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Testing the remote control. . .

The "(x_n)" sequence

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Introduction

Linear difference equations are the discrete version of linear differential equations, and have a theory that parallels the latter.

Even here, the problems can be quite difficult. E.g., going from Fibonacci's

$$F_{n+1} = F_n + F_{n-1}$$

to, say,

$$F_{n+1} = \left(1 + \sin\frac{1}{n}\right) F_n + F_{n-1}$$

is highly nontrivial, c.f. Poincaré's theorem on linear difference equations.

Destroying linearity by, say,

$$F_{n+1} = F_n^2 + F_{n-1}$$

can lead to very interesting problems.

OPs $\stackrel{\text{def}}{=}$ Orthogonal Polynomials

Let α be a positive Borel measure with infinite support in $\mathbb{R}$ and let

$$p_n(\alpha, x) = \gamma_n(\alpha)x^n + \text{l.d.t.}, \qquad n \in \mathbb{N}_0 \stackrel{\text{def}}{=} \{0, 1, 2, \ldots\},$$

denote the OPs w.r.t. α.

(l.d.t. $\stackrel{\text{def}}{=}$ lower degree terms)

Orthogonality:

$$\int_{\mathbb{R}} p_m p_n \, d\alpha = \delta_{m,n}.$$

The three-term recurrence:

$$x p_n = a_{n+1} p_{n+1} + b_n p_n + a_n p_{n-1}$$

where $a_n = \gamma_{n-1}/\gamma_n > 0$ and $b_n \in \mathbb{R}$ “describe” the symmetry of the measure w.r.t. a vertical line.

Favard's theorem

THEOREM. (Favard, 1935) Given $(a_n > 0)$ and $(b_n \in \mathbb{R})$, if (p_n) satisfy the three-term recurrence

$$xp_n = a_{n+1}p_{n+1} + b_n p_n + a_n p_{n-1}$$

then they are OPs w.r.t. some, not necessarily unique, α in $\mathbb{R}$.

Hermite polynomials

The Hermite polynomials, well, the Hermite functions, are the eigenfunctions of the Fourier transform. So there can be no doubt that they are worthy of study.

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Note. IMHO means “in my humble opinion”.

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ANSWER. IMHO, it's a tie between eigenfunction and eigenvalue.

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In the next few slides I will show how to obtain certain properties of Hermite polynomials using "nothing" but orthogonality.

The very same ideas can be applied to much more "sophisticated" weight functions.

Hermite polynomials (h_n)

$$d\alpha(x) = w(x)dx \quad \text{where} \quad w(x) \stackrel{\text{def}}{=} \exp\left(-x^2\right), \qquad x \in \mathbb{R}.$$

Orthogonality:

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The three-term recurrence:

$$x h_n = \sqrt{\frac{n+1}{2}}\, h_{n+1} + \sqrt{\frac{n}{2}}\, h_{n-1}, \qquad n = 0, 1, 2, \ldots,$$

that is,

$$a_n = \sqrt{\frac{n}{2}} \quad (n = 2, 3, \ldots) \quad \& \quad b_n = 0 \quad (n = 0, 1, 2, \ldots)$$

and

$$a_0^2 = 0 \quad \& \quad a_1^2 = \frac{\int_{\mathbb{R}} x^2 \exp\left(-x^2\right)\, dx}{\int_{\mathbb{R}} \exp\left(-x^2\right)\, dx} = \frac{\Gamma(3/2)}{\Gamma(1/2)} = \frac{1}{2}.$$

a_n for h_n

The following tricky computation of (a_n) probably goes back to Shohat and then Freud, probably innocently, repeated the same argument; it is based on the the property of w that

$$w(x)' = -2\,x\,w(x) \quad \text{i.e.} \quad y' = -2\,x\,y.$$

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On one hand,

$$\int_{\mathbb{R}} (h_n h_{n-1})'\,w = \int_{\mathbb{R}} \left(h_n' h_{n-1} + h_n h_{n-1}'\right) w \overset{\text{orth}}{=} \int_{\mathbb{R}} h_n' h_{n-1} w =$$

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$$\int_{\mathbb{R}} \left(n\,\gamma_n\,x^{n-1} + \text{l.d.t.}\right) h_{n-1} w \overset{\text{orth}}{=} n\,\gamma_n \int_{\mathbb{R}} x^{n-1} h_{n-1} w \overset{\text{orth}}{=}$$

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$$n\,\frac{\gamma_n}{\gamma_{n-1}} \int_{\mathbb{R}} \left(\gamma_{n-1}\,x^{n-1} + \text{l.d.t.}\right) h_{n-1} w \overset{\text{orth}}{=} n\,\frac{\gamma_n}{\gamma_{n-1}} \int_{\mathbb{R}} h_{n-1}^2 w = \frac{n}{a_n}.$$

a_n for h_n

On the other hand, by integration by parts,

$$\int_{\mathbb{R}} (h_n h_{n-1})' \, w \overset{\text{ibp}}{=} -\int_{\mathbb{R}} h_n h_{n-1} w' = 2\int_{\mathbb{R}} h_n h_{n-1} x \, w \overset{\text{orth}}{=} 2a_n$$

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because, in general,

$$\int_{\mathbb{R}} x \, p_n p_{n-1} \, d\alpha \overset{\text{orth}}{=} \int_{\mathbb{R}} x \, p_n \left(\gamma_{n-1} \, x^{n-1} + \text{l.d.t.}\right) \, d\alpha \overset{\text{orth}}{=}$$

$$\frac{\gamma_{n-1}}{\gamma_n} \int_{\mathbb{R}} p_n \left(\gamma_n \, x^n + \text{l.d.t.}\right) \, d\alpha = \frac{\gamma_{n-1}}{\gamma_n} \int_{\mathbb{R}} p_n^2 \, d\alpha = \frac{\gamma_{n-1}}{\gamma_n} = a_n .$$

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So

$$\frac{n}{a_n} = \int_{\mathbb{R}} (h_n h_{n-1})' \, w = 2a_n .$$

b_n for h_n

The coefficient $b_n = 0$ because w is even. In general, from the recurrence formula,

$$b_n = \int_{\mathbb{R}} x\, p_n^2\, d\alpha.$$

From the so-to-speak Hermitian point of view,

$$b_n = \int_{\mathbb{R}} x\, h_n^2\, w = -\frac{1}{2}\int_{\mathbb{R}} h_n^2\, w' \overset{\text{ibp}}{=} \frac{1}{2}\int_{\mathbb{R}} \left(h_n^2\right)'\, w = \int_{\mathbb{R}} h_n\, h_n'\, w \overset{\text{orth}}{=} 0.$$

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HOMEWORK. Find the a_n's and the b_n's for the weight function $w(x) \overset{\text{def}}{=} \exp\left(-\beta_2\, x^2 + \beta_1\, x + \beta_0\right)$ with $x \in \mathbb{R}$ where $\beta_2 > 0$.

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HOMEWORK. What if $x \in \mathbb{R}^+$ for w above instead of $x \in \mathbb{R}$?

DE for h_n

Let $k < n-1$ & let r_k be a polynomial of degree at most k. Then

$$\int_{\mathbb{R}} h_n' \, r_k \, w \overset{\text{orth}}{=} \int_{\mathbb{R}} (h_n \, r_k)' \, w \overset{\text{ibp}}{=} -2 \int_{\mathbb{R}} (h_n \, r_k) \, x \, w =$$

$$-2 \int_{\mathbb{R}} h_n \, (r_k \, x) \; w \overset{\text{orth}}{=} 0$$

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so that

$$h_n' = \text{const} \times h_{n-1}.$$

Thus, we have an Appell sequence, and by comparing leading coefficients

$$h_n' = \sqrt{2\, n}\, h_{n-1}$$

and, therefore,

$$h_n'' = \sqrt{2\, n}\, h_{n-1}' = \sqrt{2\, n}\sqrt{2\, (n-1)}\, h_{n-2} = 2\sqrt{n(n-1)}\, h_{n-2}.$$

DE for h_n

Replacing h_{n-1} and h_{n-2} by

$$h'_n = \sqrt{2\,n}\,h_{n-1} \quad \& \quad h''_n = 2\sqrt{n(n-1)}\,h_{n-2}$$

in the recurrence formula

$$x h_{n-1} = \sqrt{\frac{n}{2}}\,h_n + \sqrt{\frac{n-1}{2}}\,h_{n-2}, \qquad n = 0, 1, 2, \ldots,$$

we get the differential equation for the Hermite polynomials

$$2\,x\,y' = 2\,n\,y + y'', \qquad y = h_n.$$

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HOMEWORK. Using “nothing” but orthogonality, find the Rodrigues’ formula for h_n. Well, first google “rodrigues formula”.

$\exp\left(-c/4\,x^4 - K/2\,x^2\right)$

Let

$$w(x) \stackrel{\text{def}}{=} \exp\left(-\frac{c}{4}\,x^4 - \frac{K}{2}\,x^2\right), \qquad x \in \mathbb{R},$$

where $c > 0$ & $K \in \mathbb{R}$ (interesting case) or $c = 0$ & $K \in \mathbb{R}^+$. Then the weight w satisfies a DE similar to the Hermite weight:

$$w(x)' = \left(-c\,x^3 - K\,x\right)\,w(x) \quad \text{i.e.} \quad y' = -\left(c\,x^3 + K\,x\right)\,y.$$

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Note.

$$(-\log w)'' = 3cx^2 + 2K$$

so that

$$-\log w \text{ is convex } \iff K \geq 0.$$

It is unknown if convexity matters but the case $K < 0$ is "orders of magnitude" harder than $K \geq 0$.

The recurrence coefficients

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The b_n's are all 0 (even w). The a_n's can be computed using exactly the same idea as for the Hermite case, albeit it is a little more complicated because the recurrence formula must be used three times on the right-hand side. We get

$$\frac{n}{a_n} = c\, a_n \left(a_{n+1}^2 + a_n^2 + a_{n-1}^2\right) + K\, a_n, \qquad n \in \mathbb{N},$$

where

$$a_0^2 = 0 \quad \& \quad a_1^2 = \frac{\int_{\mathbb{R}} x^2 \exp\left(-\frac{c}{4}\, x^4 - \frac{K}{2}\, x^2\right)\, dx}{\int_{\mathbb{R}} \exp\left(-\frac{c}{4}\, x^4 - \frac{K}{2}\, x^2\right)\, dx}\,.$$

For $K = 0$,

$$a_1^2 = \frac{\Gamma\left(3/4\right)}{\Gamma\left(1/4\right)}\,.$$

Note. By their very nature, all the a_n's are positive for $n \in \mathbb{N}$.

The (a_n)'s

The case $K = 0$ was found by Shohat in 1939 and by Freud in the early 1970s, and the general case by Stan Bonan and PN in 1984. Neither Shohat nor Freud studied it except that Fred proved the following in 1976:

$$w(x) = \exp\left(-x^4/4\right) \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{a_n}{n^{1/4}} = \frac{1}{3^{1/4}}$$

It turned out that his completely elementary ad hoc technique became a successful tool for proving limits in similar situations.

The Freud Kunstgriff

We have

$$n = a_n^2 \left(a_{n+1}^2 + a_n^2 + a_{n-1}^2 \right) \quad \Longrightarrow \quad a_n^4 \le n$$

and then

$$0 \le \ell \stackrel{\text{def}}{=} \liminf_{n\to\infty} \frac{a_n^2}{\sqrt{n}} \le L \stackrel{\text{def}}{=} \limsup_{n\to\infty} \frac{a_n^2}{\sqrt{n}} < \infty$$

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so that dividing both sides by n and going to $\liminf$ with $a_n^2/\sqrt{n}$

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and going to $\limsup$ with $a_n^2/\sqrt{n}$

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from which

$$L^2 + 2L\ell \leq \ell^2 + 2L\ell \quad \Longrightarrow \quad L \leq \ell \quad \Longrightarrow \quad L = \ell. \qquad \blacksquare$$

The Freud conjecture

Based on this, Freud made the following conjecture in 1976 that fundamentally shaped the theory of orthogonal polynomials for the next 30+ years.

$$w(x) = \exp\left(-|x|^\alpha\right) \quad \Longrightarrow \quad \exists \lim_{n\to\infty} \frac{a_n}{n^{1/\alpha}} \in (0,\infty)$$

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The conjecture has been solved but that's another story; google "freud conjecture".

$\exp\left(-x^4\right)$

Returning to $w(x) = \exp\left(-x^4\right)$, Shohat (1939) & Bonan (1983) proved that (p_n) form a generalized Appel sequence:

$$p_n' = \frac{n}{a_n}\,p_{n-1} + 4\,a_n\,a_{n-1}\,a_{n-2}\,p_{n-3},$$

and Shohat (1939) & I (1984) found the DE

$$z'' + f_n z = 0$$

where

$$z \stackrel{\text{def}}{=} p_n\sqrt{\frac{w}{\varphi_n}} \qquad \text{with} \qquad \varphi_n(x) \stackrel{\text{def}}{=} a_{n+1}^2 + a_n^2 + x^2$$

and

$$f_n(x) \stackrel{\text{def}}{=} 4\,a_n^2\left[4\,\varphi_{n-1}(x)\varphi_n(x) + 1 - 4\,a_n^2\,x^2 - 4\,x^4 - 2\,x^2\,\varphi_n^{-1}(x)\right] + 6\,x^2 - 4\,x^6 + (1 - 4\,x^4)\varphi_n^{-1}(x) - 3\,x^2\,\varphi_n^{-2}(x).$$

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$$+ 6\, x^2 - 4\, x^6 + (1 - 4\, x^4)\varphi_n^{-1}(x) - 3\, x^2\, \varphi_n^{-2}(x).$$

The (x_n)'s

Let $x_n \stackrel{\text{def}}{=} a_n^2$. Then

$$\frac{n}{c} = x_n\,(x_{n+1} + x_n + x_{n-1}) + \frac{K}{c}\,x_n, \qquad x_0 = 0, \quad n \in \mathbb{N},$$

where

$$x_1 = \frac{\int_{\mathbb{R}} x^2 \exp\left(-\frac{c}{4}\,x^4 - \frac{K}{2}\,x^2\right)\,dx}{\int_{\mathbb{R}} \exp\left(-\frac{c}{4}\,x^4 - \frac{K}{2}\,x^2\right)\,dx}\,.$$

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From another point of view and with a different interpretation of facts: $\exists x_1 > 0$ such that if (x_n) is defined by

$$\frac{n}{c} = x_n\,(x_{n+1} + x_n + x_{n-1}) + \frac{K}{c}\,x_n, \qquad x_0 = 0, \quad n \in \mathbb{N}, \quad (*)$$

then

$$x_n > 0, \qquad \forall n \in \mathbb{N}.$$

Note. $(*)$ is no longer associated with the original OPs.

The (x_n)'s; WLOG $c = 1$

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Confession. Okay, I admit that this French mathematician was, in fact, a German living in France and thinking like a Frenchman (Bernhard Beckermann).

Generalizations

The equation

$$n = x_n\,(x_{n+1} + x_n + x_{n-1}) + K\,x_n$$

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- Go to several dimensions. . .

Each of them leads to a principal problem: no explicit solutions exist.

A theorem and a proof

Theorem. $\forall K \geq 0$, if $x_0 \in \mathbb{R}$, then $\exists! x_1 > 0$ such that $(x_n)_{n\in\mathbb{N}}$ defined by

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Note. If $x_0 = 0$, then the existence of a positive solution follows from the explicit solution coming from $\exp\left(-x^4/4 - Kx^2/2\right)$. Otherwise, it can be proved using either some fixed point arguments or by constructing positive sequences $(x_n)_{n=1}^{N}$ satisfying the equation up to N and then letting $N \to \infty$ and justifying that the latter is legitimate.

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In what follows, I will prove the uniqueness part that I found back in 1983 and still think as one of the most beautiful proofs I ever dreamt up.

Uniqueness

If $K \geq 0$ and (x_n) is positive, then from

$$n = x_n\left(x_{n+1} + x_n + x_{n-1}\right) + K\,x_n, \quad n \in \mathbb{N}, \qquad (\aleph)$$

we have

$$n \geq x_n^2, \quad n \in \mathbb{N}.$$

Let (α_n) and (β_n) be two positive solutions of $(\aleph)$, that is, of

$$\frac{n}{x_n} = x_{n+1} + x_n + x_{n-1} + K, \quad n \in \mathbb{N},$$

and let $\varepsilon_n \overset{\text{def}}{=} \beta_n - \alpha_n$, Then,

$$-\frac{n}{\alpha_n \beta_n}\varepsilon_n = \frac{n}{\beta_n} - \frac{n}{\alpha_n} = \varepsilon_{n+1} + \varepsilon_n + \varepsilon_{n-1}, \quad n \in \mathbb{N},$$

(K is gone), that is,

$$\varepsilon_{n+1} + \left(1 + \frac{n}{\alpha_n \beta_n}\right)\varepsilon_n + \varepsilon_{n-1} = 0, \quad n \in \mathbb{N}.$$

Uniqueness

Since $\frac{n}{\alpha_n \beta_n} \geq 1$, we get

$$2\left|\varepsilon_n\right| \leq \left|\varepsilon_{n+1}\right| + \left|\varepsilon_{n-1}\right|, \quad n \in \mathbb{N}.$$

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so that,

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that is, for $1 \leq M \leq N$,

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i.e.,

$$\left|\varepsilon_M\right| - \left|\varepsilon_{M-1}\right| \leq \left|\varepsilon_{N+1}\right| - \left|\varepsilon_N\right|, \quad 1 \leq M \leq N.$$

We are getting there…

Uniqueness

From

$$|\varepsilon_M| - |\varepsilon_{M-1}| \leq |\varepsilon_{N+1}| - |\varepsilon_N|\,, \quad 1 \leq M \leq N,$$

we get

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that is,

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so that, letting $P \to \infty$,

$$|\varepsilon_M| - |\varepsilon_{M-1}| \leq 0, \qquad 1 \leq M.$$

Uniqueness

Therefore, $(|\varepsilon_M|)_{M=1}^{\infty}$ is a decreasing nonnegative sequence, and, since $\varepsilon_0 = \beta_0 - \alpha_0 = x_0 - x_0 = 0$, we have $\varepsilon_M \equiv 0$ for all $M \in N$, that is, $\beta_M = \alpha_M$ for all $M \in N$.

■

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Explanation of difficulties

This theorem has several proofs; all are more or less elementary.

Theorem. Let $x_0 = 0$ and $K \geq 0$. Then $\exists ! x_1 > 0$ such that $(x_n)_{n\in\mathbb{N}}$ defined by

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Why? I have only intuitive reasons related to the convexity of

$$-\log \exp \left(-x^4/4 - K x^2/2 \right).$$

Progress report

Theorems for both the existence and uniqueness of positive solutions have been extended to

$$\ell_n = x_n \left(\sigma_{n,1}\, x_{n+1} + \sigma_{n,0}\, x_n + \sigma_{n,-1}\, x_{n-1}\right) + \kappa_n\, x_n$$

recently under rather mild conditions of the coefficients except that although κ_n can be negative, we must have $\liminf \kappa_n \geq 0$ in the uniqueness results. I am sure that this limitation is due rather to the methods used and not to the nature of the equation.

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Even probably $\mathrm{O}\left(\sqrt{n}\right)$ solutions are unique.

Das Ende