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Testing the remote control. . .

Curriculum Vitae

Born: on Stalin Blvd, Budapest, 19xx

College: Leningrad State University, USSR, 1966–1971

Last weekend: a hotel on Leningrader Straße in Dresden

Currently: Karl-Marx-Stadt, Deutsche Demokratische Republik

Home: Upper Arlington, Ohio, U.S.A.

Oops, needs an update . . .

Curriculum Vitae; final (???) version

Born: on Andrássy Blvd, Budapest, 19xx

College: Saint Petersburg State University, Russia, 1966–1971

Last weekend: a hotel on St. Petersburger Straße in Dresden

Currently: Chemnitz, Bundesrepublik Deutschland

Home: Upper Arlington, Ohio, U.S.A.

A Potpourri of OPs*

(a subjective & opinionated discourse)

Paul Nevai

paul@nevai.us

(telecommuting to)

King Abdulaziz University

Jeddah, The Kingdom of Saudi Arabia

(but living and working in Columbus, Ohio, USA)

*OPs $\stackrel{\text{def}}{=}$ Orthogonal Polynomials. Potpourri comes from the word putrid via French & Latin.

Dedication

First, let me dedicate this talk to Gerhard Riege who was my father's "best" friend. As a 15 year old boy, I visited Gerhard and his family in Jena in 1963 and spent a great Summer there. This was my one and only visit to East(ern) Germany up until now.

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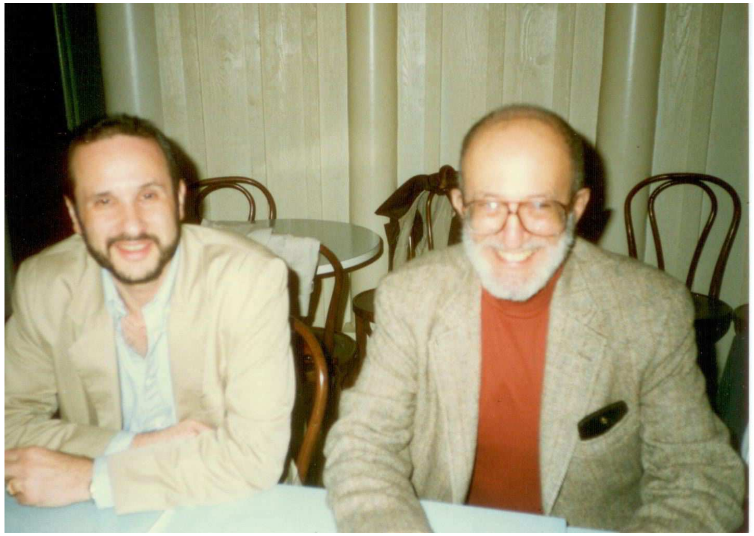
Gerhard was the Rektor of Universität Jena who, after the unification of the two Germanies became a member of the Bundestag representing the Partei des Demokratischen Sozialismus. When the Stasi files were opened up and his (minor) collaboration with it became public, he committed suicide in 1992. Yet another chapter in Germany's tragic (but self-inflicted) history in the twentieth century.

`http://de.wikipedia.org/wiki/Gerhard_Riege`

Figure 1: G. Riege, 1930–1992

Paul Halmos, 1916–2006

According to Paul Halmos, all talks must contain a proof.



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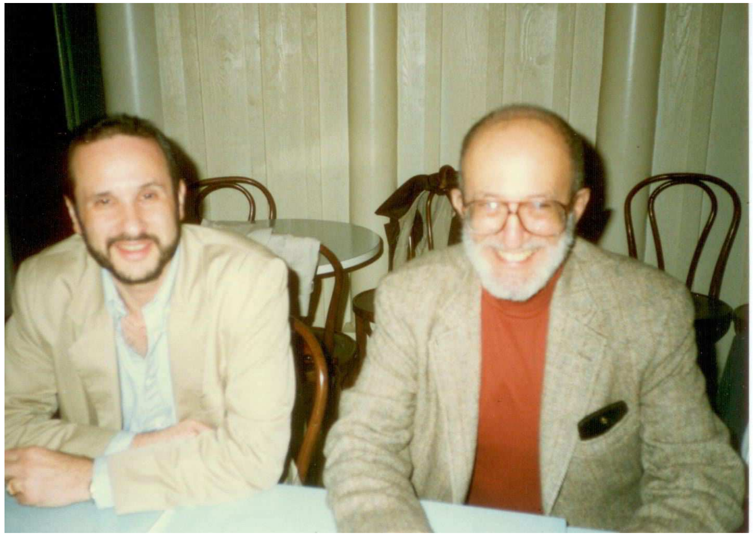
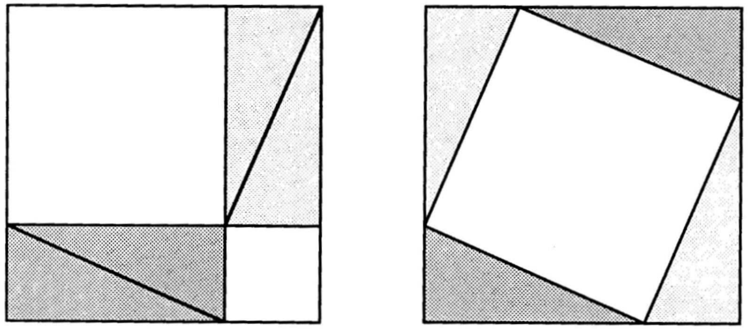


So let's get it over with; here is the 2-dimensional version of Riesz-Fisher; cf. Proofs without Words by R. B. Nelsen.



OPs

Let $\mathbb{P}_n$ denote the set of polynomials of degree at most $n-1$ (sorry for the "$n-1$") with $n \in \mathbb{N}$.
Given a finite positive Borel measure α with infinite support in, say, $\mathbb{C}$, consider the L^2 extremal problem

$$\frac{1}{\gamma_n(d\alpha)} \stackrel{\text{def}}{=} \left\{ \min_{Q \in \mathbb{P}_n} \int |t^n + Q(t)|^2 \, d\alpha(t) \right\}^{\frac{1}{2}} .$$

Then there is a unique polynomial $Q^\#$ that minimizes the right-hand side. Let $p_n(d\alpha, x) = \gamma_n x^n + Q^\#(x)$. Then, as it turns out and is easily verifiable, the polynomials in the sequence $(p_n(d\alpha))$ are *orthogonal polynomials* (OPs) w.r.t. α, that is,

$$\int p_m \overline{p_n} \, d\alpha = \delta_{mn} , \qquad m, n \in \mathbb{N}.$$

OPs

In this general setting, the theory is rather under-studied, under-developed, under-understood, and under-published, since $\mathbb{C}$ doesn't possess certain properties that allow to capitalize on the orthogonality property to obtain fundamental algebraic and analytic properties of OPs.

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On the other hand, two special subsets of $\mathbb{C}$, namely the real line $\mathbb{R}$ and the unit circle $\mathbb{D}$ lead us the beautiful theories.

Whereas one can associate George (György) Pólya as the father and Gábor Szegő as the mother of the latter (called OPUC)*, the former has way too many potential fathers and mothers to even try to establish paternity and maternity.

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Briefly, the magic properties are that in $\mathbb{R}$ inner products lack conjugation, and one has $\overline{z} = 1/z$ on $\mathbb{D}$.

*Tell story about Pólya and Szegő.

Gábor Szegő, 1895–1985

(Kunhegyes, Hungary, and also in St. Louis & Palo Alto, USA)

Pólya-Szegő, $\max(1887, 1895)$–1985

(Berlin, 1925; from *The Pólya Picture Album*)

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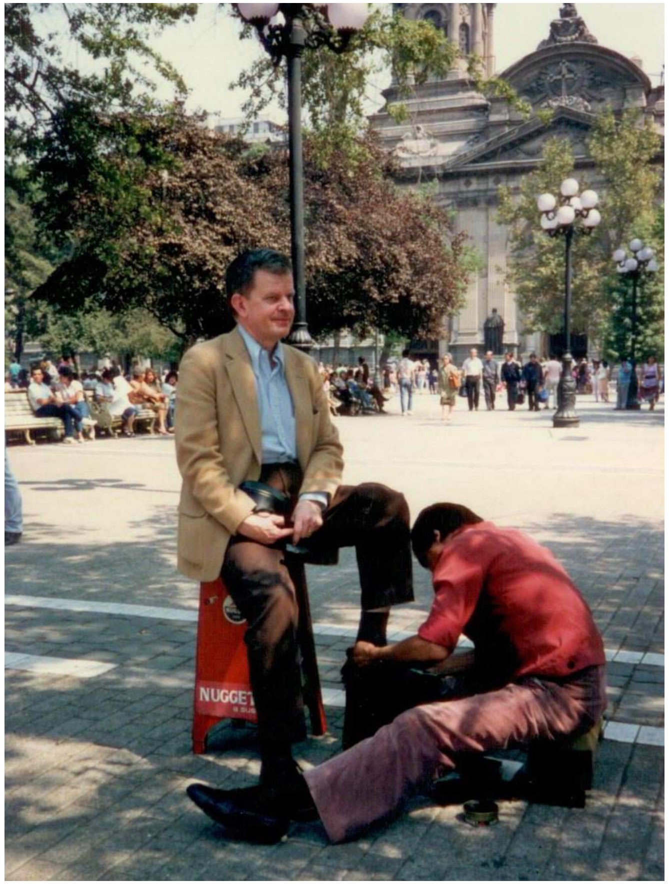
- A crucial step in solving bieberbach's* conjecture was the use of an inequality of Askey-Gasper on Jacobi polynomials.

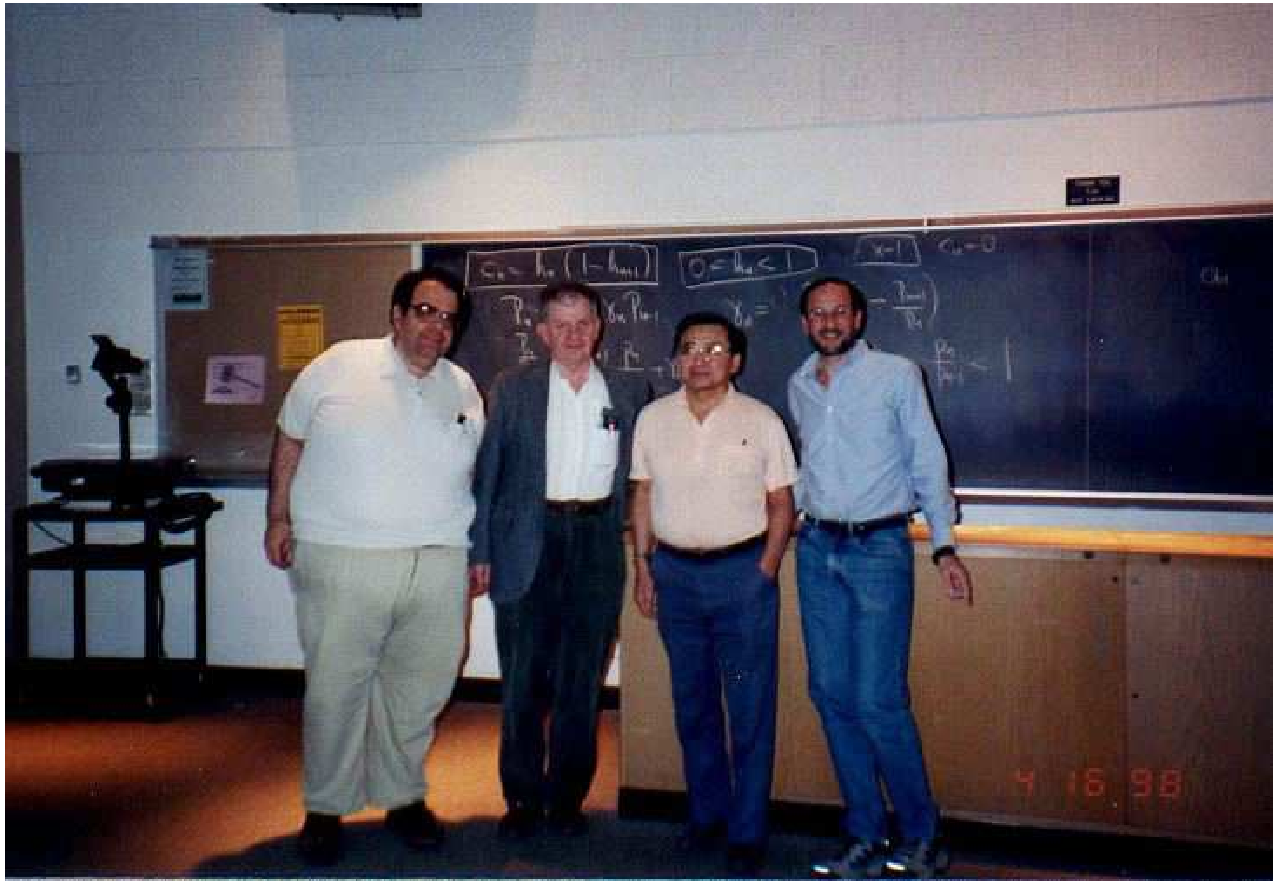
*nazis don't deserve to be capitalized.

Richard Allen Askey, 1933–2053



NOTE. The ugly American's shoes buffed by the Chilean proletariat; Santiago de Chile, March, 1989.

Mourad El-Houssieny Ismail, 1944–2064



NOTE. Ismail-Askey-Chihara-Nevai, April, 1998.

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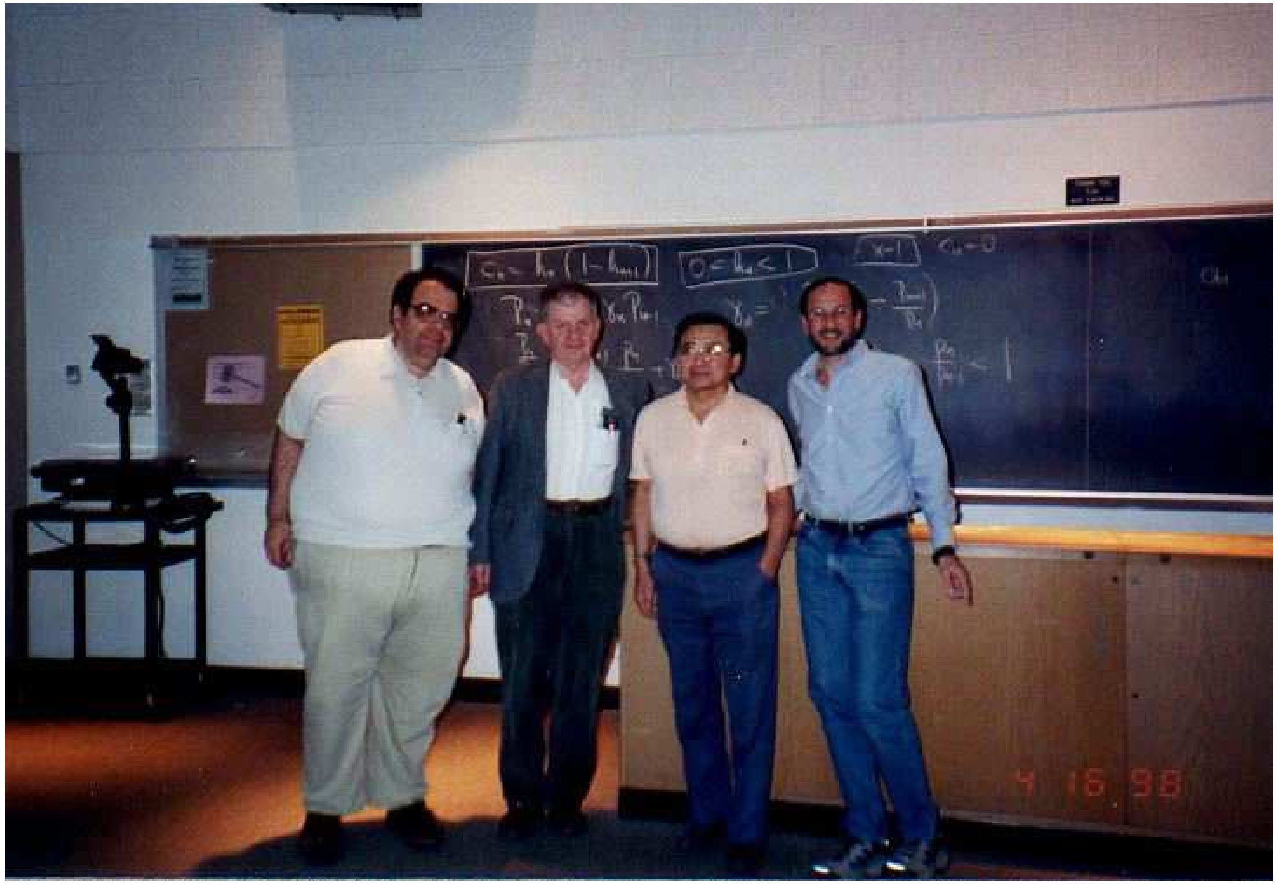

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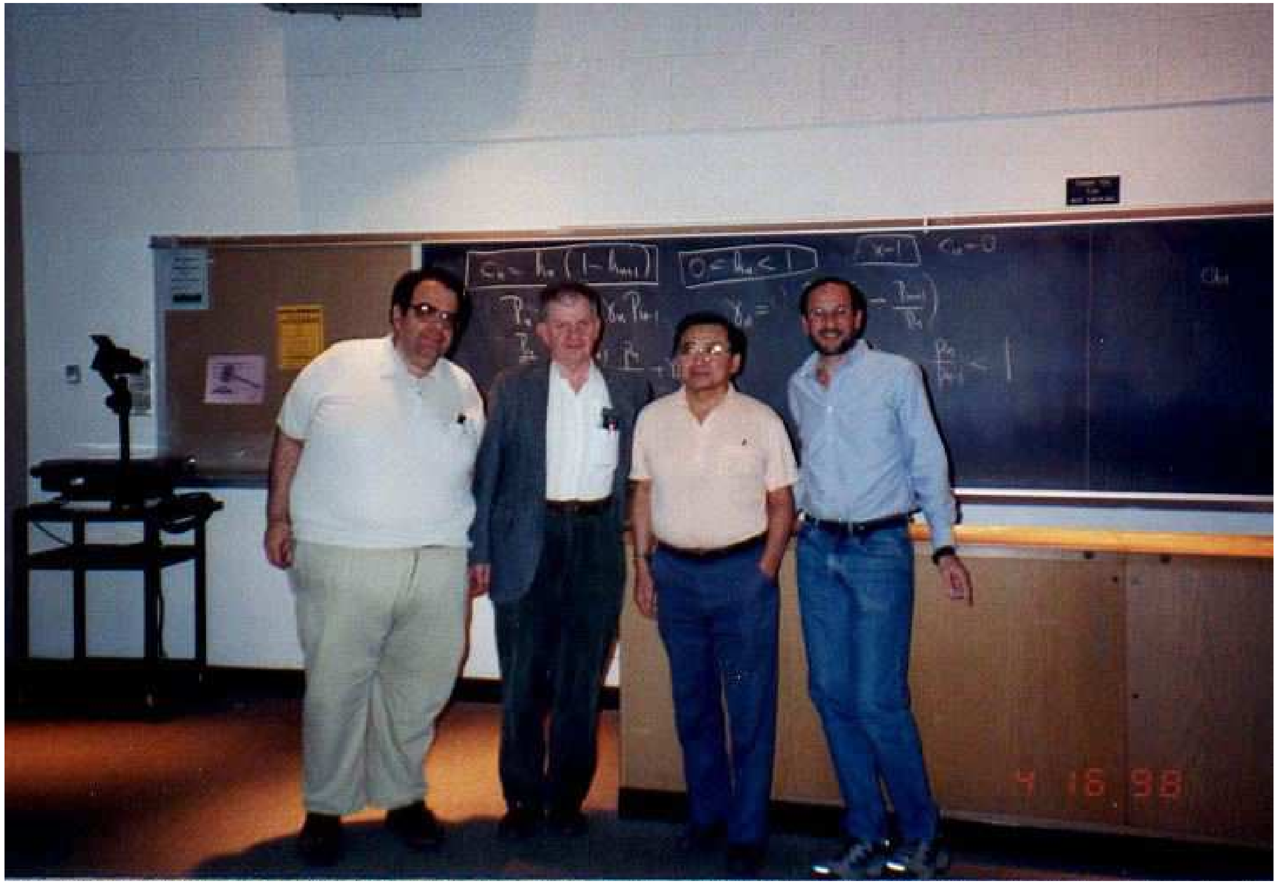

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NOTE. Scary, isn't it?

Why OPs?

- Extremal problems
- Moment problem
- Continued fractions
- Jacobi matrices
- Toeplitz matrices
- Multiplication operator
- Hankel matrices
- Hessenberg matrices
- Random matrices
- Representation theory

History of OPs

Here is a very personal, very one-sided, and very arguable history of OPs.

brute force $\Longrightarrow$ special functions $\Longrightarrow$ real analysis $\Longrightarrow$ complex analysis $\Longrightarrow$ continued fractions $\Longrightarrow$ linear algebra $\Longrightarrow$ harmonic analysis $\Longrightarrow$ operator theory $\Longrightarrow$ scattering theory $\Longrightarrow$ difference equations $\Longrightarrow$ potential theory $\Longrightarrow$ matrix theory $\Longrightarrow$ Lax–Levermore theory $\Longrightarrow$ Riemann–Hilbert methods $\Longrightarrow$ spectral analysis

Of course, there is a huge overlap, mixing, and multiplicity.

From now on,

α is supported in $\mathbb{R}$ and

$\mathrm{supp}(\alpha)$ is an infinite set.

CFs

Given a monic polynomial Q of degree n, its reverse, $x^n Q(1/x)$ is 1 at 0, so it is natural to view Q as being 1 at ∞. Hence, there comes the natural generalization of the extremal problem to

$$\lambda_n(d\alpha, x) \stackrel{\text{def}}{=} \min_{\substack{P \in \mathbb{P}_n \\ P(x)=1}} \int |P|^2 \, d\alpha \,, \qquad x \in \mathbb{C}.$$

This λ_n is called the Christoffel function. It can be expressed in terms of the OPs as

$$\lambda_n(d\alpha, x) = \frac{1}{\sum_{k=0}^{n-1} \left| p_k^2(d\alpha, x) \right|}.$$

NOTE. The term Christoffel function probably originates from Géza Freud (1971?) although the terminology Christoffel number is older (Szegő in 1939?); I found Christoffel coefficients in V. L. Goncharov's 1934 book (in Russian).

Elwin Bruno Christoffel, 1829–1900

(from www-history.mcs.st-and.ac.uk)

CFs

The unique extremal polynomial is

$$\frac{K_n(d\alpha, x, \cdot)}{K_n(d\alpha, x, x)}$$

where K_n is the reproducing kernel, that is,

$$K_n(d\alpha, x, \cdot) = \sum_{k=0}^{n-1} \overline{p_k(d\alpha, x)} p_k(d\alpha, \cdot).$$

As it turns out, for all $x \in \mathbb{R}$,

$$((x - \cdot) K_n(d\alpha, x, \cdot))_{n=1}^{\infty}$$

are also OPs (not normalized), alas with the wrong degree; they are called quasi-OPs and they play an important role in Marcel Riesz's approach to the moment problem.

Marcel Riesz, 1886–1969

(from www-history.mcs.st-and.ac.uk)

Historical remarks

I consider 1814 the starting point for OPs when Johann Carl Friedrich Gauß, in his *Methodus nova integralium valores per approximationem inveniendi*, proved that if α is the Lebesgue measure in $[-1, 1]$, and if (x_{kn}) are the roots of the corresponding OPs (Legendre), then for all polynomials $P \in \mathbb{P}_{2n}$, one has the (Gauß-Jacobi) quadrature formula

$$\int_{\mathbb{R}} P\,d\alpha = \sum_{k=1}^{n} P(x_{kn})\lambda_n(x_{kn})$$

NOTE. The significance of this formula is that it's “obvious” for $P \in \mathbb{P}_n$ and it no longer holds for all $P \in \mathbb{P}_{2n+1}$.

NOTE. Of course, OPs themselves go back way before Gauß, see, e.g., Legendre (1782).

Historical remarks

General Theory

- Carl Gustav Jacob Jacobi, 1804–1851.
- Pafnuty Lvovich Chebyshev, 1821–1894.
- Jean Gaston Darboux, 1842–1917.
- Thomas Joannes Stieltjes, 1856–1894.
- Andrey Andreyevich Markov, 1856–1922.
- Felix Hausdorff, 1868–1942.
- Hans Ludwig Hamburger, 1889–1956.
- The Hungarians, the Russians (Soviets), the Americans, the Spaniards, the Italians, the Germans, the Arabs, the Chinese. . .

Algebraic properties

- Zeros of p_n are real, simple, and are in the convex hull of $\mathrm{supp}(\alpha)$.
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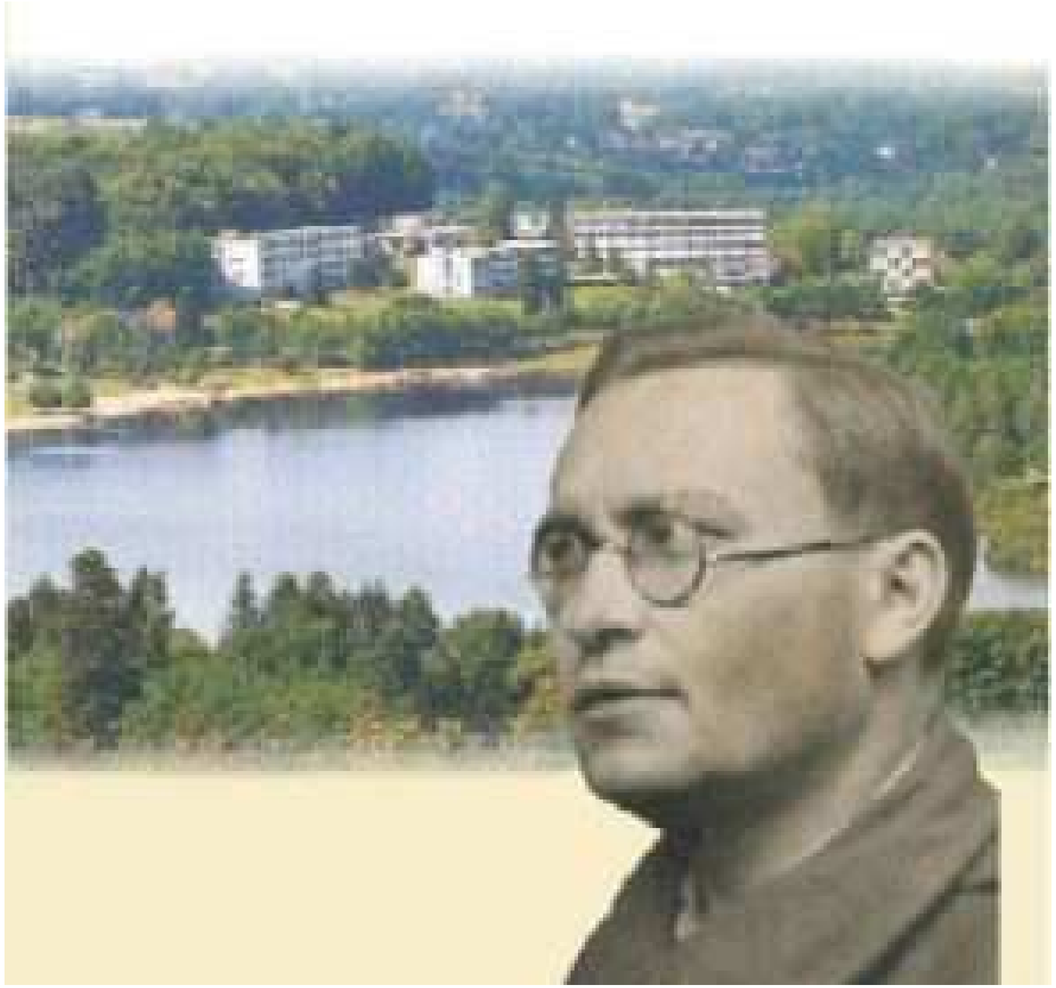
THEOREM. (Favard, 1935) Given $(a_n > 0)$ and $(b_n \in \mathbb{R})$, if (p_n) satisfy the three-term recurrence, then they are OPs w.r.t. some α in $\mathbb{R}$.

NOTE. Whether or not the above measure is unique is a totally different ball game.

Jean Favard, 1902–1965



(from Lycée Jean Favard)

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NOTE. $\exists$ close relationship to (discrete) scattering theory.

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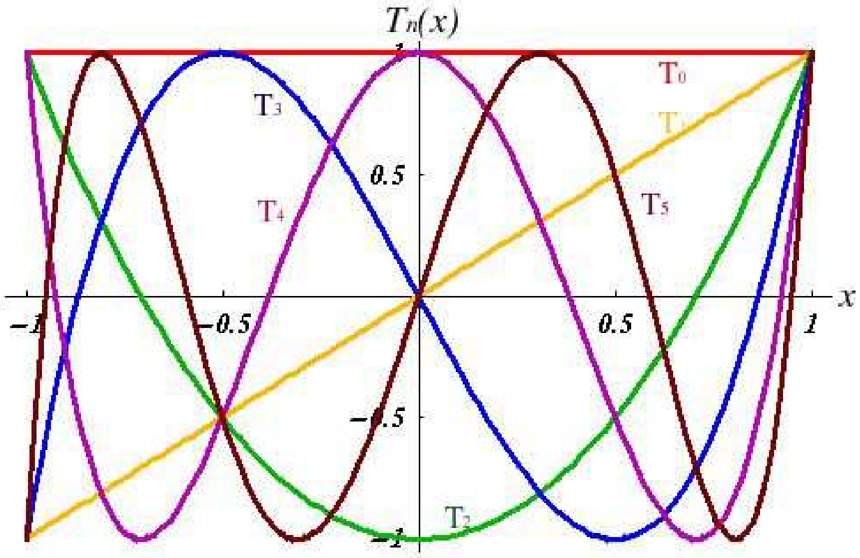

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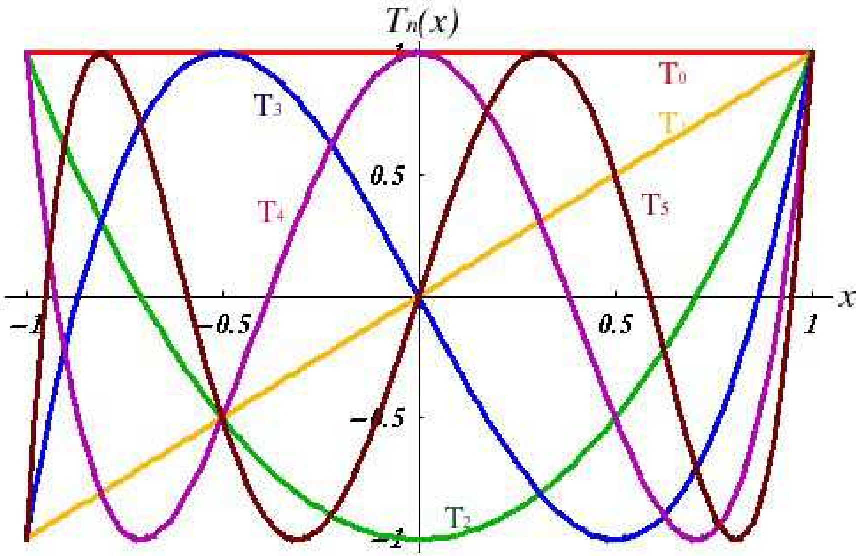

• The eigenfunctions of the Fourier transform are Hermite OPs multiplied by $\exp\left(-\frac{x^2}{2}\right)$. PEIWK.

More examples

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• $a_n \equiv 1$ and $b_n \equiv 0$ gives the second kind Chebyshev polynomials in $[-2, 2]^*$. PEIWK.

• $a_1 \neq 1$ but $a_n \equiv 1$ for all $n > 1$ and $b_n \equiv 0$. The fun begins. The OPs are linear combos of first and second kind Chebyshev polynomials. PEIWK. In particular, there might be a unique point outside $[-2, 2]$ where the OPs are in ℓ^2.

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More examples

- $a_n = 1 + \frac{C}{n^2}$ ($C < 0$) and $b_n \equiv 0$. Practically nothing is well-known, although quite a lot is known. For instance, $\mathrm{supp}(\alpha) = [-2, 2]$, α is absolutely continuous in $(-2, 2)$ but not necessarily at ± 2, and α' is positive & continuous in $(-2, 2)$. This is already quite serious math, i.e., TIAQSM.

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*I forgot the exact value of C^* but it is known; ask Ted or Mourad.

More examples (cont.)

In the last two examples, there are $a \in \mathbb{R}$ and $\mathrm{const} > 0$ such that

$$\alpha'(x) > \mathrm{const}\left(4 - x^2\right)^a , \qquad x \in (-2, 2)$$

(α is *super-Jacobi* or *super-Gegenbauer*).

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A. You decide. I say it's a tie. The mystery person is Gábor Szegő.

Fibonacci

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$$xp_n = a_{n+1}p_{n+1} + b_np_n + a_np_{n-1}$$

or

$$a_{n+1}p_{n+1} = (x - b_n)p_n - a_np_{n-1}$$

or

$$P_{n+1} = (x - b_n)P_n - a_n^2P_{n-1}$$

where P_n is the monic version of p_n.

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Fibonacci:

$$F_{n+1} = F_n + F_{n-1}, \qquad F_0 \stackrel{\text{def}}{=} 0 \quad \& \quad F_1 \stackrel{\text{def}}{=} 1.$$

No wonder that they might be related by a general theory. Indeed, they are. Namely, by the theory of higher order homogeneous linear difference equations with variable coefficients.

Fibonacci

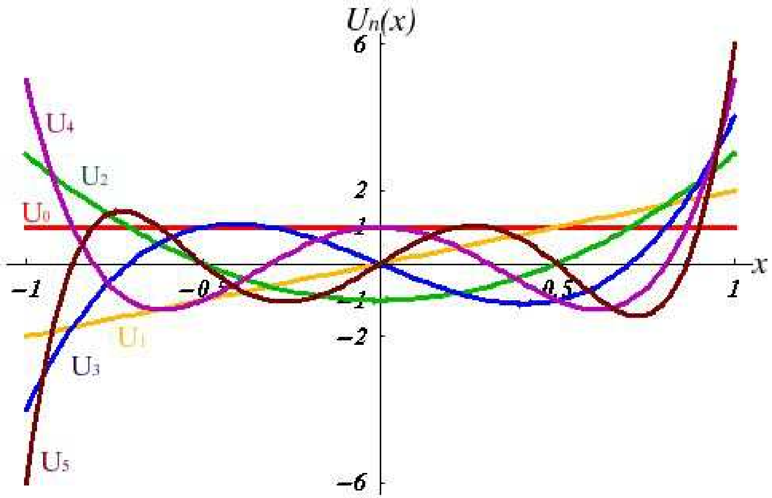
Interesting formula:

$$F_n = \frac{1}{i^{n-1}} U_{n-1}\left(\frac{i}{2}\right), \qquad i \stackrel{\text{def}}{=} \exp(0.5i\pi),$$

where

$$U_n(x) = \frac{\sin\left((n+1)\theta\right)}{\sin\theta}, \quad x = \cos\theta, \quad x \in [-1,1],$$

is the second kind Chebyshev polynomial which is orthogonal in $[-1,1]$ w.r.t. to the weight function $\sqrt{1-x^2}$; cf. Ted Rivlin's book on *Chebyshev polynomials*, p. 61.



2nd kind Chebyshev $\Longrightarrow$ Fibonacci

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta,$$

so that

$$U_{-1}(x) = 0 \quad \& \quad U_0(x) = 1 \quad \& \quad U_1(x) = 2\,x$$

and by $\sin(n\theta \pm \theta) = \ldots$

$$U_{n+1}(x) = 2\,x\,U_n(x) - U_{n-1}(x)$$

or

$$U_{n+1}(x/2) = x\,U_n(x/2) - U_{n-1}(x/2)$$

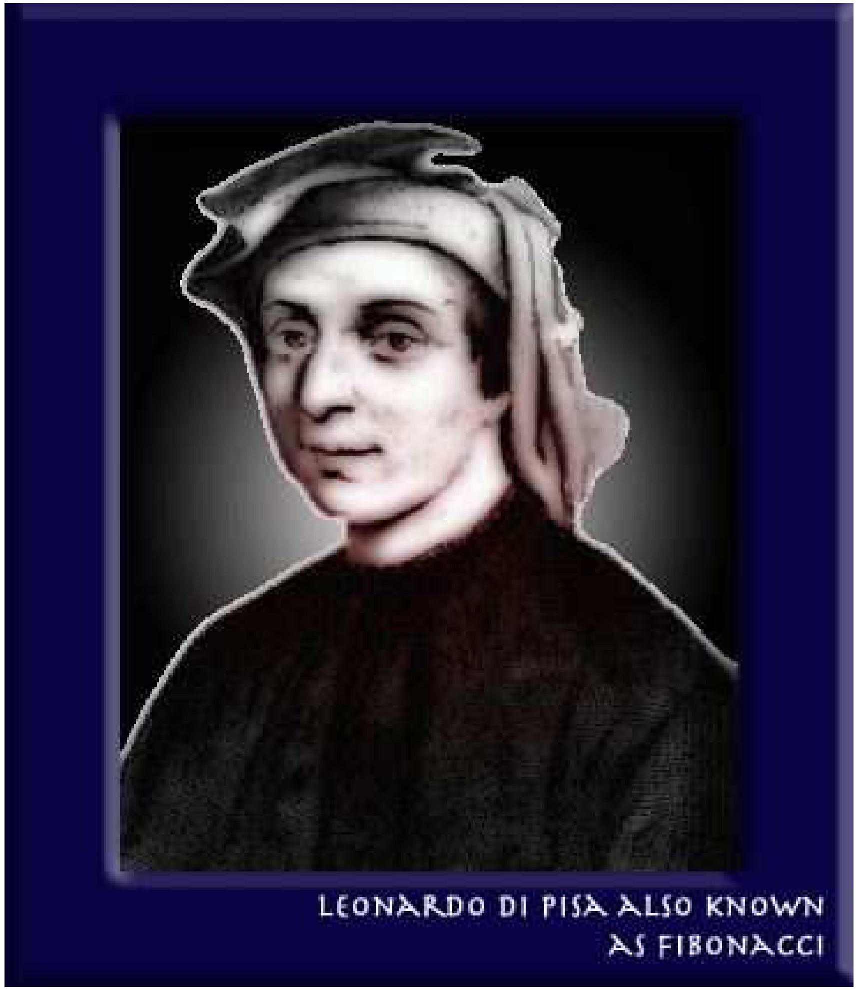
or

$$\frac{U_{n+1}(x/2)}{i^{n+1}} = \frac{x}{i}\,\frac{U_n(x/2)}{i^n} - \frac{1}{i^2}\,\frac{U_{n-1}(x/2)}{i^{n-1}}$$

so that

$$\frac{U_{n+1}(i/2)}{i^{n+1}} = \frac{U_n(i/2)}{i^n} + \frac{U_{n-1}(i/2)}{i^{n-1}}$$

Leonardi di Pisa, 1170–1250



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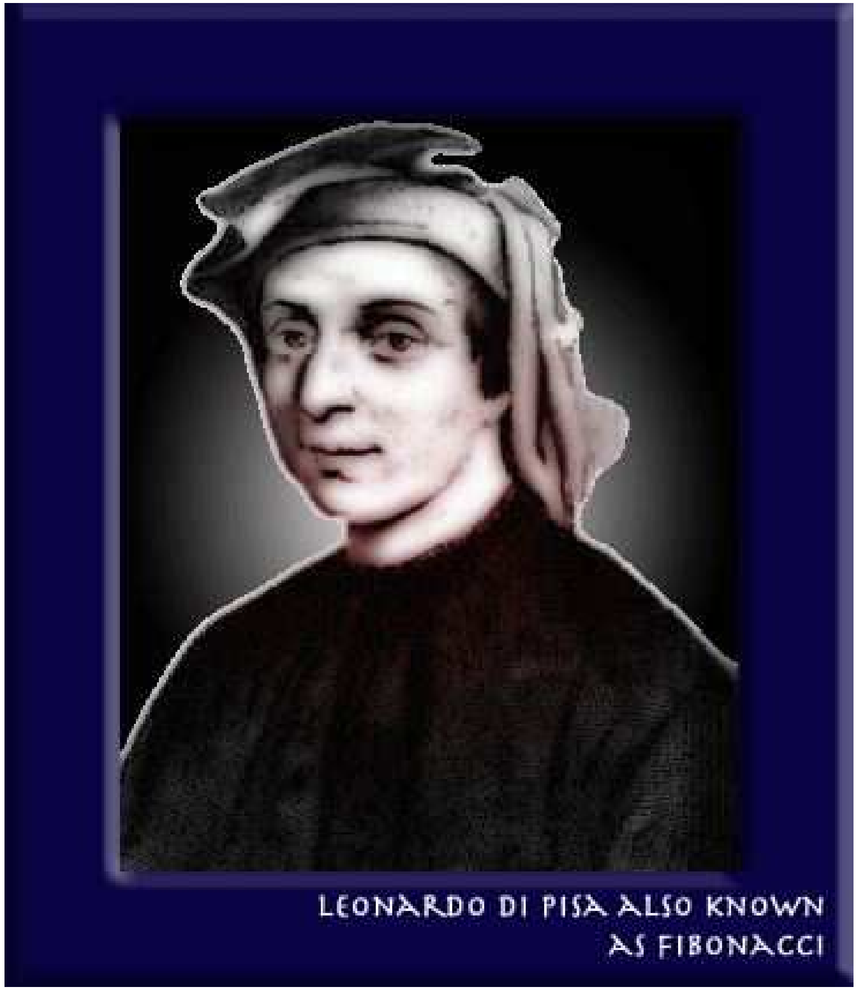

(from www.mingl.org/matematika/people)

A puzzle

The boy mathematician tells the girl mathematician

I love you.

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The boy mathematician tells the girl mathematician

I love you.

The girl mathematician dumps the boy mathematician.

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Question. Why?

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The girl mathematician dumps the boy mathematician.

Question. Why?

Answer. Because he should have said

I love you and only you.

Poincaré's marvelous theorem

THEOREM. Given $k > 0$, suppose that $(f_n)_{n=1}^{\infty}$ satisfies

$$f(n+k) + \sum_{j=0}^{k-1} a_{jn} f(n+j) = 0$$

where the limits $\lim_{n\to\infty} a_{jn} = a_j$, $0 \le j \le k-1$, exist, and the roots, say, $\zeta_1, \ldots, \zeta_k$, of the limiting characteristic equation

$$z^k + \sum_{j=0}^{k-1} a_j z^j = 0$$

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(see Henri Poincaré's 1885 paper titled *Sur les équations linéaires aux différentielles et aux différences finies*).

Jules Henri Poincaré, 1854–1912

(from `th.physik.uni-frankfurt.de/~jr`)

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THEOREM. Given $k > 0$, consider the difference equation

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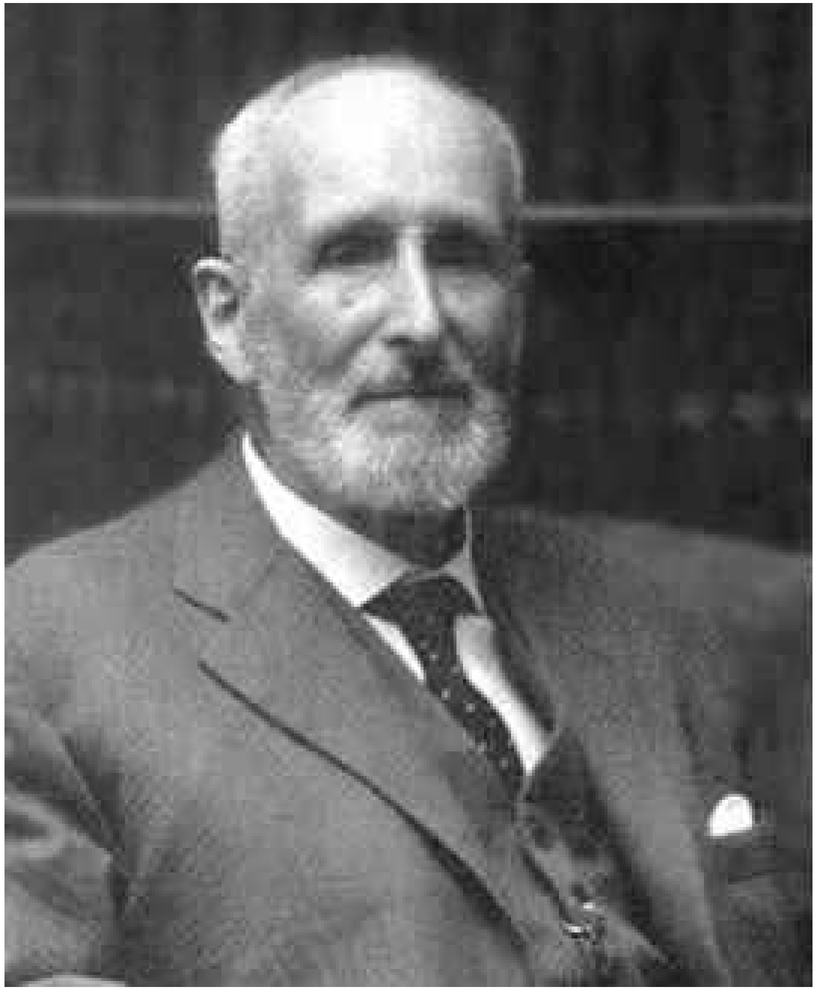
(see Oskar Perron's 1909 paper titled *Über einen Satz des Herrn Poincaré*).

Oskar Perron, 1880–1975



(from www.ub.uni-heidelberg.de)

Matrix version of Poincaré

THEOREM. (A. Máté-PN, 1990) Let $k \in \mathbb{N}$. Let $(\mathbf{A}_n) \in \mathbb{C}^{k\times k}$ be a sequence of matrices such that

$$\lim_{n\to\infty} \mathbf{A}_n = \mathbf{A}$$

exists. Suppose that all the eigenvalues of the matrix $\mathbf{A}$ have different absolute values. Write $(\mathbf{v}_j)_1^k \in \mathbb{C}^{1\times k}$ for the eigenvectors of $\mathbf{A}$. Let the sequence of column vectors $(\mathbf{u}_n) \in \mathbb{C}^{1\times k}$ be such that

$$\mathbf{u}_{n+1} = \mathbf{A}_n \mathbf{u}_n\,, \qquad n \in \mathbb{N}\,.$$

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Then there is $n_0 \in \mathbb{N}$ such that either $\mathbf{u}_n = 0$ for $n \geq n_0$, or $\mathbf{u}_n \neq 0$ for $n \geq n_0$, and, in the latter case, there are $\ell \in \mathbb{N}$ with $1 \leq \ell \leq k$ and a sequence $(\theta_n) \in \mathbb{C}$ such that

$$\lim_{n\to\infty} \theta_n \mathbf{u}_n = \mathbf{v}_\ell\,.$$

Matrix Poincaré $\Longrightarrow$ Poincaré

- Similarly to ODEs, scalar linear difference equations can be rewritten as a matrix equation where, apart from the last row, almost all entries are 0 except for the superdiagonal that consists of 1's.
- As it turns out, the matrix version of Poincaré's theorem is not only a genuine generalization, but, for some mysterious reason, has a simpler proof than that of the original.
- There exist extensions when the roots or eigenvalues can have equal sizes or allowed to have multiplicities.
- What about non-homogeneous equations?

Poincaré to OPs

THEOREM. If the OPs satisfy

$$xp_n = a_{n+1}p_{n+1} + b_n p_n + a_n p_{n-1}$$

with

$$\lim_{n\to\infty} a_n = a \geq 0 \quad \& \quad \lim_{n\to\infty} b_n = b \in \mathbb{R},$$

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ANSWER. Alfred Pringsheim; cf. The Pólya Picture Album.

Ludwig Otto Blumenthal, 1876–1944

(from J. Approx. Th.; MS by Paul Butzer & Lutz Volkmann)

The road backward

THEOREM. Let $c \le d$. Let $[c,d] \subset \operatorname{supp}(\alpha)$ and let the derived set of $\operatorname{supp}(\alpha)$ be $[c,d]$. If $\alpha' > 0$ a.e. in $[c,d]$, and if the OPs w.r.t. α satisfy

$$xp_n = a_{n+1}p_{n+1} + b_n p_n + a_n p_{n-1}$$

then

$$\lim_{n\to\infty} a_n = \frac{d-c}{4} \quad \& \quad \lim_{n\to\infty} b_n = \frac{c+d}{2}$$

(E. A. Rakhmanov, 1982 & 1986, A. Máté-PN-V. Totik, 1985, S. A. Denissov, 2004, V. Totik-PN, 2004, etc.).

NOTE. If $c = d$, then, of course, $\alpha' > 0$ a.e. in $[c,d]$; this is a special case of a theorem of M. G. Krein; see, e.g., Ted Chihara's book.

Mark Grigorievich Krein, 1907–1989

(from `wolffund.org.il`)

The perfect theorem

THEOREM. Let $\operatorname{supp}(\alpha) = [-1, 1]$. Then

$$\log \alpha'(\cos \cdot) \in L^1[(0, \pi)]$$

if and only if the recurrence coefficients (a_n) and (b_n) satisfy

$$\sum (2a_n - 1) < \infty \quad \& \quad \sum b_n < \infty$$

and

$$\sum (2a_n - 1)^2 < \infty \quad \& \quad \sum b_n^2 < \infty$$

(discovered mostly G. Szegő, but see & read also works by J. A. Shohat and Ya. L. Geronimus, the 1915–1940 period).

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NOTE. This work of Szegő gave rise, among others, to the theory of H^p spaces (Frigyes (aka Frédéric) Riesz) and to prediction theory (Andrey Nikolaevich Kolmogorov).

OPs issues (growth)

- The granddaddy of all OPs is the Chebyshev polynomial

$$T_n(x) = \cos(n\theta), \quad x = \cos\theta, \quad x \in [-1, 1]$$

and the grandma is the second kind Chebyshev polynomial

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin\theta}, \quad x = \cos\theta, \quad x \in [-1, 1]$$

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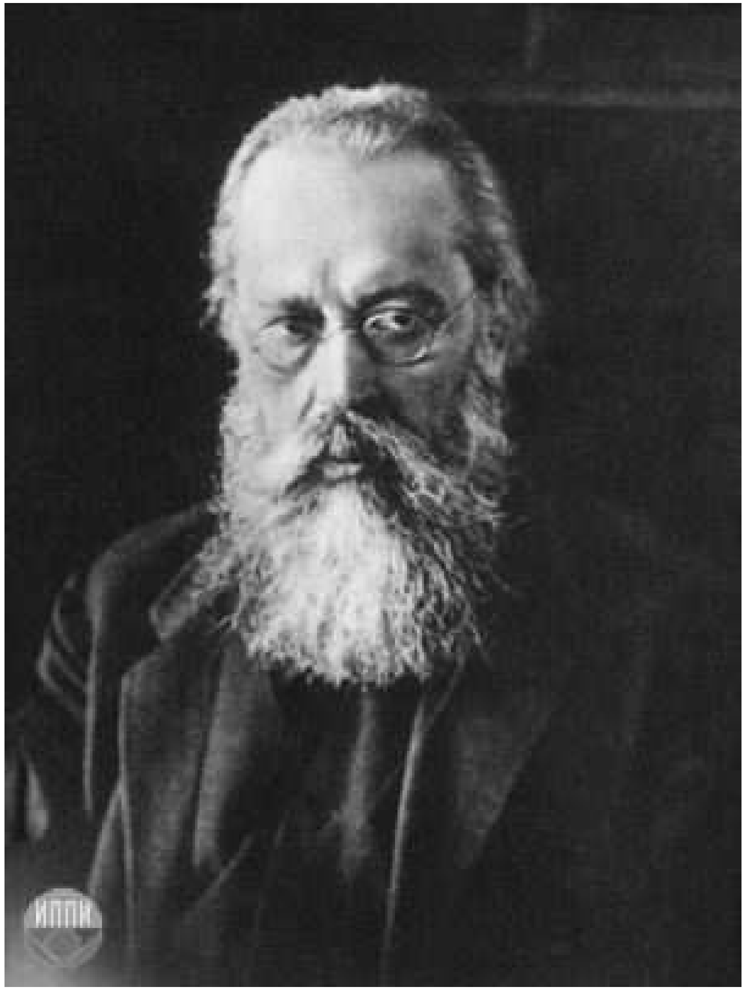
CONJECTURE. (V. A. Steklov, 1921) Roughly speaking, if the OPs live on a finite interval, are orthogonal w.r.t. an absolutely continuous measure α and $\alpha' \geq \mathrm{const} > 0$ there, then the OPs are uniformly bounded at every interior point.

Vladimir Andreevich Steklov, 1864–1926



(from www-history.mcs.st-and.ac.uk)

OPs issues (growth)

Then came the shocking...

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OPs issues (growth)

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On the other hand. . .

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Reminder:

$$\frac{\sum_{k=0}^{n-1} p_k^2(d\alpha, x)}{n} = \frac{1}{n\lambda_n(d\alpha, x)}$$

so that p_n^2 is (C,1) bounded if and only if $n\,\lambda_n$ is bounded away from zero.

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THEOREM. (A. Máté-PN, 1980) Roughly speaking, if the OPs are orthogonal w.r.t. α and on an interval, say, Δ, one has $\log \alpha' \in L^1(\Delta)$, then

$$\liminf_{n\to\infty} n\,\lambda_n(d\alpha, x) > 0 \quad \text{for a.e.} \quad x \in \Delta\,.$$

Evguenii Rakhmanov, 1952–2072

Taken in September, 1986, in Segovia, Estatuto de Autonomía de Castilla y León.

OPs issues (growth)

Let me lash out at the OPs community. . .

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If it is known that OPs are not bounded in general but under very general conditions they are (C,1) bounded, then how come that (C,γ), $0 < \gamma < 1$, boundedness has never been studied for general OPs although there are more than plenty papers dedicated to relentless transliteration of summability issues of classical trigonometric series to special OPs series when for one or another reason the OPs can be shown to behave similarly to classical trigonometric functions.

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What about some weighted L^p with some or any (C,γ)?

OPs issues (CFs)

THEOREM. (A. Máté-PN-V. Totik, 1991) Roughly speaking, if the OPs live in $[-1,1]$ and are orthogonal w.r.t. α such that $\log \alpha' \in L^1([-1,1])$, then

$$\lim_{n\to\infty} n\,\lambda_n(d\alpha, x) = \pi\sqrt{1-x^2}\,\alpha'(x) \quad \text{for a.e.} \quad x \in \Delta\,.$$

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How frustrating...

Paul Erdős, 1913–1996

(from `www-history.mcs.st-and.ac.uk`)

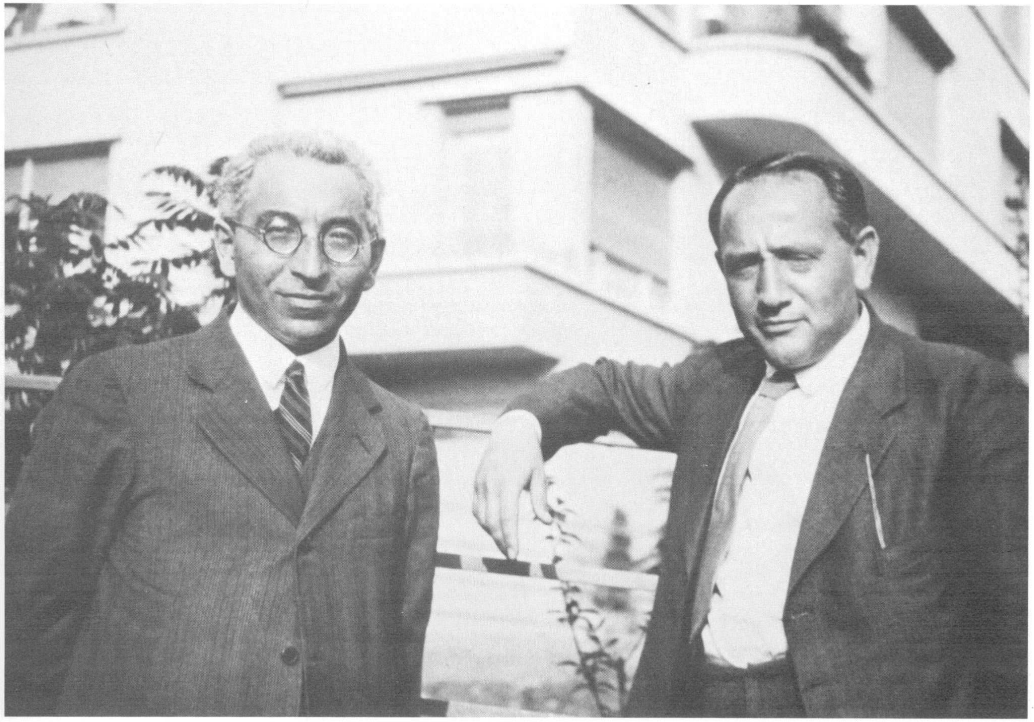
Paul Erdős, 1913–1996

(from www-history.mcs.st-and.ac.uk)

Paul Turán, 1910–1976

(by Paul Halmos)

James Alexander Shohat, 1886–1944



Here is a picture of Shohat taken with me in Zürich.

(with George Pólya; from The Pólya Picture Album)

OPs & CFs issues

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On the other hand, one could speculate whether this blunder by Geronimus was, in fact, a major catalyst for things to come in OPs for the next 25+ years; see, e.g., Rakhmanov's Theorem, MNT, etc.

OPs & CFs issues

The second error is that he "proved"

$$\lim_{n\to\infty} n\,\lambda_n(d\alpha, x) = \pi\sqrt{1-x^2}\,\alpha'(x)$$

under quite weak conditions; in particular, $\alpha' > 0$, a.e. would suffice; see Some asymptotic properties of orthogonal polynomials, Soviet Math. Dokl., 165 (1965), 1387–1389, and Vestnik Kharkov. Gos. Univ., 32 (1966), 40–50.

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For details, I recommend my case study paper on Freud in JAT 48 (1986), 3–167; cf. Chapter 4.6.

Yakov Lazarevich Geronimus, 1898–1984

(from Leonid Golinskii)

Mathematics in the USSR

NOTE. In my not necessarily humble* opinion, the main culprit was the unusual setup of mathematics culture in the (thanks G·d former) Soviet Union that has led to some unfortunate consequences. It will take generations to cure the ills, if ever. I want to point out four painful aspects of this.

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BTW, pravda, as you know it, means truth.

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- Fierce but rather irrational competition between research groups that even led to sometimes comical fistfights at international conferences; cf. Moscow vs. Leningrad or Sergey B. Stechkin yelling at Géza Freud in Poznan in August, 1972, or the historic words of Allan Pinkus at Varna: I don't know and I don't care.

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- The pathological and all-encompassing superiority complex, imperialism, nationalism, chauvinism, and, perhaps most characteristically, vicious and passionate anti-Semitism.

Mathematics in the USSR

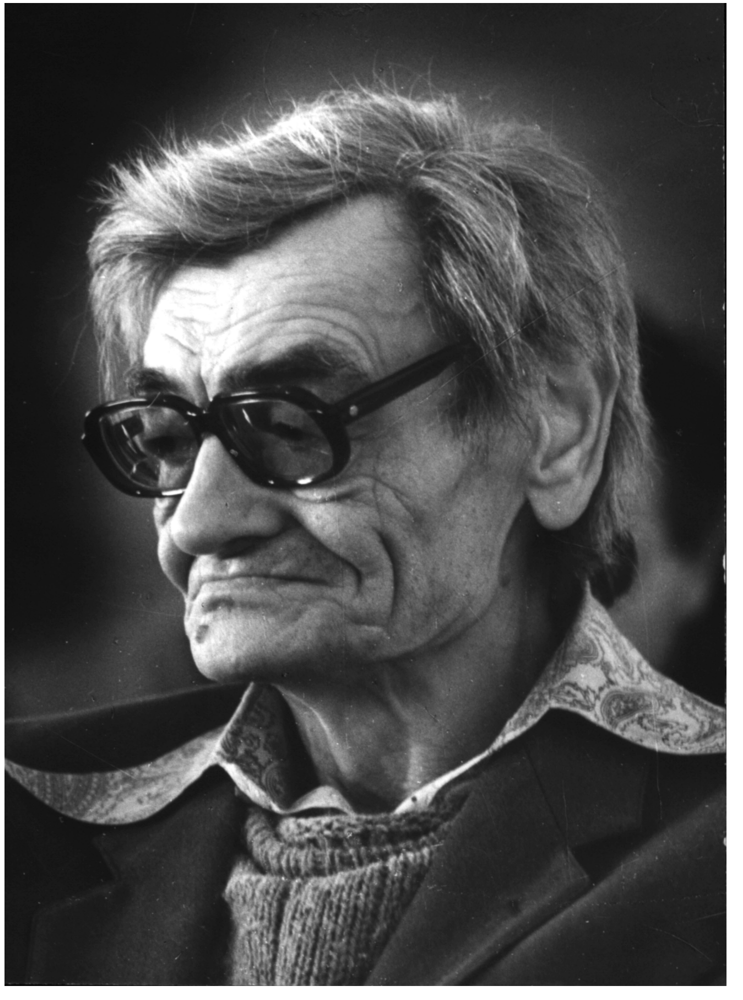
Recommended literature:

- G. G. Lorentz, Mathematics and politics in the Soviet Union from 1928 to 1953, J. Approx. Theory, Volume 116, Number 2, June 2002, 169–223; cf `math.nevai.us/LORENTZ`.
- *Golden years of Moscow mathematics*, Smilka Zdravkovska & Peter L. Duren, eds., Amer. Math. Soc., 2007.
- Google luzin affair, e.g. `www.gap-system.org/~history/Extras/Luzin.html`.
- For a post-Soviet story see Sergey Khrushchev's `mathforum.org/kb/plaintext.jspa?messageID=45118`.

Géza Freud, 1922–1979

(The Ohio State University, Columbus, Ohio, October, 1976)

Sergey Borisovich Stechkin, 1920–1995



(from Vitaly Arestov, www.imm.uran.ru)

Turán's inequality

THEOREM. Let (P_n) be the Legendre polynomials in $[-1,1]$ normalized by $P_n(1)=1$. Then, for the Turán determinants,

$$P_n^2(x) - P_{n-1}(x)\,P_{n+1}(x) > 0\,, \qquad x \in (-1,1)\,,$$

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Eventually, it was realized that the background for the positivity is that the Turán determinants converge to a positive limit. This lead to results of the type

$$\lim_{n\to\infty} \left[p_n^2(x) - p_{n-1}(x)\,p_{n+1}(x)\right] = \frac{2}{\pi}\,\frac{\sqrt{1-x^2}}{\alpha'(x)}\,, \qquad x \in (-1,1)\,,$$

under certain analytic conditions on α.

Turán's inequality

Since convergence implies convergence of (C,1) means,

$$(\mathrm{C},1) \qquad \lim_{n\to\infty}\left[p_n^2(x) - p_{n-1}(x)\,p_{n+1}(x)\right] = \frac{2}{\pi}\,\frac{\sqrt{1-x^2}}{\alpha'(x)} \qquad (\bigstar)$$

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THEOREM. If $\mathrm{supp}(\alpha) = [-1,1]$ and $\log\alpha'(\cos(\cdot)) \in L^1([0,\pi])$, then $(\bigstar)$ holds almost everywhere in $[-1,1]$.

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NOTE. Smoothness condition replaced by growth.

NOTE. Uniform convergence on intervals of continuity.

NOTE. Allows evaluation or estimation of the measure if the behavior of the OPs is known.

NOTE. Mass-points of the measure can be recovered from the rather general formula $\alpha(\{x\}) = 1/\left(\sum_{k=0}^{\infty} p_k^2(x)\right)$.

OPs issues (growth)

Now some bad news...

OPs are normalized, so, automatically,

$$\sup_{n\in\mathbb{N}} \left\{ \int p_n^2 \, d\alpha \right\}^{\frac{1}{2}} < \infty$$

However, there is not a single direct result either of the type

$$\sup_{n\in\mathbb{N}} \left\{ \int p_n^2 \, d\beta \right\}^{\frac{1}{2}} < \infty \qquad \text{(here } (p_n) \text{ are OPs w.r.t. } \alpha\text{)}$$

or

$$\sup_{n\in\mathbb{N}} \left\{ \int |p_n|^p \, d\alpha \right\}^{\frac{1}{p}} < \infty, \quad p > 2,$$

under certain general size (and not smoothness) conditions on α, β, or p.

OPs issues (growth)

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Indirect results allow to study the measure α associated with the OPs provided that

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For instance, if both measures are supported in $[-1,1]$, the measure β is absolutely continuous w.r.t. to α, and $\alpha' > 0$ a.e. in $[-1,1]$, then this implies

$$\left\{ \int_{-1}^{1} \left(\alpha'(t)\sqrt{1-t^2} \right)^{-\frac{p}{2}} \; \beta'(t)\; dt \right\}^{\frac{1}{p}} < \infty.$$

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The lack of progress happened despite such celebrities working in the general theory of OPs:

Christian Berg, Percy Deift, Géza Freud (dead), Andrei Aleksandrovich Gonchar (dead), Sergey Khrushchev, Arno B. J. Kuijlaars, Guillermo López Lagomasino, Doron Lubinsky, Andrei Martínez-Finkelshtein, Fedor Nazarov, Evgenii Mikhailovich Nikishin (dead), Franz Peherstorfer (dead), Evguenii Rakhmanov, Ed Saff, Peter Sarnak, Barry Simon, Herbert Stahl (dead), Vilmos Totik, Walter Van Assche, Alexander Volberg, and Harold Widom.

NOTE. On the average, there is at least one major international conference dedicated specifically to OPs every other year attracting 200+ participants.

OPs issues (zeros)

There have been dozens if not hundreds of papers & books dedicated to zero distribution of OPs. One of the initial steps was made by Erdős–Turán who proved, using a marvelous inequality of Remez, that, if $\mathrm{supp}(\alpha) = [-1, 1]$ and $\alpha' > 0$ there, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_{kn}) = \frac{1}{\pi} \int_0^{\pi} f(\cos t)\, dt$$

for $f \in \mathrm{C}$, that is, the zeros are arcsin–distributed.

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In fact, under the weaker condition that the recurrence coefficients (a_n) and (b_n) converge, say, to $1/2$ and 0, resp., much more is true. Namely, for differentiable functions f,

$$(\mathrm{C}, -1) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_{kn}) = \frac{1}{\pi} \int_0^{\pi} f(\cos t)\, dt.$$

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With this optimistic prediction [smiley], let us move on to…

OPs issues (zeros)

Jacobi matrices are real, symmetric, tridiagonal matrices with positive entries on the off-diagonals. OPs connection:

$$\left[\left[\int x p_j p_k d\alpha\right]\right]_{j,k=0}^{\infty}$$

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is a Jacobi matrix, and, vice versa, characteristic poly's of truncated Jacobi matrices are OPs. Hence, zeros of OPs are eigenvalues of truncated Jacobi matrices. Replacing the function x above by φ leads to Hankel matrices.

PROBLEM. Study the behavior of the eigenvalues of

$$\left[\left[\int \varphi p_j p_k d\alpha\right]\right]_{j,k=0}^{\infty}$$

(initiated by Ulf Grenander and Gábor Szegő; see their 1958 book *Toeplitz forms and their applications*).

OPs issues (eigenvalues)

THEOREM. Let α be supported in $[-1,1]$ and let $\alpha' > 0$ a.e. there. Let $\varphi \in L^\infty_\alpha$. Let G be a continuous function in an interval containing the essential range of φ. Then the eigenvalues (Λ_{kn}) of the $n \times n$ truncated Hankel matrix

$$\left[\left[\int \varphi p_j p_k d\alpha\right]\right]_{j,k=0}^{n-1}$$

satisfy

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} G(\Lambda_{kn}) = \frac{1}{\pi} \int_0^\pi G \circ \varphi(\cos t)\, dt$$

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NOTE. I don't know why, but in many papers, incl. mine, such Hankel matrices are called Toeplitz matrices.

Hermann Hankel, 1839–1873

(from www-history.mcs.st-and.ac.uk)

Otto Toeplitz, 1881–1940

(from owpdb.mfo.de/person_detail?id=4231)

Fourier series

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- Jean Baptiste Joseph Fourier (blame him; see the introduction to Godfrey Harold Hardy's *Divergent Series*)
- Georg Friedrich Bernhard Riemann ($\exists$ Riemann integrals & $\exists$ uniform convergence because of FS)
- Marie Ennemond Camille Jordan, Johann Peter Gustav Lejeune Dirichlet, Ulisse Dini (from Pisa), Rudolf Otto Sigismund Lipschitz (the convergence guys)
- Georg Ferdinand Ludwig Philipp Cantor ($\exists$ ∞-sets because of FS)
- Paul David Gustav du Bois-Reymond (oh, my G·d, a divergent FS of a continuous function)

- Henri Lebesgue ($\exists$ measure theory because of FS)
- Lipót (aka Leopold) Fejér ($\exists$ summability theory & $\exists$ great Hungarian math because of FS)
- Lebesgue ($\exists$ Lebesgue–points because of FS)
- Fejér & Lebesgue ($\exists$ Banach–Steinhaus because of FS)
- Andrei Nikolaevich Kolmogorov (oh, no, this time an everywhere divergent FS)
- Nikolai Nikolaevich Luzin & Lennart Axel Edvard Carleson (OK, at least a.e. conv. in L^2, I mean, in L^p for $p > 1$)
- Carleson, Jean-Pierre Kahane & Yitzhak Katznelson (for every set of Lebesgue-measure 0, $\exists$ a continuous function whose FS diverges there)

OPs issues (series)

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The basic tools go back to Géza Freud, the guy who probably coined the term Christoffel function (and also happened to have been my advisor + crazy as hell).

OPs issues (interpolation)

Although this was the primary reason why I became interested in OPs, and this is how I became familiar with the works of J. A. Shohat, G. Pólya, G. Szegő, A. Zygmund, A. Marcinkiewicz, P. Erdős, P. Turán, Ya. L. Geronimus, G. Freud, and R. Askey, I need to skip this subject today.

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Let me just make a claim: weighted mean convergence of Lagrange (and related) interpolation is a area rich of both high quality esthetic beauty and technical challenges.

Antoni Zygmund, 1900–1992

NOTE. Zygmund, at the age of 80+, actually read my AMS Memoir on orthogonal polynomials.

Józef Marcinkiewicz, 1910–1940

(from `www-history.mcs.st-and.ac.uk`)

NOTE. Murdered by the Soviets (google “Katyn massacre”).

Generalizations

A straightforward generalization of Christoffel functions could be given by

$$\min_{P \in \mathbb{P}_n} \frac{\|P\|_1}{\|P\|_2} \qquad \text{or} \qquad \max_{P \in \mathbb{P}_n} \frac{\|P\|_1}{\|P\|_2}$$

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Many such generalizations actually predate CFs.

Objects (inequalities) whose studies were initiated by Bernstein, Favard, Kolmogorov, Landau, Markov, Schoenberg, Riesz, Totik, etc., and, especially, Nikolskiĭ, are all special cases of Christoffel functions; e.g.,

$$\max_{P \in \mathbb{P}_n} \frac{\|P\|_p}{\|P\|_q} \approx n^{\frac{2}{p} - \frac{2}{q}}, \qquad 0 < q \le p \le \infty .$$

Sergey Mikhailovich Nikolskiĭ, 1905–2012

(Budapest, Hungary, August, 1995)

oops, wrong picture. . .

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The Nikolskiĭ inequality

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has played an essential (really, quintessential) role in approximation theory since the times of Edmund Landau, Dunham Jackson, and Sergei Natanovich Bernstein, that is, since the birth of the direct and inverse theorems in approximation theory in the beginning of the 20th century.

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Still, the best constants are known only in special cases although better and better estimates are coming out, mostly from the group at Ural State University in Ekaterinburg, Russia, including some very recent papers.

Edmund Landau, 1877–1938

(from `www-history.mcs.st-and.ac.uk`)

FULL NAME: Edmund Georg Hermann (Yehezkel) Landau

According to the Mathematics Genealogy Project, he has 29 students and 3544 descendants.

Dunham Jackson, 1888-1946

(from www.math.umn.edu)

Sergei Natanovich Bernstein, 1880–1968

(from www.york.ac.uk)

Richard Steven Varga 1928–2048

According to Dick Varga, all talks must end with a geat joke.

(Hangzhou, China, May, 1985)

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Here is one coming directly from Ricsi, no guarantees for quality. . .

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Then, suddenly, the soldier gently observes, che bella voce.

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(blame `translate.google.com`)

Das Ende