xi_m + i\eta_m, \eta_m > 0$ or $\eta_m < 0$, respectively. This guarantees the invariance properties (17) in dense subspaces. It remains to prove that the factorization of A_λ corresponding with (25) represents a factorization into bounded operators through an suitable intermediate space Z, i.e., where $P = \ell_0 r_+$ is bounded (cf. the end of the proof of Theorem 3.2). The positive answer is given by studying the asymptotic behavior at infinity of $\sigma_{\lambda\pm}$ in (23). We obtain:

$$\gamma_+(\xi) \approx \begin{cases} i^{1/2}\,|\xi'|^{1/2}\,\xi_m^{-1/2} & , \quad \xi_m \to +\infty\,,\, |\xi'| \to +\infty \\ \sqrt{2}\,i^{-1/2}\,|\xi'|^{-1/2}\,|\xi_m|^{1/2} & , \quad \xi_m \to -\infty\,,\, |\xi'| \to +\infty \end{cases}$$

$$\gamma_-(\xi) \approx \begin{cases} i^{1/2}\,|\xi'|^{-1/2}\,\xi_m^{1/2} & , \quad \xi_m \to +\infty\,,\, |\xi'| \to +\infty \\ i^{1/2}/\sqrt{2}\ \ |\xi'|^{1/2}\,|\xi_m|^{-1/2} & , \quad \xi_m \to -\infty\,,\, |\xi'| \to +\infty \end{cases}$$

$$\begin{aligned} |\exp[C\cdot\log\gamma_\pm(\xi)]| = &\ \ \mathcal{O}(\exp\{\Re e C\cdot\log|\gamma_\pm(\xi)|\} \\ = &\begin{cases} \mathcal{O}((|\xi_m|/|\xi'|)^{\mp\Re e C/2}) & , \quad \xi_m \to +\infty\,,\, |\xi'| \to +\infty \\ \mathcal{O}((|\xi_m|/|\xi'|)^{\pm\Re e C/2}) & , \quad \xi_m \to -\infty\,,\, |\xi'| \to +\infty\,. \end{cases} \end{aligned}$$

The asymptotic behavior of the corresponding lifted symbols is, with respect to ξ_m, the same as in the one-dimensional case given by (28) where $\delta = \Re e C = \Re e(\frac{i}{\pi}\log\frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1}) \in]0,1]$. The asymptotic behavior of $\gamma_\pm$ in the previous formulas shows that the increase orders with respect to $|\xi'|$ are inverse, i.e., $\delta - 1$ and $1 - \delta$ have to be exchanged (28). The exceptional case of $\delta = 1$ corresponds with $\lambda \in]0,1[$ and the intermediate space $Z = L^2(\mathbb{R})^2$. In all other cases we have the unisotropic Sobolev space with $z \neq (0,0)$. □

Example 4

The most prominent genuine example is the so-called Sommerfeld problem [Nob58,Mei97,MS89,Som96] with mixed Dirichlet-Neumann conditions on Σ, also denoted as Rawlins' problem (after [Raw81]) where the interface conditions (1) gain the form

$$\begin{aligned} u_0^+ = g_0 \quad , \quad -u_1^- = g_1 \quad on \quad \Sigma \\ u_0^+ - u_0^- = 0 \quad , \quad u_1^+ - u_1^- = 0 \quad on \quad \Sigma'. \end{aligned} \tag{31}$$

By analogy to the lowest dimensional case $n = 2, m = 1$ [MS89,S89] one can now conclude that, for sufficiently smooth and decreasing data, the solution in $\mathbb{R}^3$ has the asymptotic behavior

$$\nabla u(x) \quad = \quad \mathcal{O}(|(x_2, x_3)|^{-3/4}) \;\; as \;\; |x| \to 0\,. \tag{32}$$

However, there are other very efficient methods for the description of this kind of asymptotics, see [Dau88].

Theorem 8

Let $m \in \mathbb{N}$, $\Sigma = \mathbb{R}^m_+$, $k \in \mathbb{C}$, $\Im m\, k > 0$, $\lambda^{-1} \in \mathbb{C} \setminus [0,1]$. Then the inverse of the operator $T_{\lambda,\Sigma}$ of (11) is given by formulas (23), namely

(i) for $m = 1$ as composition of bounded operators in the sense of (25),

(ii) for $m \geq 2$ with the above-mentioned substitutions (ℓ extends from $\mathbb{R}^m_+$ to $\mathbb{R}^m$, $\mathcal{F}$ is the m-dimensional Fourier transformation, t defined in $\mathbb{R}^m$ and k replaced by $(k^2 - \xi'^2)^{1/2}$ as in Theorem 6) as a composition of bounded operators in the sense of (29) and (30).

Proof This is a direct consequence of the foregoing results. □

Completion of the proof of the main theorem

Now we put all together. Under the assumptions of Theorem 1 the well-posedness of the (semihomogeneous) IFP is equivalent to the bounded invertibility of B_0. We further know that $B_0 \sim T_{\lambda,\Sigma}$. The inverse of the last operator is given by Theorem 8 and bounded, as well. Therefore the resolvent operator B_0^{-1} exists and is bounded as a composition of the corresponding bounded operators, see (2), (10), (12), Theorem 3 and Theorem 8:

$$B_0^{-1} \quad = \quad \mathcal{K}\, B_-^{-1}\, T^{-1} \quad = \quad \mathcal{K}\, B_-^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \frac{a}{b} \end{pmatrix} T_{\lambda,\Sigma}^{-1} \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{c} \end{pmatrix}. \tag{33}$$

Conversely, if the IFP is wellposed, we conclude by the help of the above operator relations that $T_{\lambda,\Sigma}$ and the equivalent operator $W_{\lambda,\Sigma}$ are boundedly invertible. By Theorem 3, the operator A_λ necessarily admits a FIS. □

Decomposing systems

Now we study the "exceptional cases" where $abcd = 0$ in (5). All of them admit an explicit solution, even under more general assumptions on Σ. Most are interesting from the viewpoint of functional and asymptotic analysis and some of them can be considered as physically interesting. Two very different classes appear where (α) either an off-diagonal element vanishes or (β) an element in the diagonal vanishes.

Case $\alpha.0$: $b = c = 0$ and $ad \neq 0$. Obviously, the operator T given by (9) with symbol σ defined in (12) is just $T = r_\Sigma \operatorname{diag}(aI, dI) = \operatorname{diag}(r_\Sigma aI, r_\Sigma dI)$, i.e., it maps onto $r_\Sigma H_\Sigma^{1/2} \times r_\Sigma H_\Sigma^{-1/2} = \widetilde{H}^{1/2}(\Sigma) \times \widetilde{H}^{-1/2}(\Sigma)$ which is a dense proper subspace of the image space of T. Hence, replacing T by an operator $T^<$ with this smaller image space (and equipped with the induced norm topology), we obtain a bijective, bi-continuous operator

$$T^< \quad : \quad H_\Sigma^{1/2} \times H_\Sigma^{-1/2} \to \widetilde{H}^{1/2}(\Sigma) \times \widetilde{H}^{-1/2}(\Sigma) \,. \tag{34}$$

Decomposing systems, further cases $bc = 0$

Case $\alpha.1$: $b = 0$ and $acd \neq 0$. Similar arguments lead to the normalization

$$T^{<} \quad : \quad H_{\Sigma}^{1/2} \times H_{\Sigma}^{-1/2} \to \widetilde{H}^{1/2}(\Sigma) \times H^{-1/2}(\Sigma)\,. \tag{35}$$

It is easy to find various realizations of IFPs falling into this class of problems. For instance, imposing jump conditions on Σ', then $b = 0$ implies the condition $a_0 + b_0 = 0$ and a_1, b_1 remain arbitrary, not both vanishing. The corresponding IFP does not seem to be physically important.

Case $\alpha.2$: $c = 0$ and $abd \neq 0$. Here the normalized operator maps like

$$T^{<} \quad : \quad H_{\Sigma}^{1/2} \times H_{\Sigma}^{-1/2} \to H^{1/2}(\Sigma) \times \widetilde{H}^{-1/2}(\Sigma)\,. \tag{36}$$

Explicit inversion is simple as before. In these examples Σ is quite arbitrary, in contrast to the following.

Decomposing systems, cases $ad = 0$

Case $\beta.0$: $a = d = 0$ and $bc \neq 0$. This case leads us to the study of the operators

$$W_{t,\Sigma} \quad = \quad r_\Sigma \, \mathcal{F}^{-1} t \cdot \mathcal{F} \; : \; H_\Sigma^{1/2} \to H^{-1/2}(\Sigma) \quad , \quad W_{t^{-1},\Sigma} \quad = \quad r_\Sigma \, \mathcal{F}^{-1} t^{-} \tag{37}$$

These operators have been explicitly inverted in [CDS14 for $k \in i\mathbb{R}_+$ and a wide class of polygonal-conical domains in $\mathbb{R}^2$, i.e., elements in the set algebra generated by half-planes, finite intersection and the interior of complements. Hence our operator T can be inverted as

$$T^{-1} \quad = \quad \begin{pmatrix} 0 & b^{-1} W_{t^{-1},\Sigma}^{-1} \\ c^{-1} W_{t,\Sigma}^{-1} & 0 \end{pmatrix} . \tag{38}$$

A generalization to higher dimensions $m \geq 3$ is not hard to prove, anyway not carried out here. The case where Σ is a half-space and the wave number k just satisfies $\Im m \, k > 0$ can be treated by scalar factorization, see Example 1.

Decomposing systems, further cases $ad = 0$

Case $\beta.1$: $a = 0$ and $bcd \neq 0$. This is a simple generalization of the previous case working with triangular 2 by 2 operator matrices where the off-diagonal operators are invertible.

Case $\beta.2$: $d = 0$ and $abc \neq 0$. The same idea holds as before. This class includes the case $\lambda = 0$ that was avoided before.

Realizations of the last two classes are not completely irrelevant. If $a = 0$, we conclude $a_0 = b_0$, i.e., the first interface conditions on Σ tells us that the traces on the upper and the lower bank of Σ coincide while the second condition if arbitrary (as long as both coefficients a_1 and b_1 do not vanish simultaneously).

If $d = 0$, we conclude $a_1 = b_1$, i.e., the second interface conditions on Σ tells us that the normal derivatives on the upper and the lower bank of Σ coincide while the first condition is arbitrary (as long as both coefficients a_0 and b_0 do not vanish simultaneously).

Generalizations: 1. The inhomogeneous problem

The relationship between the semi-homogeneous problem to the full problem is clearly described by an operator matrix identity:

$$B = E \begin{pmatrix} B_0 & 0 \\ 0 & I_{Z^*} \end{pmatrix} F \tag{39}$$

where Z^* is a suitable Banach space and E, F linear homeomorphisms. Here we have the Hilbert space $Z^* = H^{1/2}(\Sigma') \times H^{-1/2}(\Sigma')$, see [S12red] for details. The relation (39) is more general than operator equivalence but, in particular, it also transfers the invertibility property, explicit representation of inverses (provided E, F or E^{-1}, F^{-1}, respectively, are known), and operator normalization in the above sense. Altogether we obtain the chain of operator relations

$$B \quad \overset{*}{\sim} \quad B_0 \quad \sim \quad T \quad \sim \quad W$$

where $B \overset{*}{\sim} B_0$ stands for an equivalent after extension relation. It implies particularly that the images are closed only simultaneously, that the kernels are isomorphic to each other and the cokernels are isomorphic, as well [BT91,CS98,S12red].

Generalizations: 2. Parameters $\lambda \in]1, \infty]$

The operator A_λ (see (19),(19)) is obviously invertible in $H^{1/2} \times H^{-1/2}$ if $\lambda \neq 1$. For $n = 2$ and $\Sigma = \mathbb{R}_+$ it is known that the corresponding Wiener-Hopf operator is not to normally solvable, but a normalization is possible by changing the spaces to $H^{s,p}, p \neq 2$, for instance, see [S89]. A higher-dimensional analogue is expected, however technically complicated.

Generalizations: 3. Small regularity

For $m = 1$ and with the help of the theory of Wiener-Hopf operators in Sobolev spaces with symbols in $C^\mu(\ddot{\mathbb{R}})$ [MoST98] it is possible to prove that in the situation of Theorem 1.1 a solution $u \in H^{1+\varepsilon}$ for data in $H^{1/2+\varepsilon} \times H^{-1/2+\varepsilon}$ and certain values of $\varepsilon \in]0, 1/2[$ which depend of λ and can be determined from (25). Further results in this direction are expected for the higher-dimensional case. Also normalization in the H^s-scale seems to be possible (replacing $s = \pm 1/2$ by $s = \pm 1/2 + \varepsilon$).

Open problems: 1. More general screens Σ

Sommerfeld problems with Dirichlet or Neumann conditions on Σ (and jump conditions on Σ') can be solved (in the present sense) for a wide class of plane screens like quarter-planes, cones or even polygonal-conical screens [CDS14] with rather sophisticated methods. At present it seems not possible to obtain corresponding results for problems with more general transmission conditions as shown in (1). The lifting of WHOs into L^2 spaces is already difficult for a quarter-plane or cone [DS92,DS93].

The minimal assumptions (i), (ii) on Σ mentioned in the introduction are really needed in general. Condition (i) is necessary for well-posedness of the IFP (otherwise data on $\overline{\Sigma} \cap \overline{\Sigma'}$ can not be determined uniquely) and (ii) is needed for the equivalent reduction to the semi-homogeneous problem. If Σ is bounded, the two assumptions are satisfied exactly for strong Lipschitz domains [HW08]. If Σ is unbounded, the situation is not so clear anymore.

Open problems: 2. Oblique derivatives

Admitting in (1) not only normal but also oblique derivatives, one obtains interface operators (9) with symbols which contain polynomial terms. Even for $m = 1$ the factorization problem is completely unsolved at present.

Open problems: 3. Sectoriality

Since A_λ acts in a symmetric space setting $X = Y = H^{1/2} \times H^{-1/2}$ (in contrast to the Sommerfeld type problems in [CDS14]) it is natural to apply the well-known idea of sectoriality [BoeSil06,S85]. However, this does not work directly for general screens, since the operator W in (19) is "highly asymmetric" with $P_1 \neq P_2$. Only in the case of a half-space Σ one can lift the operator into L^2-spaces (see Example 1) and then apply this idea.

Open problems: 4. Two-media problems

Based on the previous remark it is possible to tackle certain higher-dimensional problems as in [S85,S89], namely by considering the corresponding operator $T_{\lambda,\Sigma}$ (whose symbol contains now two different square root functions $t_j(\xi) = (\xi^2 - k_j^2)^{1/2}$) as a perturbation of an operator associated to a one-medium problem. In certain cases this leads to a solution by the help of an approximation by a Neumann series.

Acknowledgments

The work was supported by FCT - Portuguese Science Foundation through the Center for Functional Analysis and Applications at Instituto Superior Técnico, Universidade de Lisboa.

Thanks for your attention !

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