# Adaptive Wavelet Methods for the Efficient Approximation of Images

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# Adaptive Wavelet Methods for Image Approximation

## Outline

- Introduction: Adaptive wavelet transforms
    - Generalized lifting schemes
    - Geometric approaches with adaptivity costs
- Description of the EPWT algorithm
- Examples and experiments
- A hybrid method using the EPWT
- Numerical experiments
- Denoising of scattered data
- References

## Introduction

**Idea** Design adaptive approximation schemes respecting the local geometric regularity of two-dimensional functions

**Basic adaptive wavelet approaches**

a) Apply a generalized lifting scheme to the data using (nonlinear) data-dependent prediction and update operators

b) Adaptive approximation schemes using geometric image information, usually with extra adaptivity costs

# Basic adaptive wavelet approaches

**a)** Apply a generalized lifting scheme to the data using (nonlinear) data-dependent prediction and update operators

**Literature (incomplete)**

- **discrete MRA and generalized wavelets** (Harten '93)
- **second generation wavelets** (Sweldens '97)
- **edge adapted multiscale transform** (Cohen & Matei '01)
- **Nonlinear wavelet transforms** (Claypoole et al. '03)
- **adaptive lifting schemes** (Heijmans et al. '06)
- **adaptive directional lifting based wavelet transf.** (Ding et al. '06)
- **edge-adapted nonlinear MRA (ENO-EA)** (Arandiga et al. '08)
- **meshless multiscale decompositions** (Baraniuk et al. '08)
- **nonlinear locally adaptive filter banks** (Plonka & Tenorth '09)

## How does it work?

The general lifting scheme consists of three steps.

1. **Split** Split the given data $\mathbf{a} = (a(i,j))_{i,j=0}^{N-1}$ into two sets $\mathbf{a}^e$ and $\mathbf{a}^o$

2. **Predict** Find a good approximation $\tilde{\mathbf{a}}^o$ of $\mathbf{a}^o$ of the form

$$\tilde{\mathbf{a}}^o = P_1 \mathbf{a}^o + P_2 \mathbf{a}^e$$

Put

$$\mathbf{d}^o := \tilde{\mathbf{a}}^o - \mathbf{a}^o.$$

Assume that $(\mathbf{a}^e, \mathbf{a}^o) \mapsto (\mathbf{a}^e, \mathbf{d}^o)$ is invertible, i.e., $I - P_1$ is invertible.

3. **Update** Find a "smoothed" approximation of $\mathbf{a}^e$ (a low-pass filtered subsampled version of $\mathbf{a}$)

$$\tilde{\mathbf{a}}^e := U_1(\mathbf{d}^o) + U_2(\mathbf{a}^e)$$

Assume that $(\mathbf{a}^e, \mathbf{d}^o) \mapsto (\tilde{\mathbf{a}}^e, \mathbf{d}^o)$ is invertible, i.e., that $U_2$ is invertible.

## How to choose the prediction and update operators?

**Prediction operator** local approximation of $\mathbf{a}^o$ by an adaptively weighted average of "neighboring" data

**Example 1.**

- Fix a stencil at a neighborhood of $a^o(i,j)$ (adaptively)
- Compute a polynomial $p$ by interpolating/approximating the data on the stencil
- Choose $p(i,j)$ to approximate $a^o(i,j)$.

**Example 2.** Use nonlinear diffusion filters to determine the prediction operator

**Update operator** usually linear, non-adaptive

# Basic adaptive wavelet approaches

**b)** Adaptive wavelet approximation schemes using geometric image information, usually with extra adaptivity costs

**Literature (incomplete)**

- **wedgelets** (Donoho ’99)
- **bandelets** (Le Pennec & Mallat ’05)
- **geometric wavelets** (Dekel & Leviatan ’05)
- **geometrical grouplets** (Mallat ’09)
- **EPWT** (Plonka et al. 09)
- **tetrolets** (Krommweh ’10)
- **generalized tree-based wavelet transform** (Ram, Elad et al. ’11)

## Basic adaptive wavelet approaches

**wedgelets** (Donoho '99)

approximation of images using an adaptively chosen domain decomposition

**bandelets** (Le Pennec & Mallat '05)

wavelet filter bank followed by adaptive geometric orthogonal filters

**geometric wavelets** (Dekel & Leviatan '05)

binary space partition and polynomial approximations in subdomains

**geometrical grouplets** (Mallat '09)

association fields that group points, generalized Haar wavelets

**EPWT** (Plonka et al. 09)

**tetrolets** (Krommweh '10)

generalized Haar wavelets on adaptively chosen tetrolet partitions

## Comparison of basic adaptive wavelet approaches

**a)** Generalized lifting scheme with nonlinear prediction

**Advantages** invertible transform, no side information necessary
usually a justifiable computational effort

**Drawbacks** bad stability of the reconstruction scheme
only slightly better approximation results compared with linear (nonadaptive) transforms

**b)** Adaptive wavelet approximation using geometric image information

**Advantages** very good approximation results

**Drawbacks** adaptivity costs for encoding
usually high computational effort

## Description of the EPWT

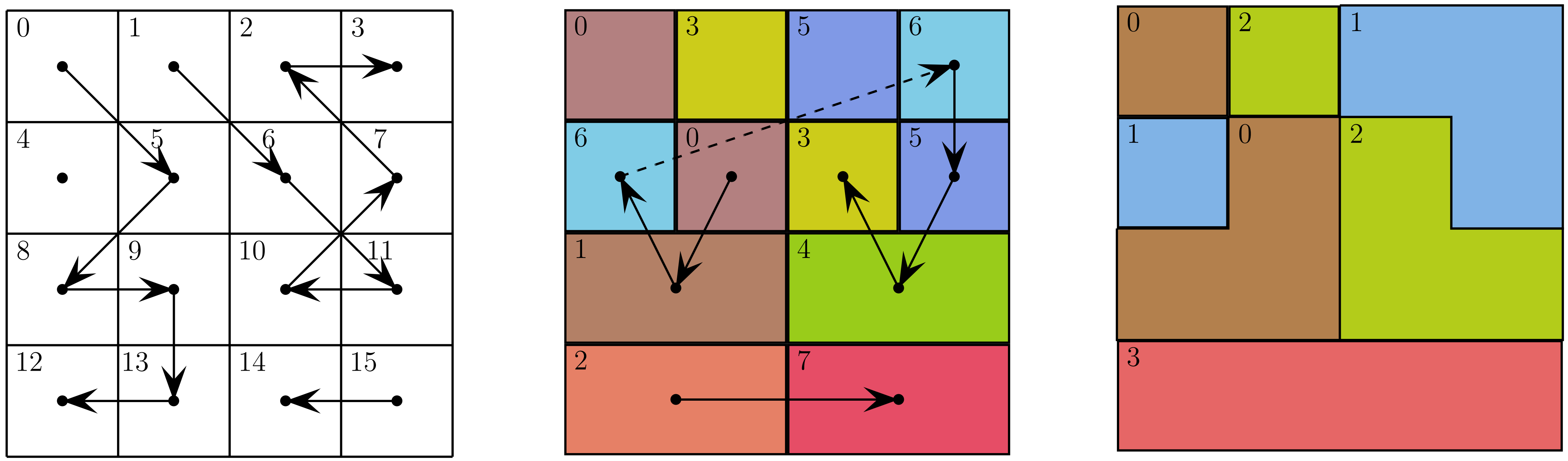
**Problem** Given a matrix of data points (image values), how to compress the data by a wavelet transform thereby exploiting the local correlations efficiently?

**Idea**

1. Find a (one-dimensional) path through all data points such that there is a strong correlation between neighboring data points.
2. Apply a one-dimensional wavelet transform along the path.
3. Apply the idea repeatedly to the low-pass filtered array of data.

## Toy Example

$$\mathbf{f} = \begin{bmatrix} 115 & 108 & 109 & 112 \\ 106 & 116 & 107 & 109 \\ 112 & 110 & 108 & 108 \\ 108 & 109 & 103 & 106 \end{bmatrix} \qquad \text{array of data.}$$



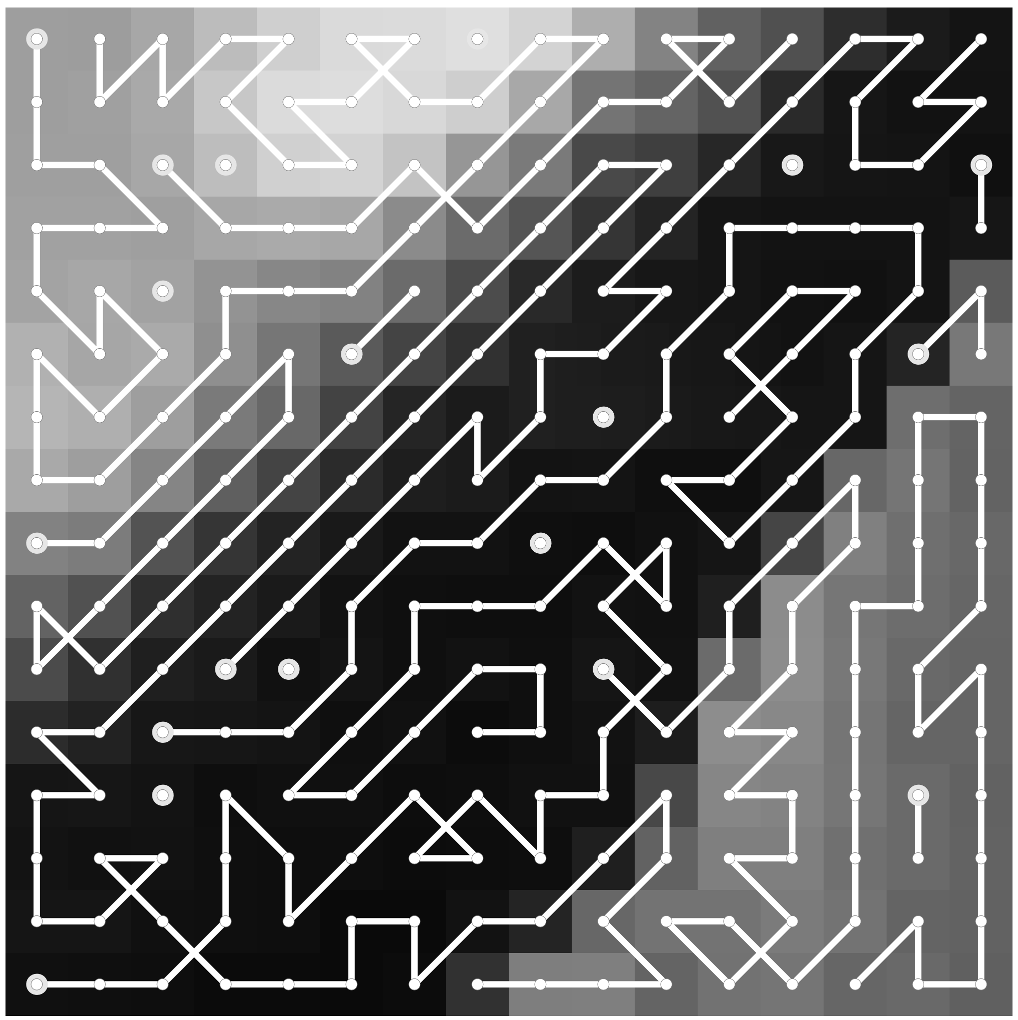

$$\mathbf{p}^4 = ((0,5,8,9,13,12),(1,6,11,10,7,2,3),(4),(15,14)),$$

$$\mathbf{f}^3 = (115.5, 111, 108.5, 107.5, 108, 109, 109, 104.5),$$

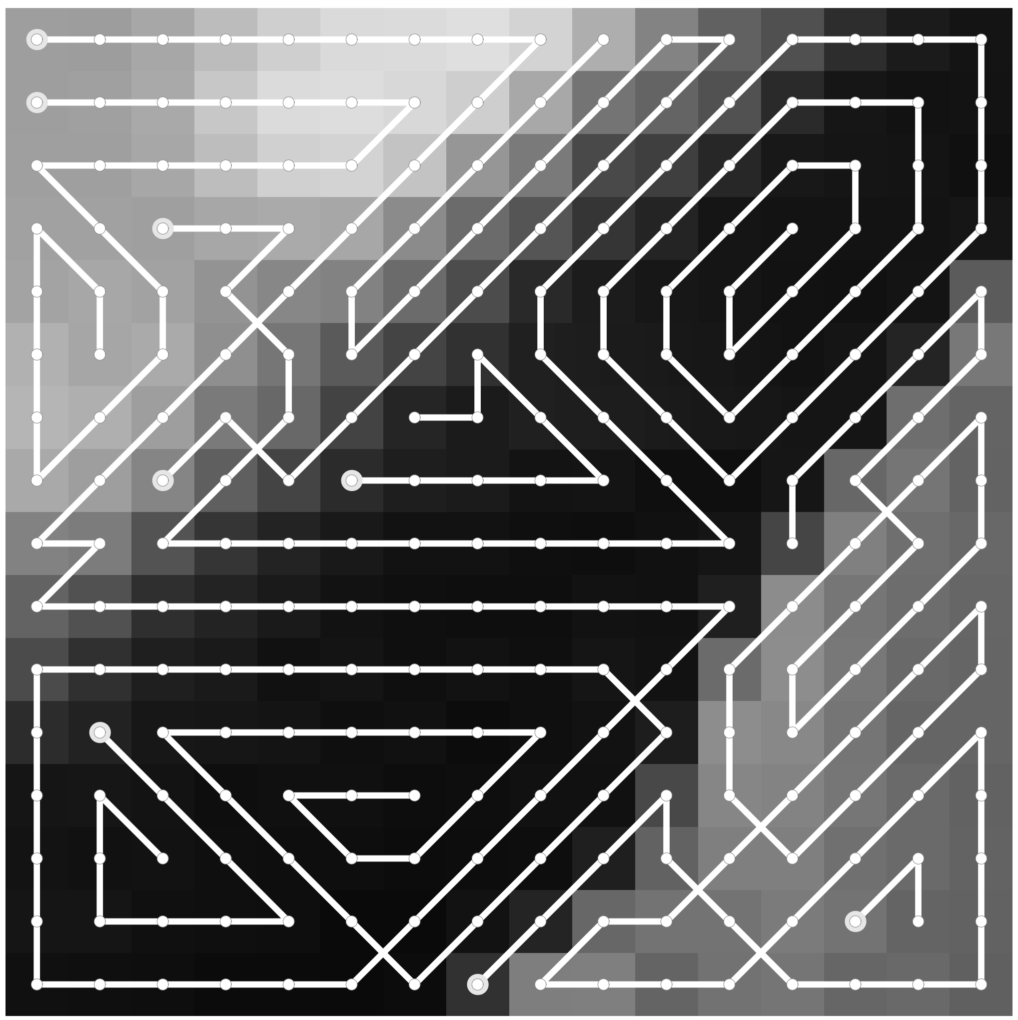
$$\mathbf{p}^3 = ((0,1,6,5,4,3),(2,7)), \qquad \mathbf{p}^2 = (0,1,2,3).$$

## The relaxed EPWT

Idea: Change the direction of the path only if the difference of data values is greater than a predetermined value $\theta$.



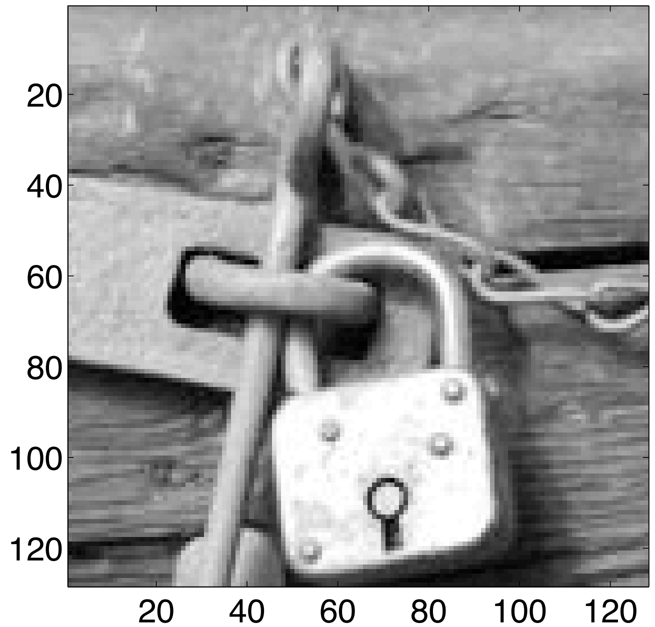
*rigorous EPWT ($\theta = 0$)*
*Entropy 2.08 bit per pixel*



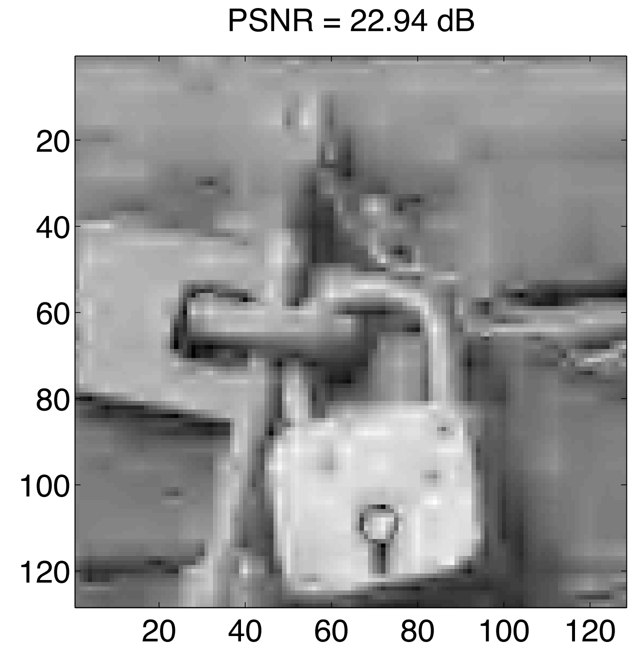
*relaxed EPWT ($\theta = 0.14$)*
*Entropy 0.39 bit per pixel*

**Test: door lock image** ($128 \times 128$)

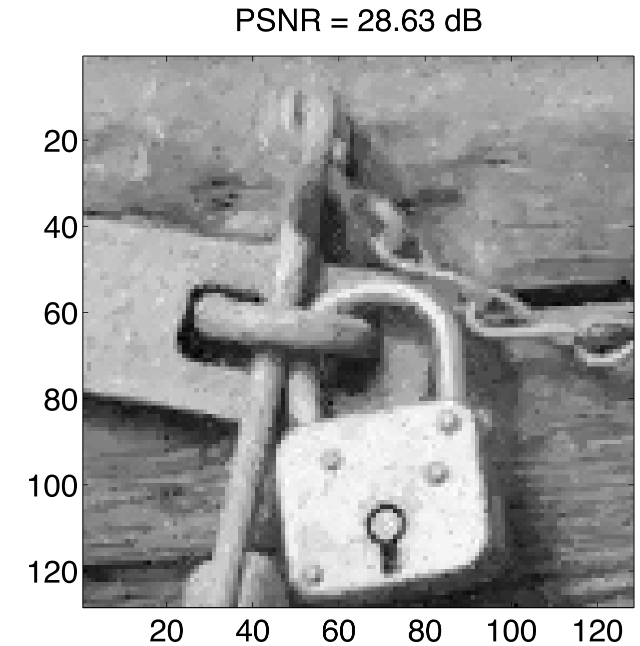
| WT | $\theta_1$ | levels | nonzero coeff | PSNR | entropy of $\tilde{\mathbf{p}}^{14}$ |
|---|---|---|---|---|---|
| tensor prod. Haar | - | 7 | 512 | 22.16 | - |
| tensor prod Daub. | - | 6 | 512 | 22.94 | - |
| tensor prod 7-9 | - | 4 | 512 | 22.49 | - |
| EPWT Haar | 0.00 | 14 | 512 | 28.04 | 2.22 |
| EPWT Haar | 0.05 | 14 | 512 | 28.37 | 1.11 |
| EPWT Haar | 0.10 | 14 | 512 | 27.74 | 0.55 |
| EPWT Daub. | 0.00 | 12 | 512 | 28.63 | 2.22 |
| EPWT Daub. | 0.05 | 12 | 512 | 29.23 | 1.11 |
| EPWT Daub. | 0.10 | 12 | 512 | 28.67 | 0.55 |
| EPWT Daub. | 0.15 | 12 | 512 | 27.65 | 0.32 |
| EPWT 7-9 | 0.00 | 10 | 512 | 28.35 | 2.22 |
| EPWT 7-9 | 0.05 | 10 | 512 | 28.99 | 1.11 |
| EPWT 7-9 | 0.10 | 10 | 512 | 28.38 | 0.55 |



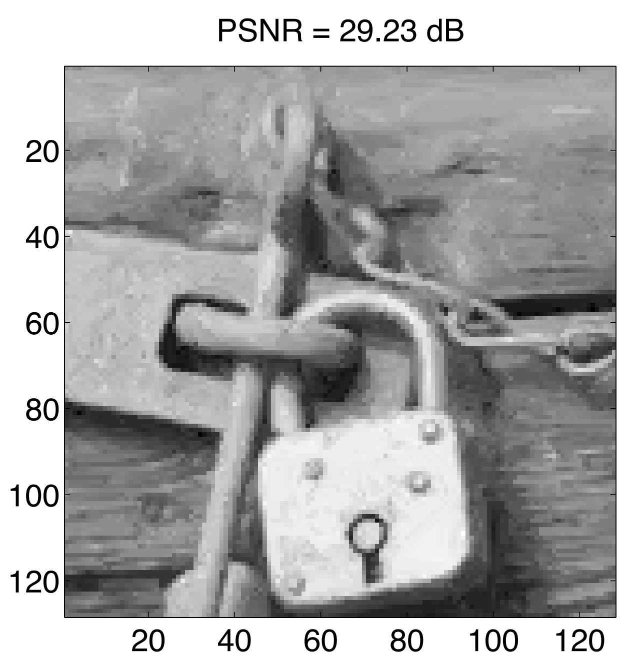

*original image*



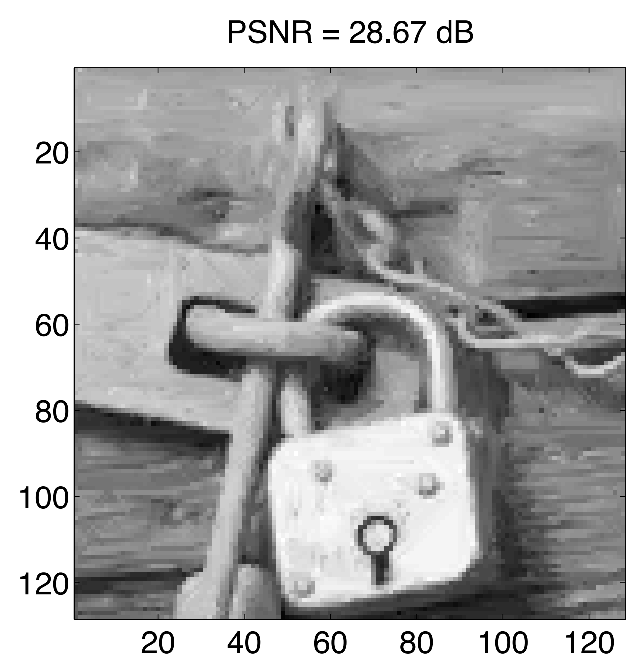

*D4, 512 coeff.*
*PSNR= 22.94*



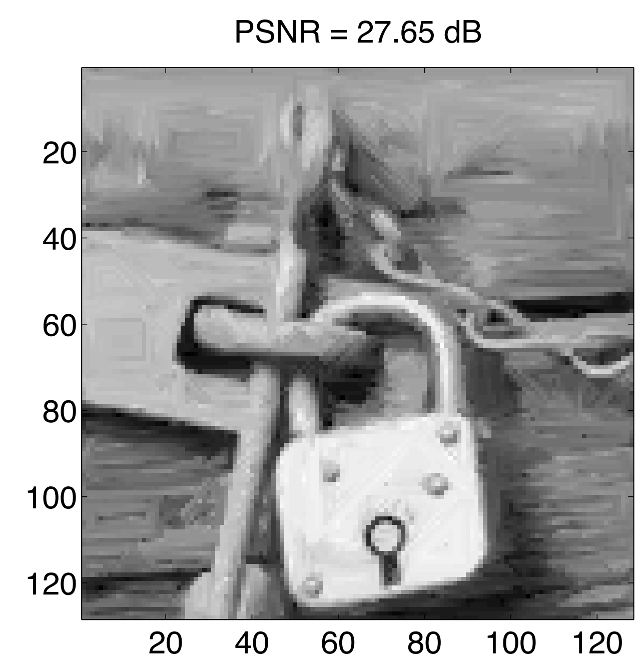

EPWT $\theta_1 = 0$
*PSNR=28.63*




EPWT, $\theta_1 = 0.05$
*PSNR=29.23*




EPWT, $\theta_1 = 0.1$
*PSNR=28.67*




EPWT, $\theta_1 = 0.15$
*PSNR = 27.65*

## Results for N-term approximation

**Theorem 1 (Plonka, Tenorth, Iske (2011))**

The EPWT (with the Haar wavelet transform) leads for suitable path vectors to an $N$-term approximation of the form

$$\|f - f_N\|_2^2 \le C\, N^{-\alpha}$$

for piecewise Hölder continuous functions of order $\alpha$ (with $0 < \alpha \le 1$) possessing discontinuities along curves of finite length.

**Theorem 2 (Plonka, Iske, Tenorth (2013))**

The application of the EPWT leads for suitably chosen path vectors to an $N$-term approximation of the form

$$\|f - f_N\|_2^2 \le C\, N^{-\alpha}$$

for piecewise Hölder smooth functions of order $\alpha > 0$ possessing discontinuities along curves of finite length.

## The hybrid method using the EPWT

**Idea**

1. Apply an image separation into a smooth image part and a remainder part containing edges and texture

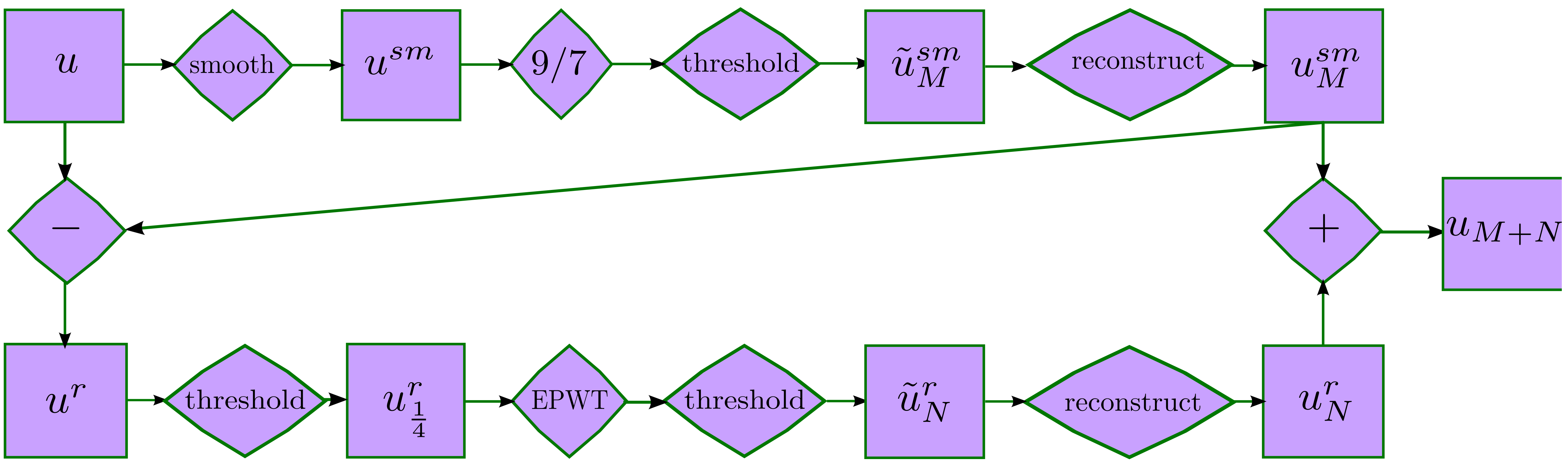
$$u = u^{sm} + u^r$$

   using e.g. a suitable smoothing filter.

2. Apply a tensor product wavelet transform to the smooth image part $u^{sm}$ to get an $N$-term approximation $u_N^{sm}$.
3. Apply the EPWT to the (shrinked) remainder $u^r$ to get am $M$-term approximation $u_M^r$.
4. Add $u_N^{sm}$ and $u_M^r$ to find a good approximation of $u$.

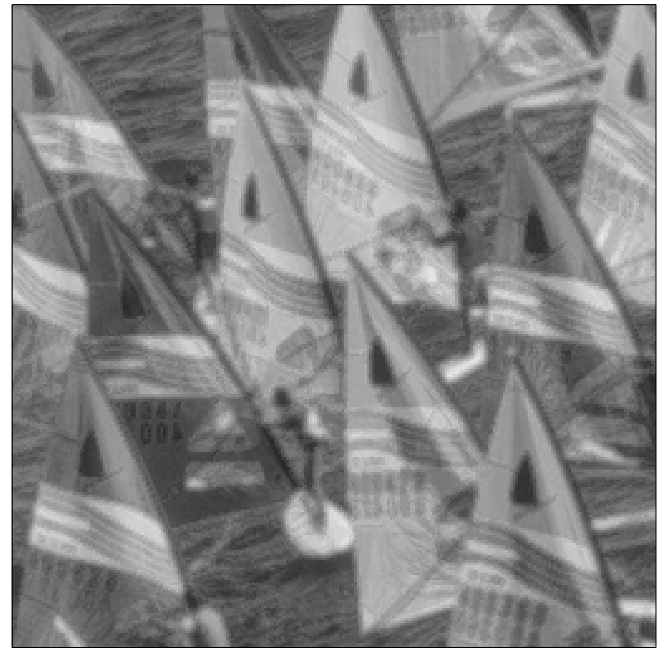
## A sketch of the hybrid method

We use the tensor-product wavelet transform for the smoothed image and the EPWT for the (shrunken) difference image.



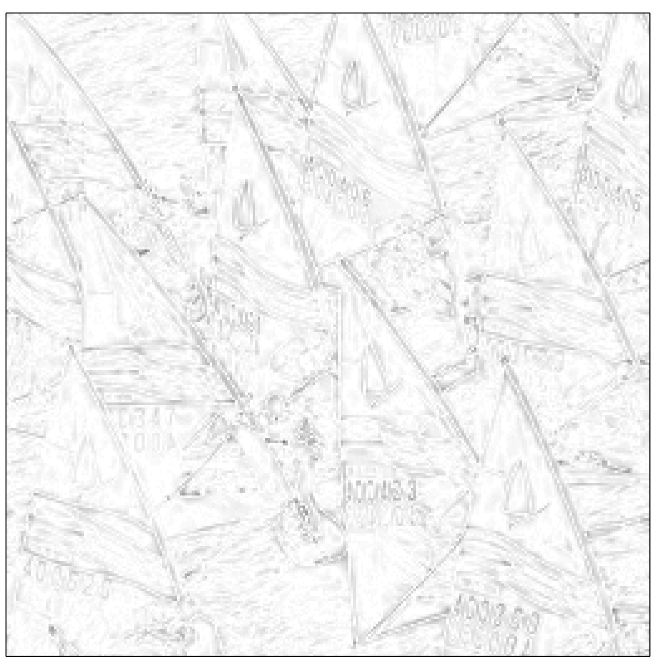
## Example

*Original image*

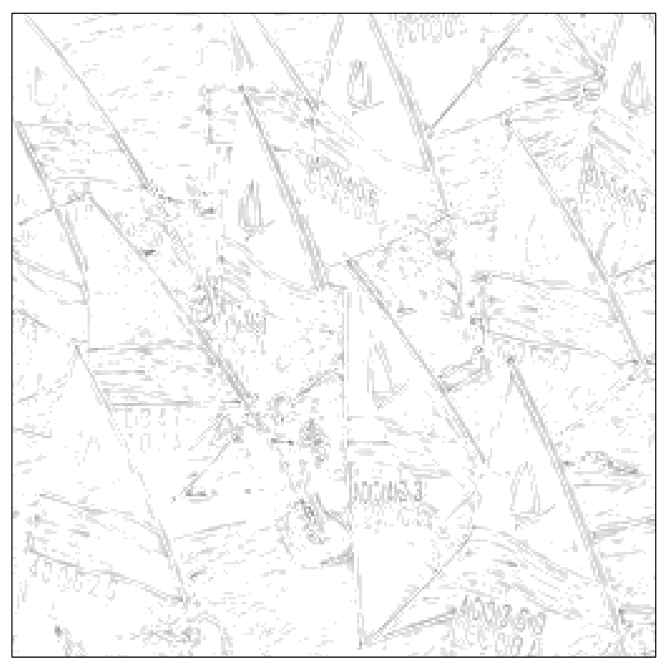


*smoothed image* $u^{sm}$

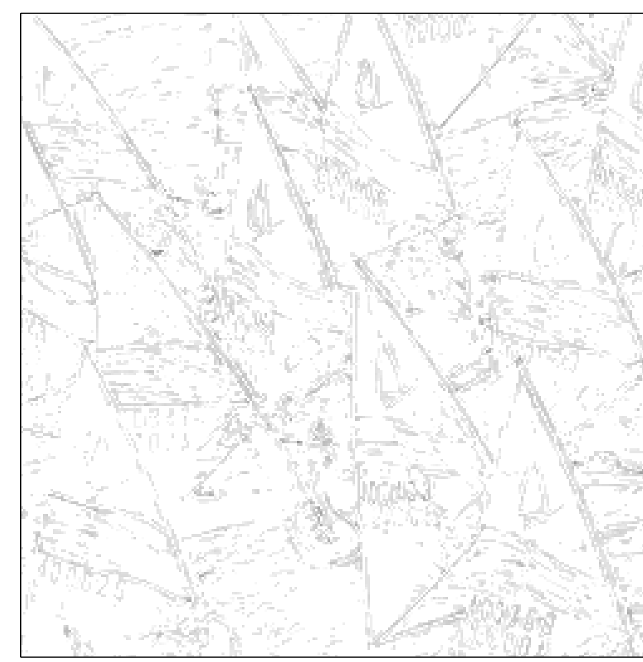
*wavelet approximation* $u^{sm}_{1200}$



*difference image* $u^r$



*shrunken difference* $u^r_{1/4}$



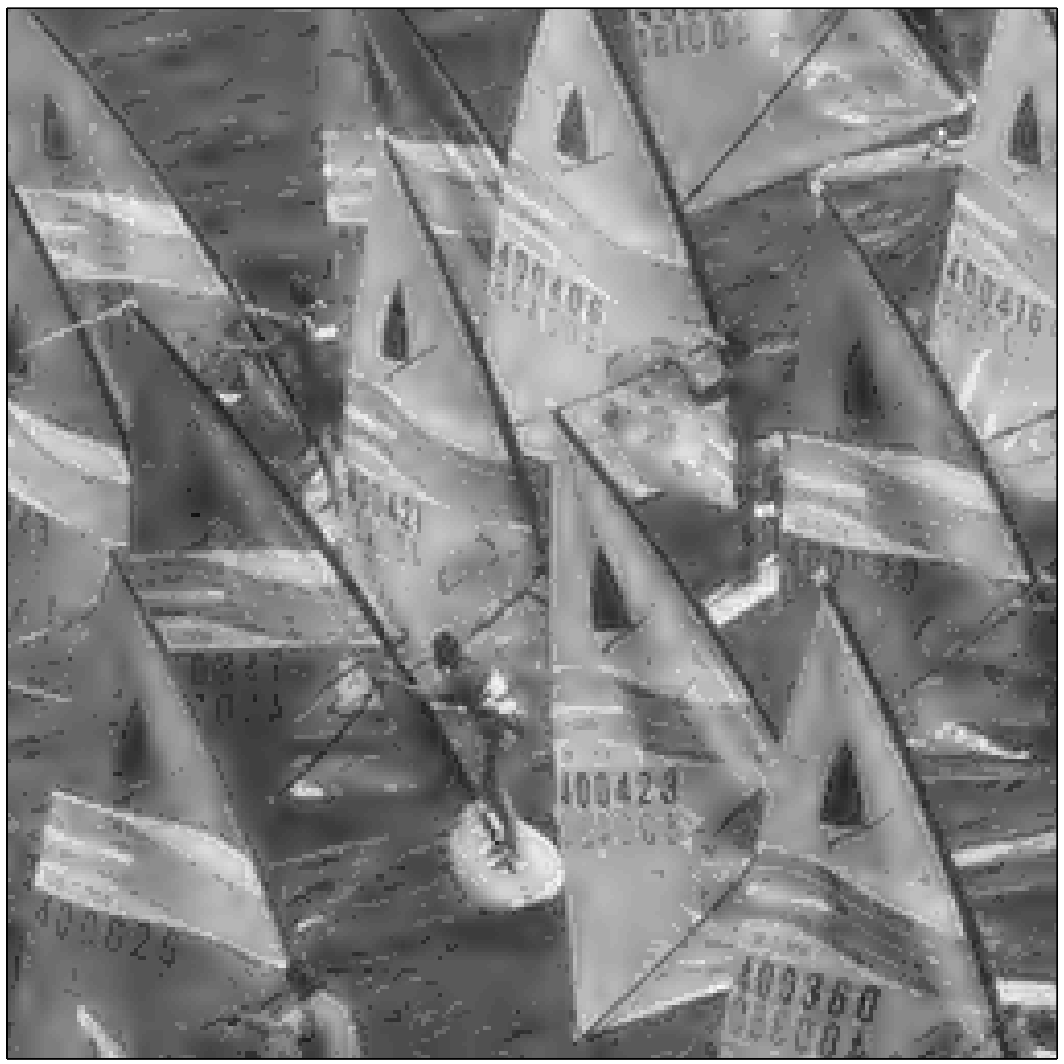
*EPWT approximation* $u^r_{800}$

## Example continued

$N$-term approximation with $N = 2000$.

(a) $u_{1200+800}$ using the new hybrid method

(b) $u_{2000}$ using the 9/7 wavelet transform with 2000 non-zero elements



*(a)*

*(b)*

## Numerical results for the hybrid method

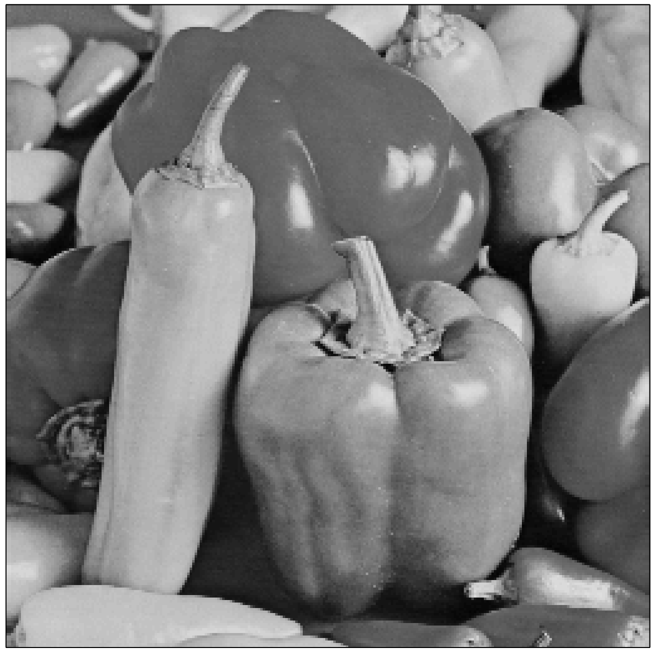
| image | nzc | 9/7 PSNR | Hybrid PSNR | entropy |
|---|---|---|---|---|
| barbara | 500 | 23.33 | 27.28 | 1.0070 |
| cameraman | 500 | 22.54 | 27.49 | 0.9893 |
| clock | 500 | 24.61 | 30.87 | 0.8742 |
| goldhill | 500 | 24.18 | 28.19 | 0.8408 |
| lena | 500 | 23.21 | 27.91 | 0.9022 |
| pepper | 500 | 23.41 | 28.03 | 0.8795 |
| sails | 500 | 21.32 | 25.42 | 0.9190 |

Hybrid: Search for suitable path vectors in each level

*Original image*

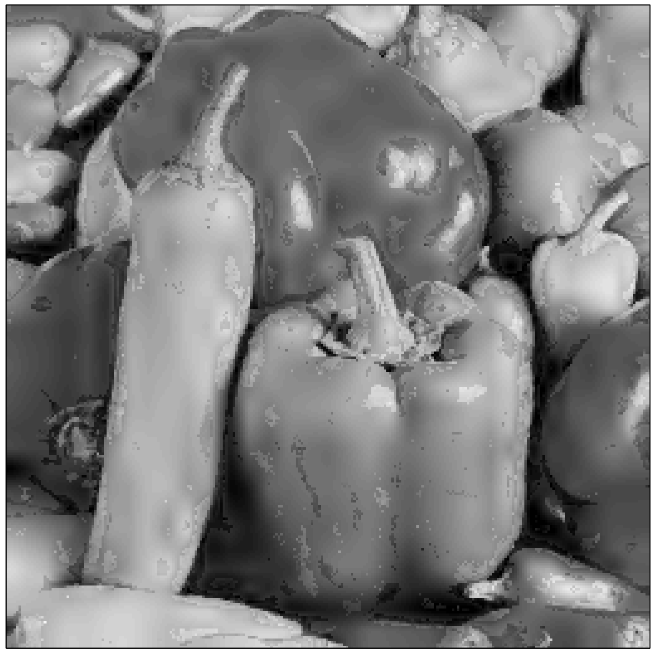
*7/9, 500 coeff.*
*PSNR= 23.21*

*Hybrid, 500 coeff.*
*PSNR=27.91*



*Original image*

*7/9, 500 coeff.*
*PSNR=23.41*



*Hybrid, 500 coeff.*
*PSNR = 28.03*

## Denoising of scattered data using the EPWT approach

**Given**

a set of $d$-dimensional points $\Gamma = \{x_1, x_2, \ldots, x_N\} \subset \mathbb{R}^d$

noisy function values $\tilde{f}(x_j) = f(x_j) + z_j, \; j = 1, \ldots, N$

where

$f : \mathbb{R}^d \to \mathbb{R}$ piecewise smooth

$z_j$ independent and $\mathcal{N}(0, \sigma_j^2)$ distributed (Gaussian noise)

**Wanted** denoised function values $f(x_j)$

## Denoising of scattered data using the EPWT approach

**Given**

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where

$f : \mathbb{R}^d \to \mathbb{R}$ piecewise smooth

$z_j$ independent and $\mathcal{N}(0, \sigma_j^2)$ distributed (Gaussian noise)

**Wanted** denoised function values $f(x_j)$

**Classical wavelet shrinkage**

wavelet decomposition

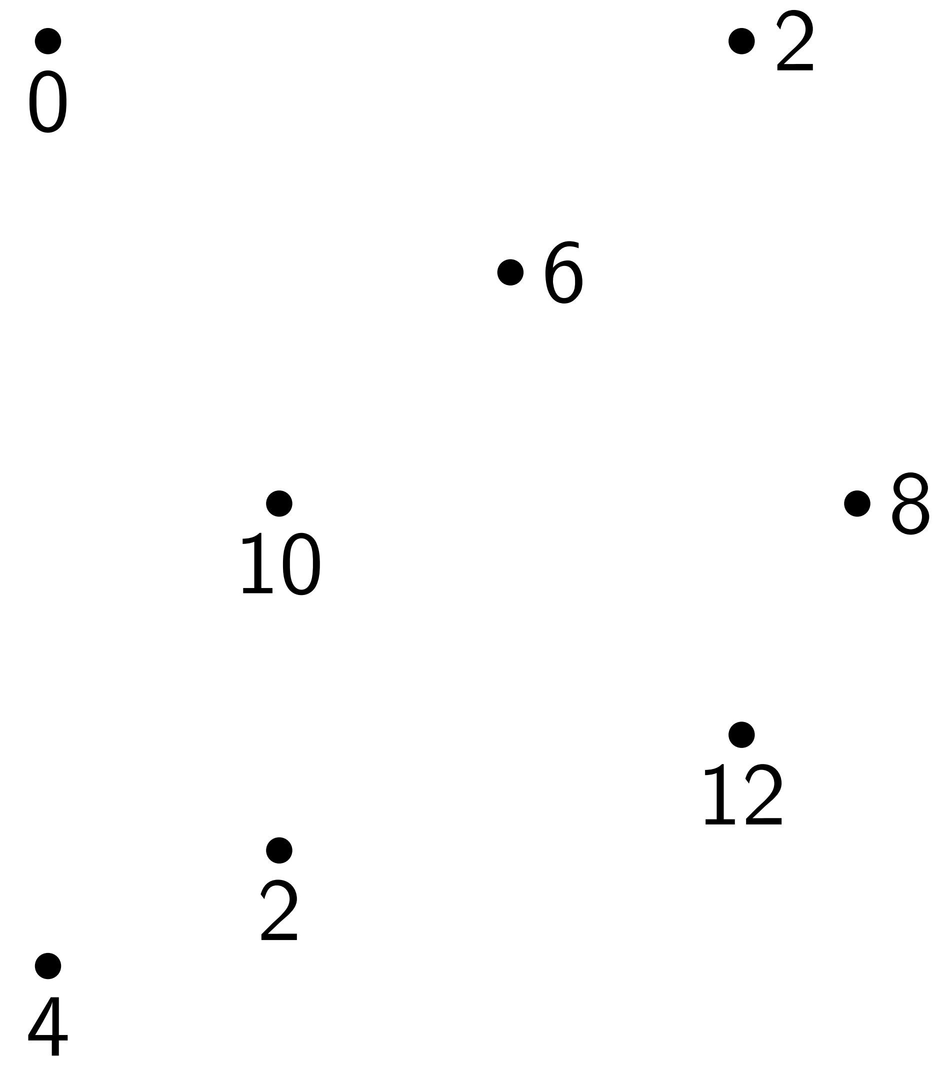
shrinkage: set small high-pass coefficients to zero

wavelet reconstruction

Analogon of cycle shift:
average [ shift $\to$ wavelet shrinkage $\to$ un-shift ]

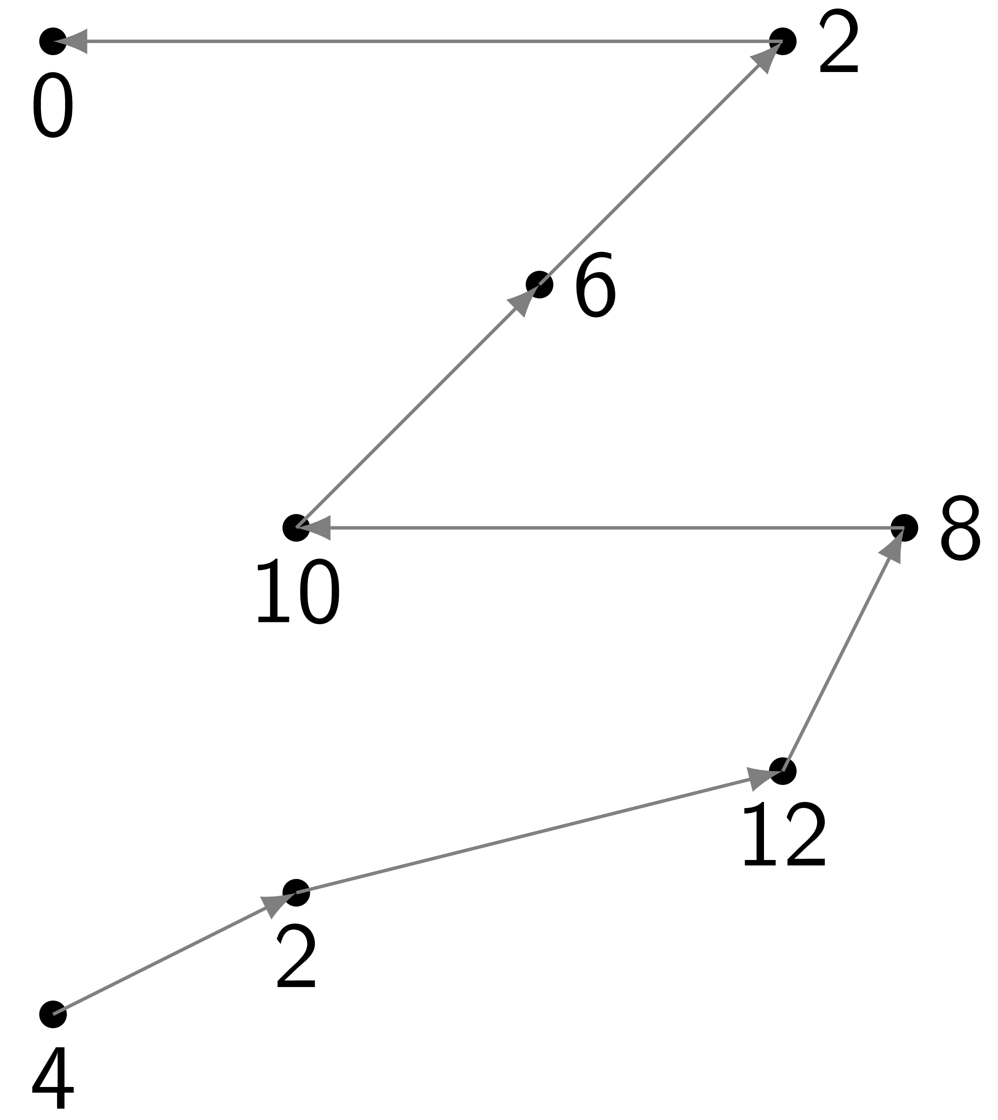
## **Denoising scheme** (wavelet decomposition and shrinkage)

- find path through all points



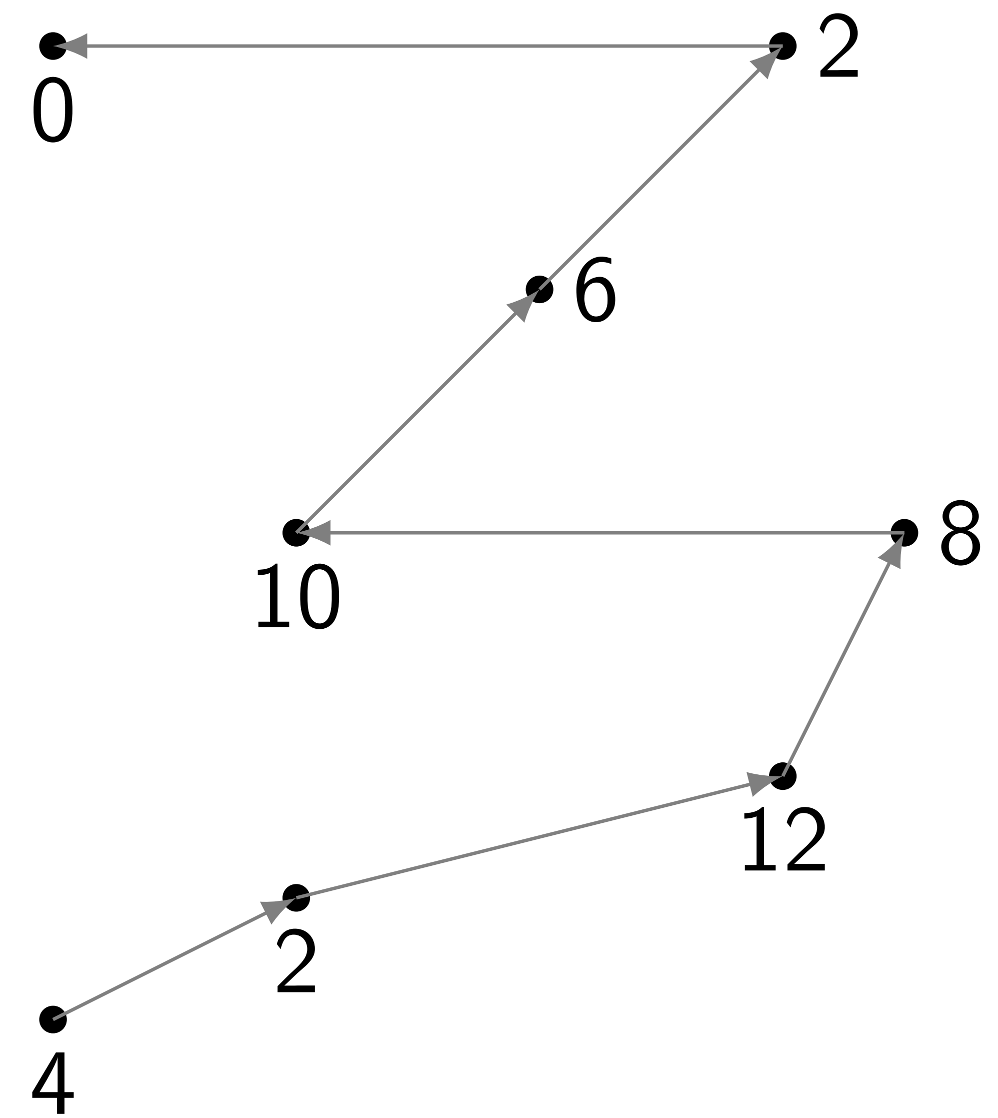
## **Denoising scheme** (wavelet decomposition and shrinkage)

- find path through all points



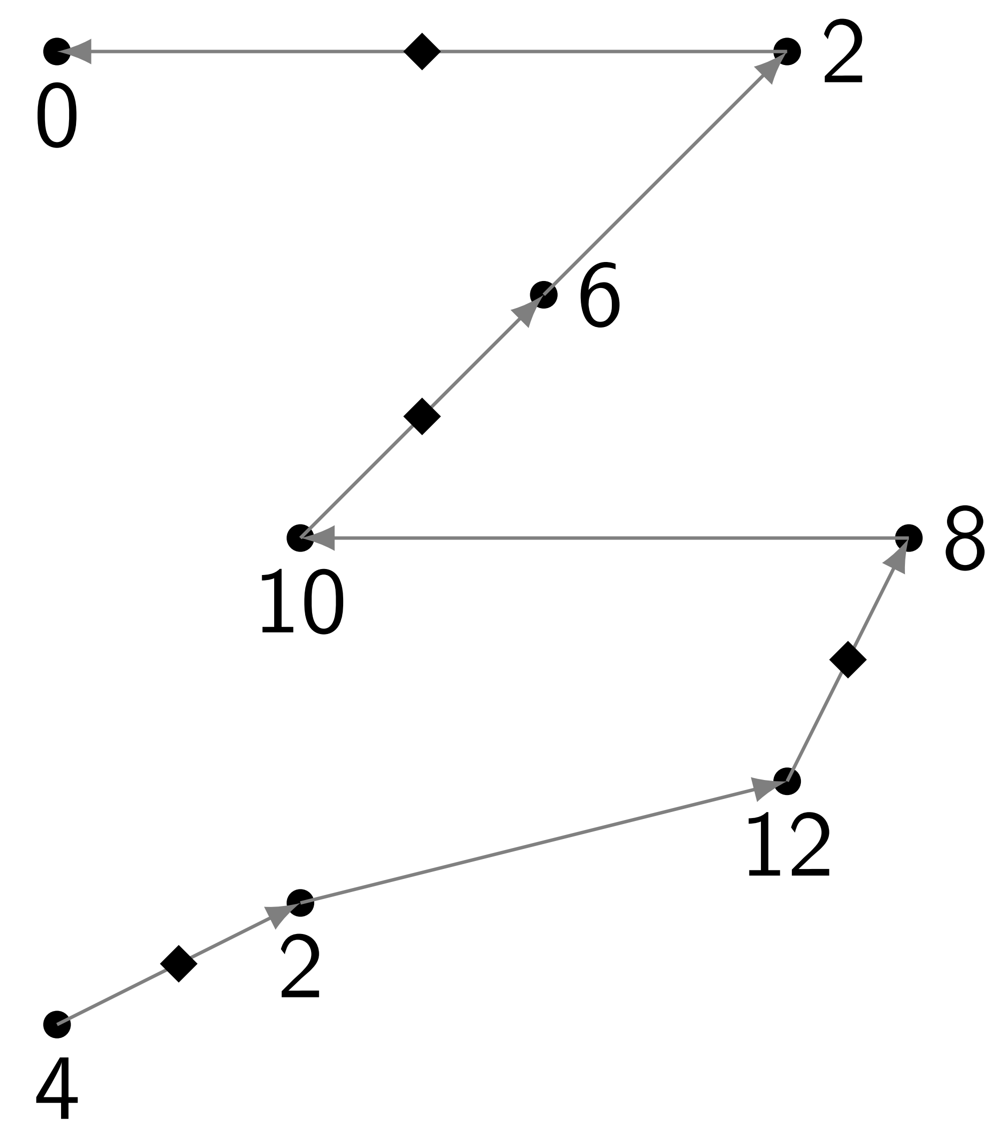
## **Denoising scheme** (wavelet decomposition and shrinkage)

- find path through all points
- apply 1D wavelet transform
  along the path
  low pass coefficients $(3, 10, 8, 1)$
  high pass coefficients $(1, 2, 2, 1)$



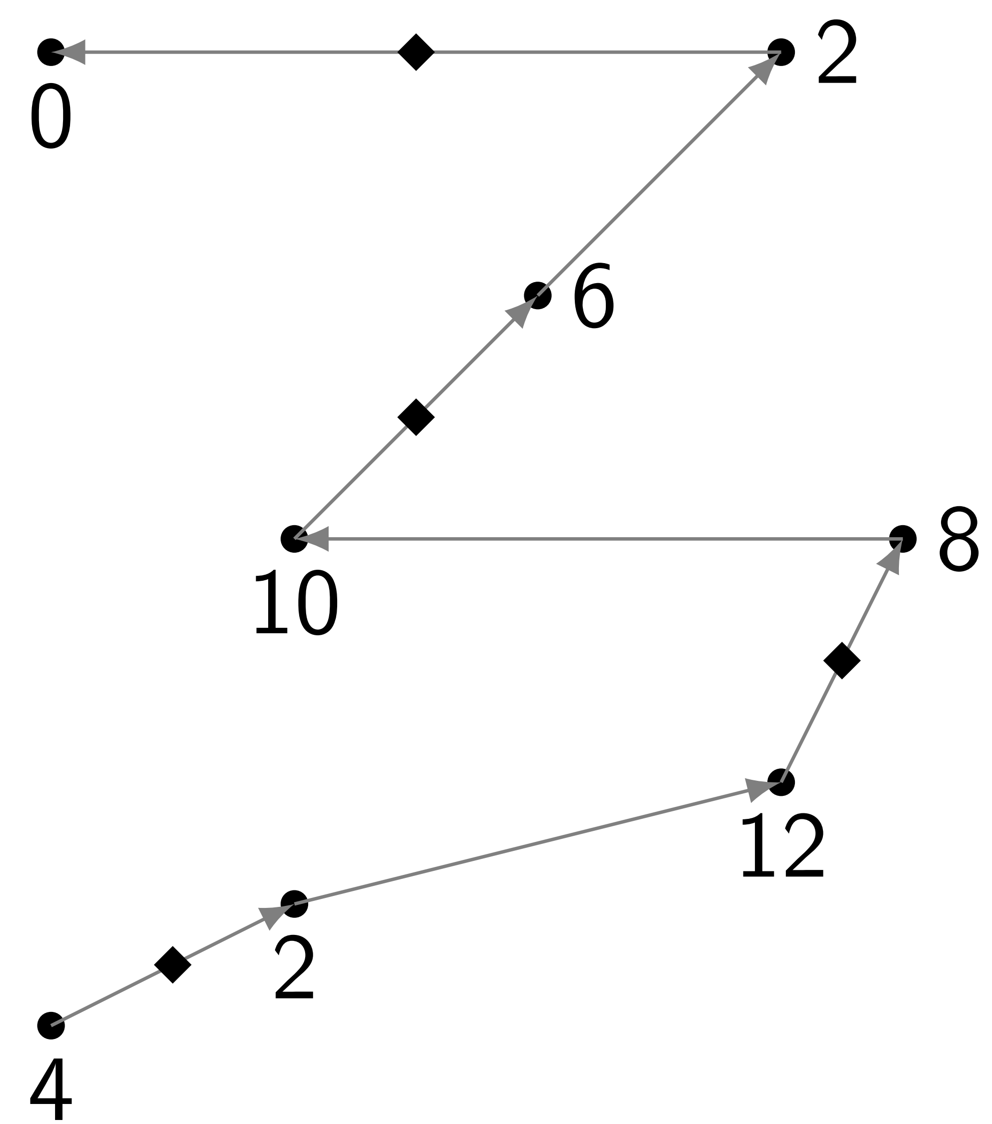
## **Denoising scheme** (wavelet decomposition and shrinkage)

- find path through all points
- apply 1D wavelet transform along the path
  low pass coefficients $(3, 10, 8, 1)$
  high pass coefficients $(1, 2, 2, 1)$
- update point set
- apply shrinkage to wavelet coefficients



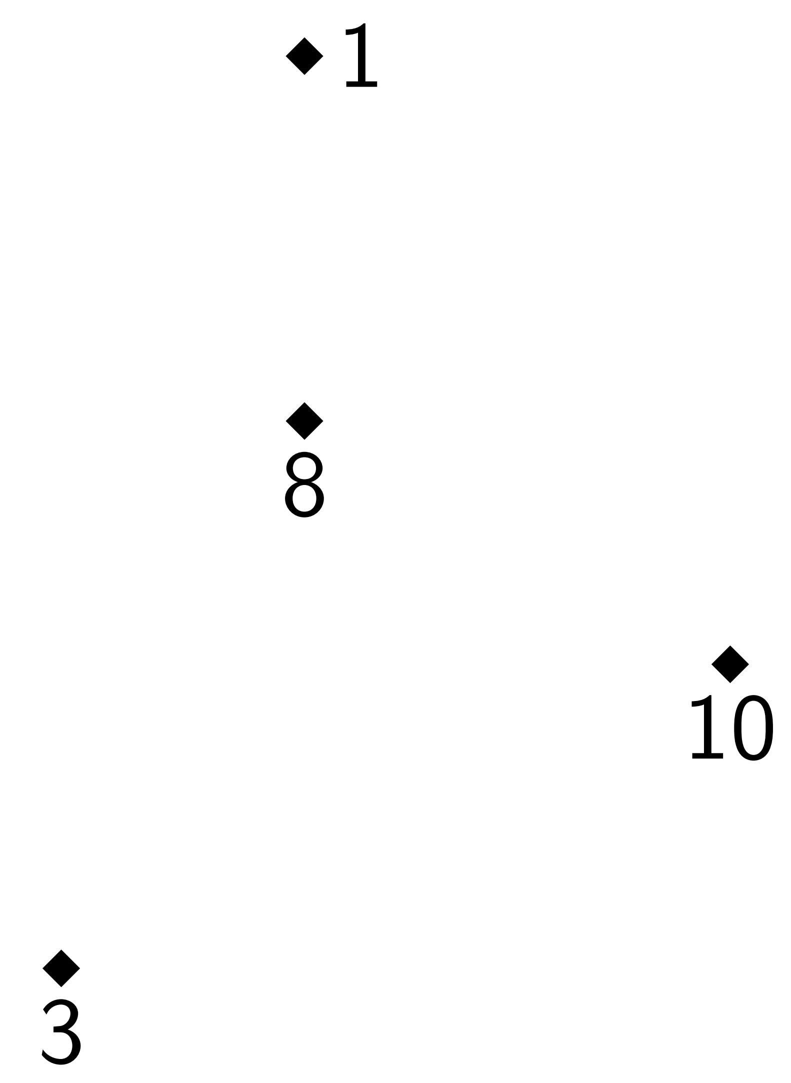
## **Denoising scheme** (wavelet decomposition and shrinkage)

- find path through all points
- apply 1D wavelet transform along the path
  low pass coefficients $(3, 10, 8, 1)$
  high pass coefficients $(1, 2, 2, 1)$
- update point set
- apply shrinkage to wavelet coefficients
- relate low pass coefficients to the updated point set



## **Denoising scheme** (wavelet decomposition and shrinkage)

- find path through all points
- apply 1D wavelet transform along the path
  low pass coefficients $(3, 10, 8, 1)$
  high pass coefficients $(1, 2, 2, 1)$
- update point set
- apply shrinkage to wavelet coefficients
- relate low pass coefficients to the updated point set
- continue at the next level



## Adaptive path reconstruction

- Choose first path index $p(1)$ randomly from $\Gamma := \{1, \ldots, N\}$.
- For $k = 1, \ldots, N-1$ choose $p(k+1)$ such that

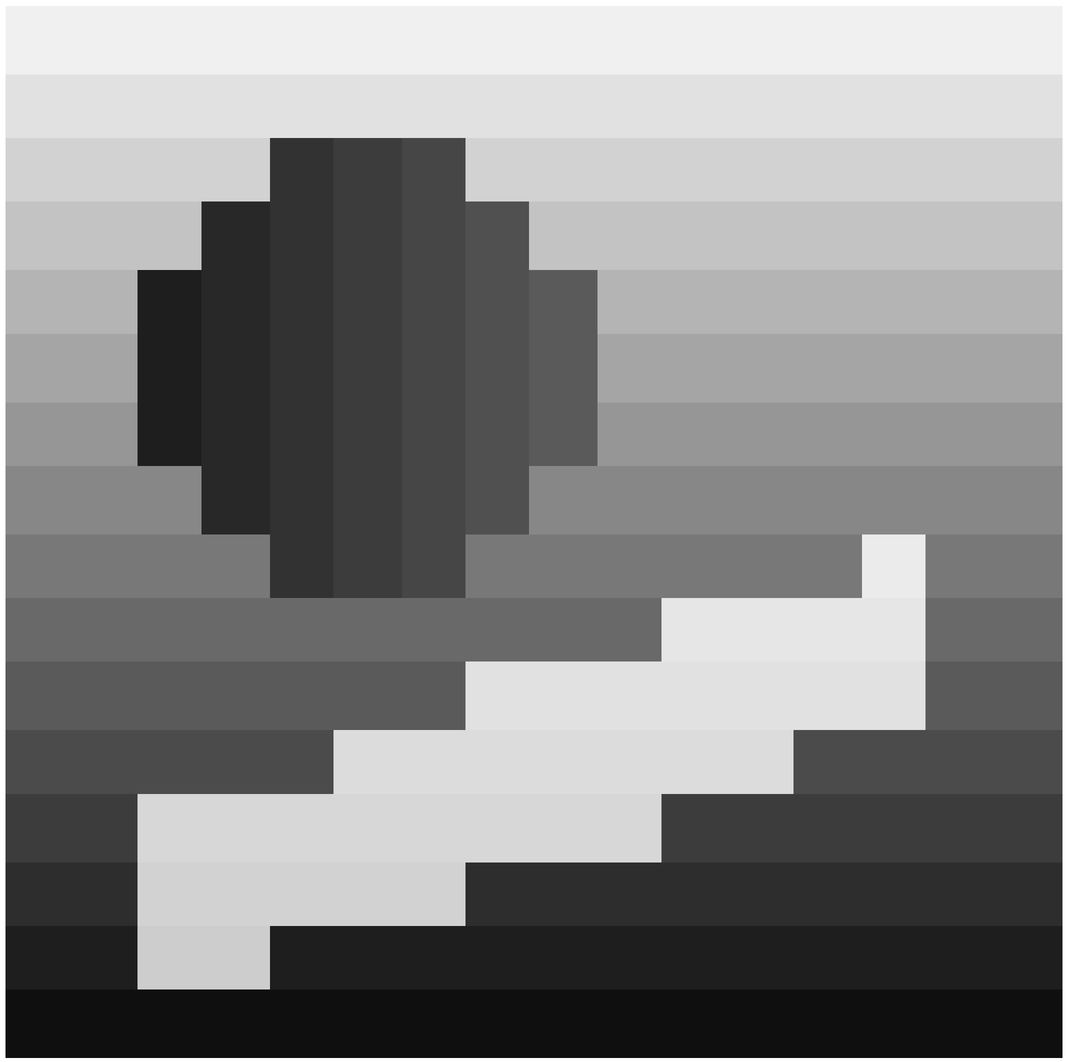
$$x_{p(k+1)} = \underset{x \in N_{C,\theta}(x_{p(k)})}{\operatorname{argmax}} \frac{\langle x_{p(k)} - x_{p(k-1)}, x - x_{p(k)} \rangle}{\|x_{p(k)} - x_{p(k-1)}\| \cdot \|x - x_{p(k)}\|}$$

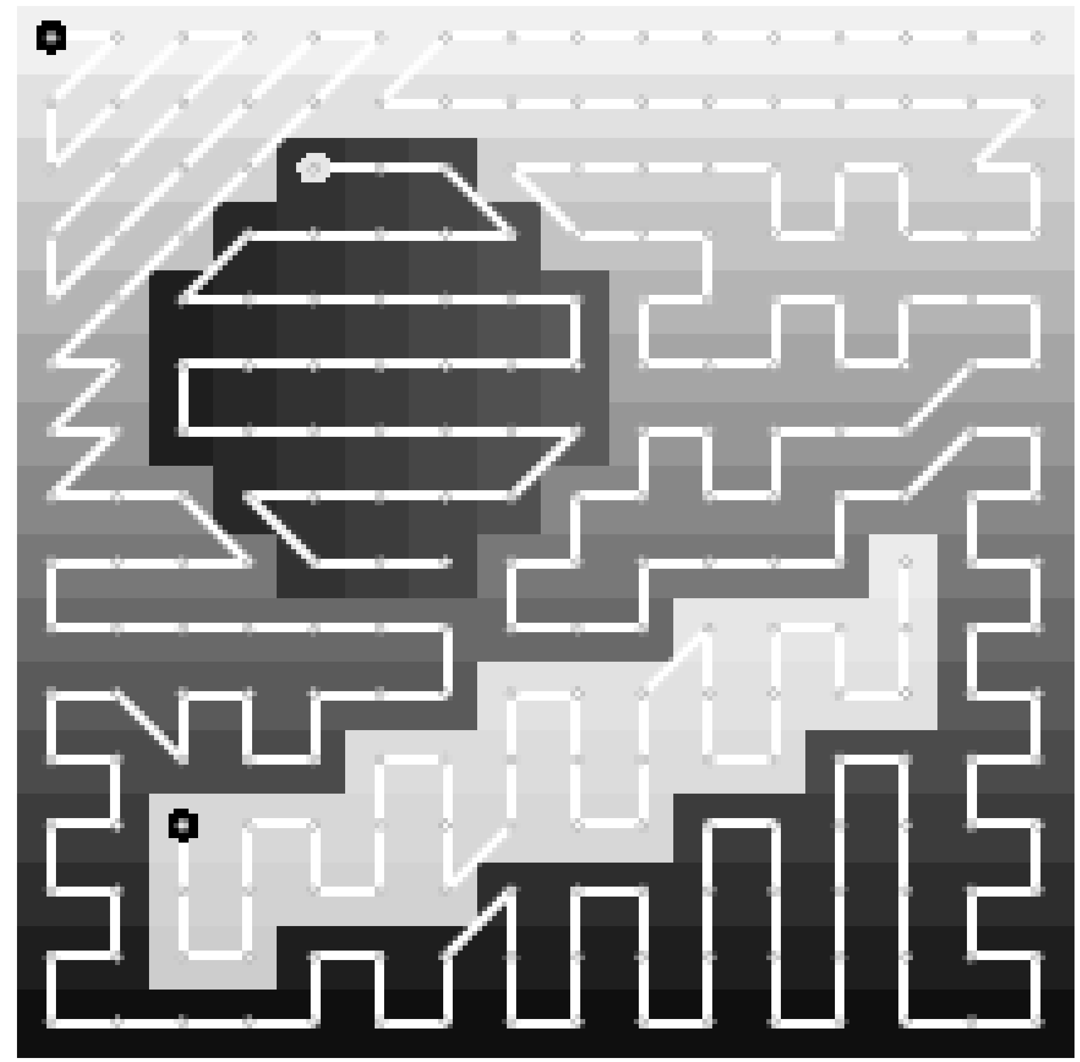
where $N_{C,\theta}(x_{p(k)})$ contains all points $x_r \in \Gamma$ fulfilling:

1. $r \notin \{p(1), \ldots, p(k)\}$
2. $\|x_r - x_{p(k)}\|_2 \leq C$
3. $|f(x_r) - f(x_{p(k)})| \leq \theta$.

If $N_{C,\theta}(x_{p(k)}) = \emptyset$, randomly choose $p(k+1)$ among the indices fulfilling 1 & 2 or only 1.

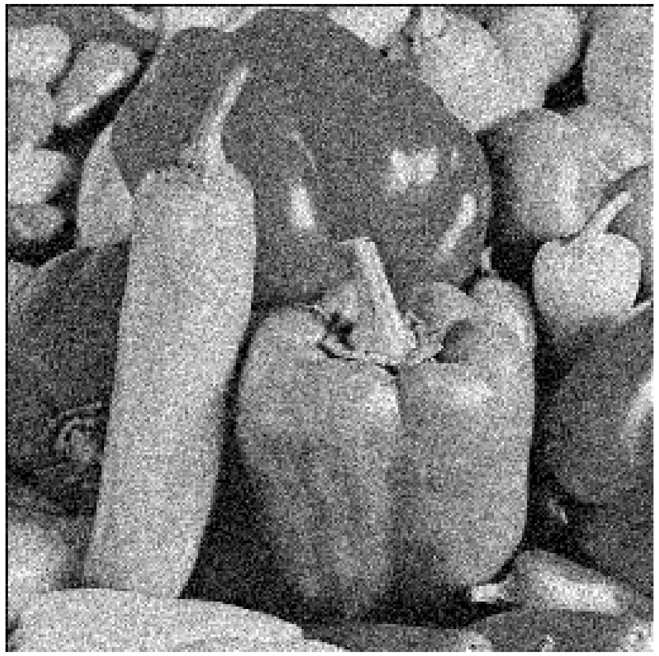
# Example: Adaptive path reconstruction





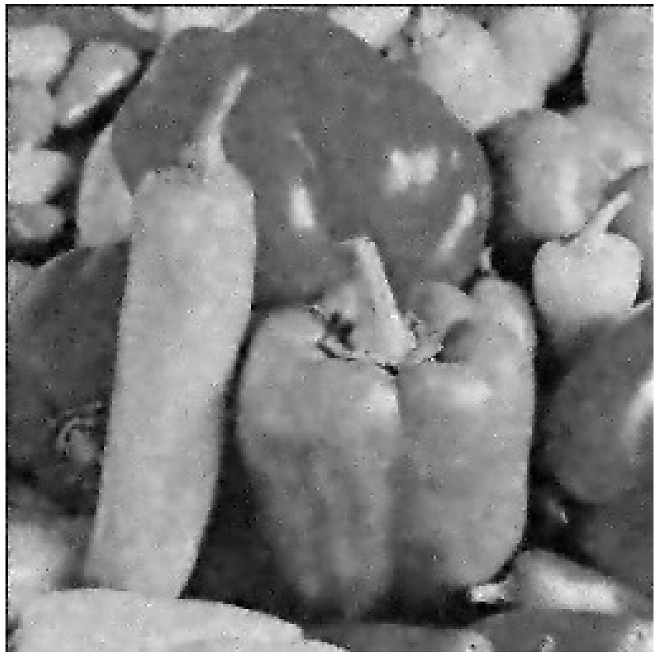
$\sigma = 0.1$

*Original image*



*noisy image*
*PSNR= 19.97*

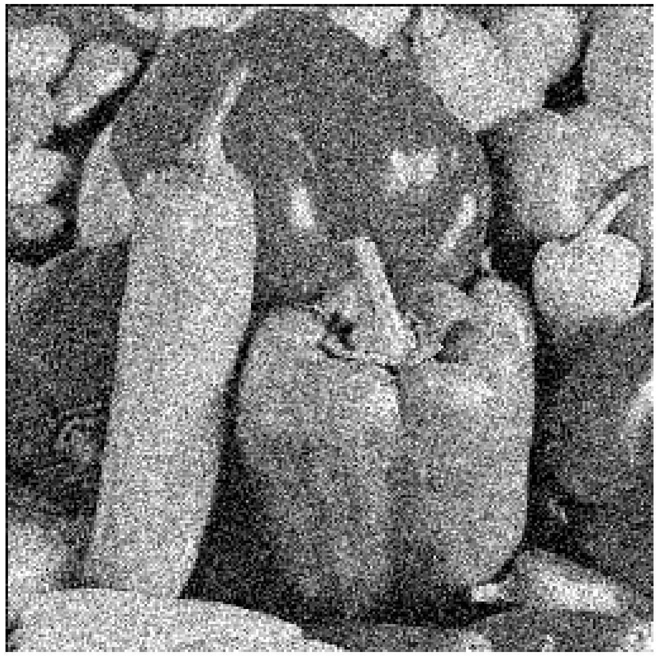
*adaptive path constr.*
*PSNR 29.01*



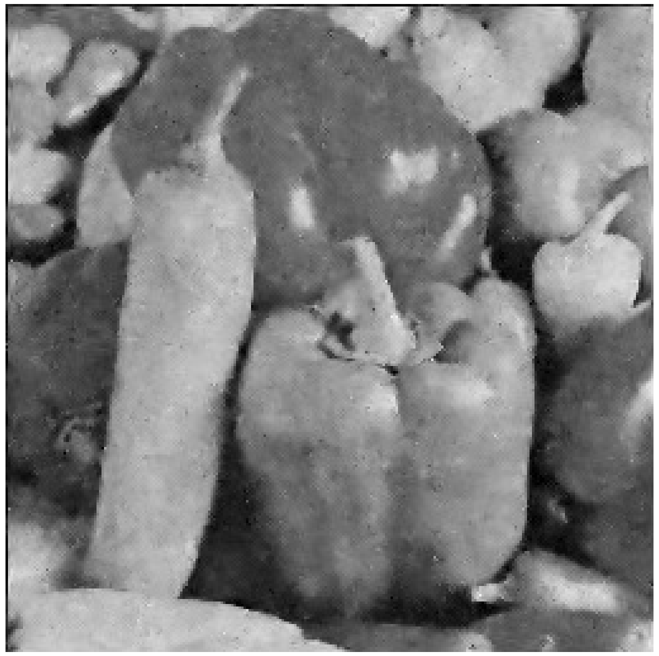
*random path constr.*
*PSNR = 27.96*

$\sigma = 0.15$

*Original image*



*noisy image*
*PSNR= 16.45*



*adaptive path constr.*
*PSNR 26.44*

*random path constr.*
*PSNR = 25.69*

# Comparison of denoising results

| | peppers | peppers | cameraman | cameraman |
|---|---|---|---|---|
| noisy image | 19.97 | 16.45 | 19.97 | 16.45 |
| tensor product wavelet shrinkage | 24.91 | 23.20 | 24.74 | 22.86 |
| with cycle spinning | 28.11 | 25.86 | 27.19 | 25.14 |
| 4-pixel scheme | 28.26 | 26.13 | 27.64 | 25.73 |
| curvelet shrinkage | 26.36 | 23.95 | 25.48 | 23.73 |
| shearlet shrinkage | 26.82 | 25.04 | 26.07 | 24.23 |
| deterministic path | **29.01** | **26.44** | **28.28** | **26.15** |
| random path | 27.96 | 25.69 | 27.44 | 24.85 |

# Denoising of non-rectangular domains

*Original image*

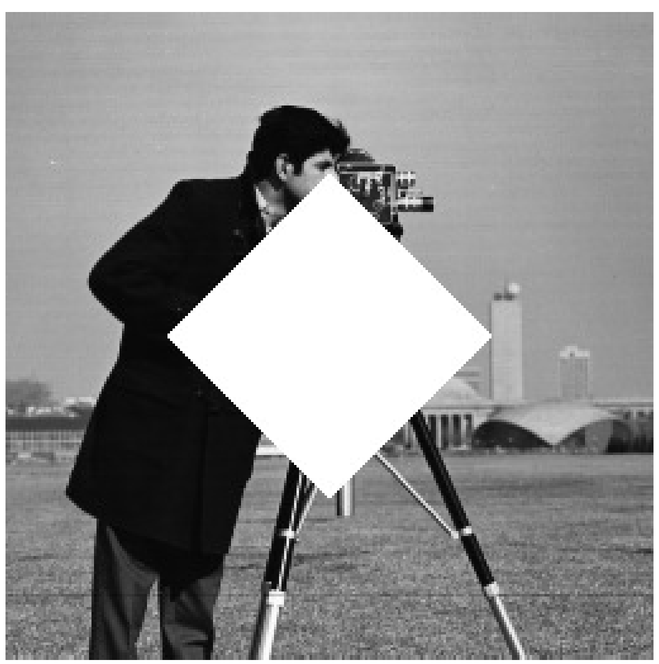
*noisy image*
*PSNR= 19.97*

*denoised image*
*PSNR=27.77*

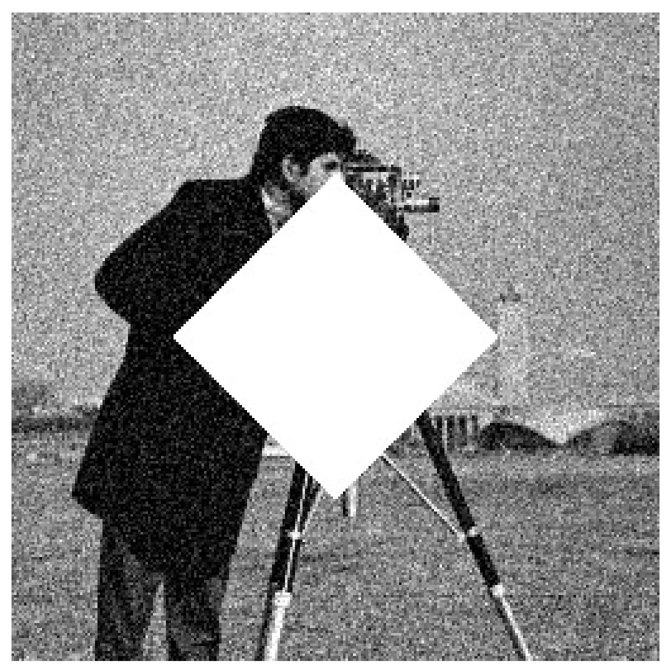
*Original image*

*noisy image*
*PSNR= 19.98*

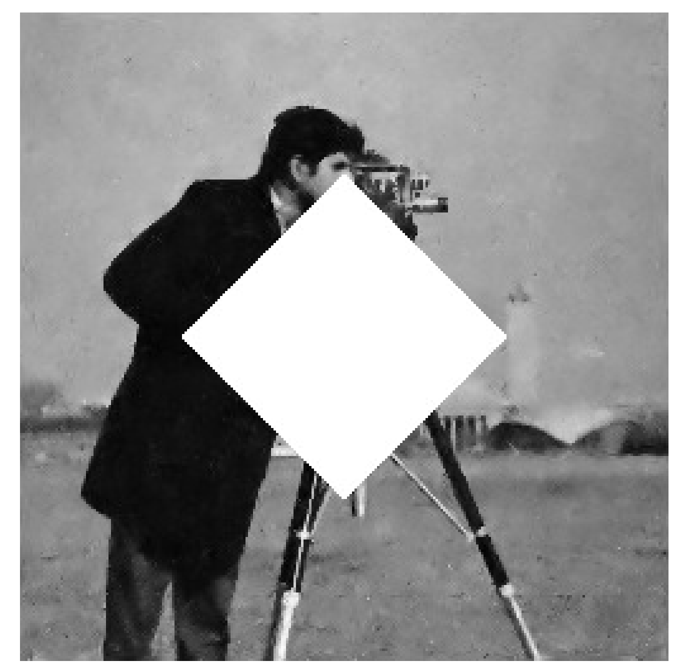
*denoised image*
*PSNR=26.31*



*Original image*



*noisy image*
*PSNR=19.96*



*denoised image*
*PSNR = 28.71*

## Literature


- Gerlind Plonka.
  *The easy path wavelet transform: A new adaptive wavelet transform for sparse representation of two-dimensional data.*
  SIAM Multiscale Modeling and Simulation 7(3) (2009), 1474-1496.

- Gerlind Plonka, Daniela Roşca.
  *Easy Path Wavelet Transform on triangulations of the sphere.*
  Mathematical Geosciences 42(7) (2010), 839-855.

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