Sparse Approximation of Signals and Images

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in collaboration with

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Sparse Approximation of Signals and Images

Outline

- Talk 1: Prony-like methods and applications I
- Talk 2: Prony-like methods and applications II
- Talk 3: Adaptive transforms for sparse image approximation

Prony-like methods and applications I

Outline

- Introduction
- Classical Prony method
- Relations to other methods
- Numerical approach
- Recovering of spline functions from a few Fourier samples
- Reconstruction of sparse linear combinations of translates of a function

Introduction

(a) Consider the following problem:

Function $$f(x) = \sum_{j=1}^{M} c_j \, e^{T_j x}$$

We have function values $f(\ell)$, $\ell = 0, \ldots, 2N-1$, $N \geq M$

We want $c_j \in \mathbb{C}$, $T_j \in [-\alpha, 0] + i[-\pi, \pi)$, $j = 1, \ldots, M$.

Problem:
Find the best M-term approximation, i.e. an optimal linear combination of M terms from the set $\{e^{T_j x} : T_j \in [-\alpha, 0] + i[-\pi, \pi)\}$.

(b) Determine the breakpoints and the associated jump magnitudes of a compactly supported piecewise polynomial if finitely many values of its Fourier transforms are given.

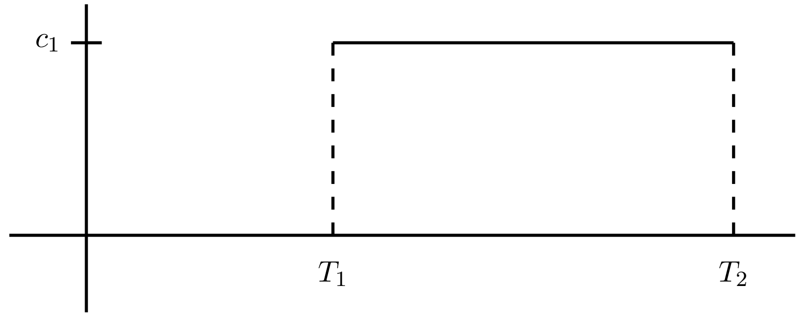
Consider

$$f(x) = \sum_{j=1}^{N} c_j^m \, B_j^m(x),$$

where B_j^m is a B-spline of order m with knots $T_j, \ldots, T_{j+m}$.

How many Fourier samples of f are needed in order to recover f completely?

Example. $f(x) = c_1^1 \, \mathbf{1}_{[T_1, T_2)}(x)$



Introduction

(c) A vector $\mathbf{x} \in \mathbb{C}^N$ is called M-sparse, if $M \ll N$ and if only M components of $\mathbf{x}$ are different from zero.

Problem.
Recover an M–sparse vector $\mathbf{x} \in \mathbb{C}^N$, if only few scalar products

$$y_k = \mathbf{a}_k^* \mathbf{x}, \qquad k = 1, \ldots, 2L$$

with $L \geq M$ with suitable chosen vectors $\mathbf{a}_k \in \mathbb{C}^N$ are given.

With $\mathbf{A}^T = (\mathbf{a}_1, \ldots, \mathbf{a}_{2L}) \in \mathbb{C}^{N \times 2L}$, find an M-sparse solution of the underdetermined system

$$\mathbf{A}\mathbf{x} = \mathbf{y}.$$

Classical Prony method

Function $$f(x) = \sum_{j=1}^{M} c_j \, e^{T_j x}$$

We have $M,\ f(\ell),\ \ell = 0, \ldots, 2M-1$

We want $c_j \in \mathbb{C}, \quad T_j \in [-\alpha, 0] + i[-\pi, \pi),\ j = 1, \ldots, M.$

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Introduce the **Prony polynomial**

$$P(z) := \prod_{j=1}^{M} \left(z - e^{T_j}\right) = \sum_{\ell=0}^{M} p_\ell \, z^\ell$$

with unknown parameters T_j and $p_M = 1$.

$$\begin{aligned} \sum_{\ell=0}^{M} p_\ell f(\ell + m) &= \sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j e^{T_j(\ell+m)} = \sum_{j=1}^{M} c_j \, e^{T_j m} \sum_{\ell=0}^{M} p_\ell \, e^{T_j \ell} \\ &= \sum_{j=1}^{M} c_j e^{T_j m} P(e^{T_j}) = 0, \qquad m = 0, \ldots, M-1. \end{aligned}$$

Reconstruction algorithm

Input: $f(\ell)$, $\ell = 0, \ldots, 2M-1$

- Solve the Hankel system

$$\begin{pmatrix} f(0) & f(1) & \ldots & f(M-1) \\ f(1) & f(2) & \ldots & f(M) \\ \vdots & \vdots & & \vdots \\ f(M-1) & f(M) & \ldots & f(2M-2) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{M-1} \end{pmatrix} = - \begin{pmatrix} f(M) \\ f(M+1) \\ \vdots \\ f(2M-1) \end{pmatrix}.$$

- Compute the zeros of the Prony polynomial $P(z) = \sum_{\ell=0}^{M} p_\ell z^\ell$ and extract the parameters T_j from its zeros $z_j = e^{T_j}$, $j = 1, \ldots, M$.
- Compute c_j solving the linear system

$$f(\ell) = \sum_{j=1}^{M} c_j e^{T_j \ell}, \qquad \ell = 0, \ldots, 2M-1.$$

Output: Parameters T_j and c_j, $j = 1, \ldots, M$.

Literature

[Prony] (1795):	Reconstruction of a difference equation
[Schmidt] (1979):	**MUSIC** (Multiple Signal Classification)
[Roy, Kailath] (1989):	**ESPRIT** (Estimation of signal parameters via rotational invariance techniques)
[Hua, Sakar] (1990):	**Matrix-pencil method**
[Stoica, Moses] (2000):	**Annihilating filters**
[Potts, Tasche] (2010, 2011):	**Approximate Prony method**

Golub, Milanfar, Varah ('99); Vetterli, Marziliano, Blu ('02); Maravić, Vetterli ('04); Elad, Milanfar, Golub ('04); Beylkin, Monzon ('05,'10); Batenkov, Sarg, Yomdin ('12, '13); Filbir, Mhaskar, Prestin ('12); Peter, Potts, Tasche ('11,'12,'13); Plonka, Wischerhoff ('13); ...

Relation to differential and difference equations

Consider a homogeneous linear differential equation with constant coefficients of the form

$$\sum_{k=0}^{M} \xi_k \, f^{(k)}(x) = 0.$$

Then a general solution is of the form

$$f(x) = \sum_{k=1}^{M} c_k \, e^{\lambda_k x},$$

where λ_k are the pairwise distinct zeros of the characteristic polynomial $\sum_{k=0}^{M} \xi_k z^k$.

Hence, the Prony method recovers solutions of differential equations from its functions values $f(l)$, $l = 0, 1, \ldots, 2M - 1$.

Analogously, consider a homogeneous difference equation with constant coefficients of the form

$$\sum_{k=0}^{M} p_k\, f(k+m) = 0, \qquad m \in \mathbb{Z}.$$

Then a general solution is of the form

$$f(x) = \sum_{k=1}^{M} c_k\, e^{\lambda_k x},$$

where λ_k are the pairwise distinct zeros of the characteristic polynomial $\sum_{k=0}^{M} p_k z^k$.

Hence, the Prony method recovers solutions of difference equations from its values $f(l)$, $l = 0, 1, \ldots, 2M-1$.

Relation to linear prediction methods

Let $\mathbf{h} = (h_n)_{n\in\mathbb{N}_0}$ be a discrete signal.

Linear prediction method: Find suitable predictor parameters $p_j \in \mathbb{C}$ such that the signal value $h_{\ell+M}$ can be expressed as a linear combination of the previous signal values h_j, $j = \ell, \ldots, \ell + M - 1$, i.e.

$$h_{\ell+M} = \sum_{j=0}^{M-1} (-p_j)\, h_{\ell+j}, \quad \ell \in \mathbb{N}_0.$$

With $p_M := 1$, this representation is equivalent to a homogeneous linear difference equation. Assuming that

$$h_k = \sum_{j=1}^{M} c_j\, z_j^k, \quad k \in \mathbb{N}_0\,,$$

we obtain the classical Prony problem where the Prony polynomial coincides with the negative value of the forward predictor polynomial.

Relation to annihilating filters

Consider the discrete signal $\mathbf{h} = (h_n)_{n\in\mathbb{Z}}$ with

$$h_n := \sum_{j=1}^{M} c_j z_j^n, \quad j \in \mathbb{Z},\ z_j \in \mathbb{C},\ |z_j| = 1.$$

The signal $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$ is called an **annihilating filter** of $\mathbf{h}$, if

$$(\mathbf{a} * \mathbf{h})_n := \sum_{\ell=-\infty}^{\infty} a_\ell\, h_{n-\ell} = 0, \quad n \in \mathbb{Z}\,.$$

Consider

$$0 = \sum_{\ell=0}^{M} a_\ell \Big(\sum_{j=1}^{M} c_j z_j^{n-\ell}\Big) = \sum_{j=1}^{M} c_j\, z_j^n \Big(\sum_{\ell=0}^{M} a_\ell z_j^{-\ell}\Big)$$

Hence, the z–transform $a(z)$ of the annihilating filter $\mathbf{a}$ and the Prony polynomial $p(z)$ have the same zeros z_j, $j = 1, \ldots, M$, since $z^M\, a(z) = p(z)$ for all $z \in \mathbb{C} \setminus \{0\}$.

Relation to Padé approximation

Consider $h(k) = \sum_{j=1}^{M} c_j\, e^{\mathrm{i}\, T_j\, k} = \sum_{j=1}^{M} c_j\, z_j^k$ with $z_j = e^{\mathrm{i}\, T_j}$.

By

$$\sum_{k=0}^{\infty} z_j^k\, z^{-k} = \frac{1}{1 - z_j/z} = \frac{z}{z - z_j}$$

we find the z-transform of $(h(k))_{k \in \mathbb{N}_0}$,

$$h(z) = \sum_{k=0}^{\infty} h(k)\, z^{-k} = \sum_{j=1}^{M} c_j\, \frac{z}{z - z_j} = \frac{a(z)}{p(z)},$$

where $p(z)$ is the Prony polynomial and $a(z)$ is a polynomial of degree M.

Hence the Prony method can be regarded as Padé approximation.

Numerical approach to Prony's method

Let $f(x) = \sum\limits_{j=1}^{M} c_j \, e^{T_j x}$.

Given: noisy samples $f_k = f(k) + e_k$, $k = 0, \ldots, 2N-1$

Given: $L \geq M$ (upper bound for M) and $N \geq L$

Wanted: M, $c_j \in \mathbb{C}$, $T_j \in [-\alpha, 0] + i[-\pi, \pi)$, $j = 1, \ldots, M$.

We consider the Hankel matrix

$$\mathbf{H}_{2N-L,L+1} = (f_{\ell+m})_{\ell,m=0}^{2N-L-1,L}$$

$$= \begin{pmatrix} f(0) & f(1) & \ldots & f(L) \\ f(1) & f(2) & \ldots & f(L+1) \\ \vdots & \vdots & & \vdots \\ h(2N-L-1) & h(2N-L) & \ldots & f(2N-1) \end{pmatrix}$$

$$= (\mathbf{f}_0, \quad \mathbf{f}_1, \quad \ldots, \quad \mathbf{f}_L)$$

For the submatrix $\mathbf{H}_{M,M} = (\mathbf{f}_0, \ldots, \mathbf{f}_{M-1})$ and $\mathbf{f}_M = (f(M+\ell))_{\ell=0}^{M-1}$ we had

$$\mathbf{H}_M\, \mathbf{p} = -\mathbf{f}_M,$$

where $\mathbf{p} = (p_0, \ldots, p_{M-1})^T$ contains the coefficients of the Prony polynomial.

Consider the companion matrix

$$\mathbf{C}_M(p) = \begin{pmatrix} 0 & 0 & \ldots & 0 & -p_0 \\ 1 & 0 & \ldots & 0 & -p_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -p_{M-1} \end{pmatrix}$$

Then

$$\mathbf{H}_M\, \mathbf{C}_M(p) = (\mathbf{f}_1, \ldots, \mathbf{f}_M) =: \mathbf{H}_M(1)$$

and the eigenvalues of $\mathbf{C}_M(p) = \mathbf{H}_M^{-1}\mathbf{H}_M(1)$ are the wanted zeros $z_j = e^{T_j}$ of the Prony polynomial.

ESPRIT for equispaced sampling (Potts, Tasche '13)

Input: L, $f(\ell)$, $\ell = 0, \dots, 2N-1$, where $L \le N$

- Compute the SVD
$$\mathbf{H}_{2N-L,L+1} = \mathbf{U}_{2N-L}\,\mathbf{D}_{2N-L,L+1}\,\mathbf{W}_{L+1}$$
of the rectangular Hankel matrix $\mathbf{H}_{2N-L,L+1}$. Determine the approximate rank M of $\mathbf{H}_{2N-L,L+1}$.
- Put $\mathbf{W}_{M,L}(s) := \mathbf{W}_{L+1}(1:M,\, 1+s:L+s)$, $\quad s = 0,\, 1$ and
$$\mathbf{F}_M := \left(\mathbf{W}_{M,L}(0)^{\mathrm{T}}\right)^{\dagger} \mathbf{W}_{M,L}(1)^{\mathrm{T}},$$
and compute the eigenvalues $z_j = e^{T_j}$, $j = 1, \dots, M$, of the matrix $\mathbf{F}_M$.
- Compute c_j solving the overdetermined linear system
$$f(\ell) = \sum_{j=1}^{M} c_j e^{T_j \ell}, \qquad \ell = 0, \dots, 2N-1.$$

Output: Parameters T_j and c_j, $j = 1, \dots, M$.

Prony-like methods and applications II

Outline

- Recovering of spline functions from a few Fourier samples
- Reconstruction of sparse linear combinations of translates of a function
- Recapitulation: Classical Prony method
- Reconstruction of M-sparse polynomials
- The generalized Prony method
- Application to the translation and the dilation operator
- Recovery of sparse sums of orthogonal polynomials
- Recovery of sparse vectors

The Prony method

Function: $P(\omega) = \sum_{j=1}^{N} c_j \, e^{-\mathrm{i}\omega T_j}$

Wanted: $c_j \in \mathbb{R} \setminus \{0\}$ and $T_1 < T_2 < \ldots < T_N$

Given: $P(h\ell)$, $\ell = 0, \ldots, N$ with $h > 0$, $|hT_j| < \pi \;\forall j = 1, \ldots, N$

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Given: $P(h\ell)$, $\ell = 0, \ldots, N$ with $h > 0$, $|hT_j| < \pi \;\forall j = 1, \ldots, N$

Idea: Consider the Prony polynomial

$$\Lambda(z) := \prod_{j=1}^{N} \left(z - e^{-\mathrm{i}hT_j}\right) = \sum_{\ell=0}^{N} \lambda_\ell z^\ell$$

with unknown frequencies T_j.

$$\sum_{\ell=0}^{N} \lambda_\ell P(h(\ell - m)) = \sum_{\ell=0}^{N} \lambda_\ell \sum_{j=1}^{N} c_j e^{-\mathrm{i}h(\ell-m)T_j} = \sum_{j=1}^{N} c_j e^{\mathrm{i}hmT_j} \sum_{\ell=0}^{N} \lambda_\ell e^{-\mathrm{i}h\ell T_j}$$

$$= \sum_{j=1}^{N} c_j e^{\mathrm{i}hmT_j} \Lambda(e^{-\mathrm{i}hT_j}) = 0, \qquad m = 0, \ldots, N.$$

Prony method

Input: $P(\ell h)$, $\ell = 0, \ldots, N$, step size h.

- Form the Toeplitz matrix

$$\mathbf{T}_{N+1} = \left(P(h(\ell - m))\right)_{m,\ell=0}^{N} \in \mathbb{R}^{(N+1)\times(N+1)} \text{ with } P(-\ell h) = \overline{P(\ell h)}.$$

- Solve $\mathbf{T}_{N+1}\lambda = \mathbf{0}$, where $\lambda_N = 1$.
- Compute the zeros of the polynomial $\Lambda(z) = \sum_{\ell=0}^{N} \lambda_\ell z^\ell$ on the unit circle and extract the frequencies T_j from $z_j = e^{-\mathrm{i}hT_j}$, $j = 1, \ldots, N$.
- Compute c_j from $P(\ell h) = \sum_{j=1}^{N} c_j e^{-\mathrm{i}\ell h T_j}$, $\ell = 0, \ldots, N$.

Output: frequencies T_j and coeffcients c_j, $j = 1, \ldots, N$.

Problem (b) Reconstruction of step functions and splines

Consider

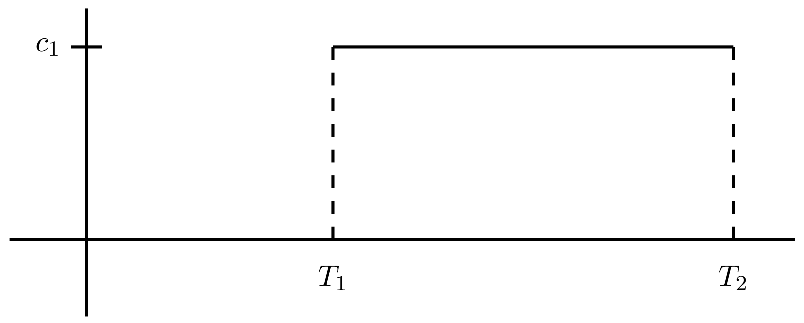
$$f(x) = \sum_{j=1}^{N} c_j^1 \, \mathbf{1}_{[T_j, T_{j+1})}(x).$$

How many Fourier samples are necessary to recover $f(x)$ (i.e. $c_1^1, \ldots, c_N^1 \in \mathbb{R}$ and $T_1 < T_2 < \ldots < T_{N+1}$) uniquely?

Let

$$\widehat{f}(\omega) = \int_{\mathbb{R}^d} f(x) e^{-\mathrm{i}\omega \cdot x} \mathrm{d}x$$

Example. $f(x) = c_1^1 \, \mathbf{1}_{[T_1, T_2)}(x)$



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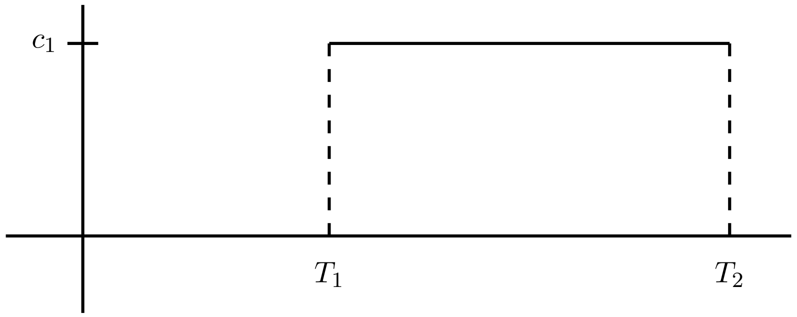


$$\begin{aligned} \widehat{f}(0) &= c_1^1 (T_2 - T_1) \\ \widehat{f}(1) &= c_1^1 \int_{T_1}^{T_2} e^{-\mathrm{i}x} \mathrm{d}x = c_1^1 (\sin T_2 - \sin T_1) + \mathrm{i} c_1^1 (\cos T_2 - \cos T_1) \end{aligned}$$

Step functions

Theorem 1 (Plonka, Wischerhoff '13)

Let $-\infty < T_1 < T_2 < \ldots < T_{N+1} < \infty$,
and let $c_j^1 \in \mathbb{R}$ for $j = 1, \ldots, N$ with $c_j^1 \neq c_{j+1}^1$ for $j = 1, \ldots, N-1$.
Let $h > 0$ satisfy $|hT_j| < \pi$ für $j = 1, \ldots, N+1$.

Then $f(x) = \sum\limits_{j=1}^{N} c_j^1 \, \mathbf{1}_{[T_j, T_{j+1})}(x)$ can be recovered uniquely from $N+1$ Fourier samples $\widehat{f}(\ell h), \ell = 1, \ldots, N+1$.

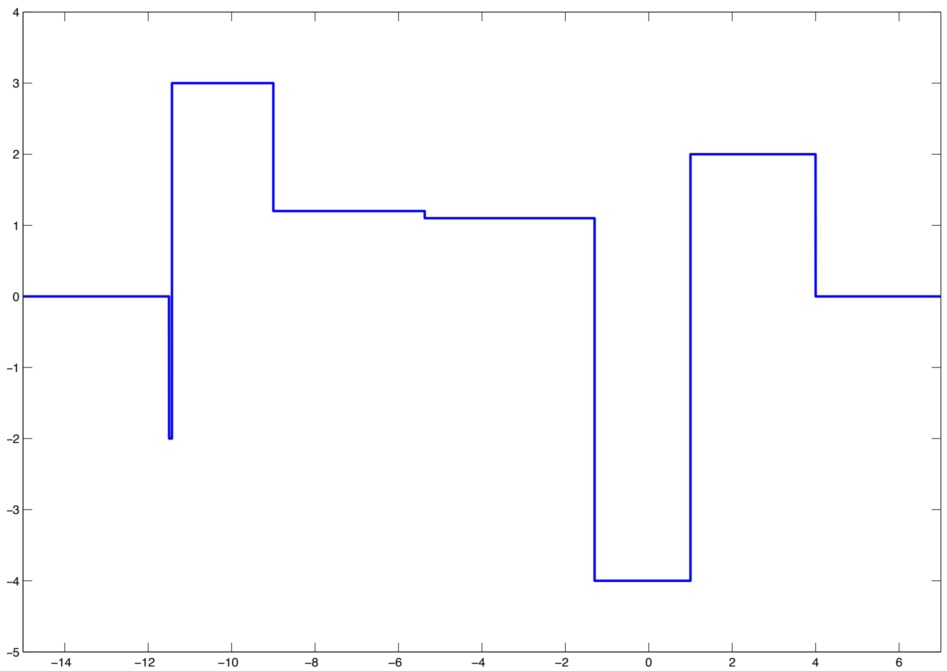
Proof.

$$\widehat{f}(\omega) = \sum_{j=1}^{N} \frac{c_j^1}{\mathrm{i}\omega}\left(e^{-\mathrm{i}\omega T_j} - e^{-\mathrm{i}\omega T_{j+1}}\right) = \frac{1}{\mathrm{i}\omega} \sum_{j=1}^{N+1} c_j^0 e^{-\mathrm{i}\omega T_j}$$

with $c_j^0 = c_j^1 - c_{j-1}^1$.

Apply the Prony method.

Example



j	T_j	$\lvert T_j^* - T_j \rvert \approx$	c_j	$\lvert c_j^* - c_j \rvert \approx$
1	-11.5	$9.81 \cdot 10^{-13}$	-2	$6.24 \cdot 10^{-11}$
2	-11.43	$4.867 \cdot 10^{-13}$	3	$1.91 \cdot 10^{-14}$
3	-9	$5.329 \cdot 10^{-15}$	1.2	$2.864 \cdot 10^{-14}$
4	-5.37	$1.51 \cdot 10^{-14}$	1.1	$3.153 \cdot 10^{-14}$
5	-1.3	$1.554 \cdot 10^{-15}$	-4	$4.441 \cdot 10^{-14}$
6	1	$1.998 \cdot 10^{-15}$	2	$6.306 \cdot 10^{-14}$
7	4	$3.997 \cdot 10^{-15}$		

Step function with $N = 6$

Table: Parameter of the original function and reconstruction error with $h = 0.27$.

Reconstruction of spline functions

Consider

$$f(x) = \sum_{j=1}^{N} c_j^m \, B_j^m(x),$$

where B_j^m is the B-spline of order m with the knots $T_j, \ldots, T_{j+m}$.

Reconstruction of spline functions

Consider

$$f(x) = \sum_{j=1}^{N} c_j^m \, B_j^m(x),$$

where B_j^m is the B-spline of order m with the knots $T_j, \ldots, T_{j+m}$.

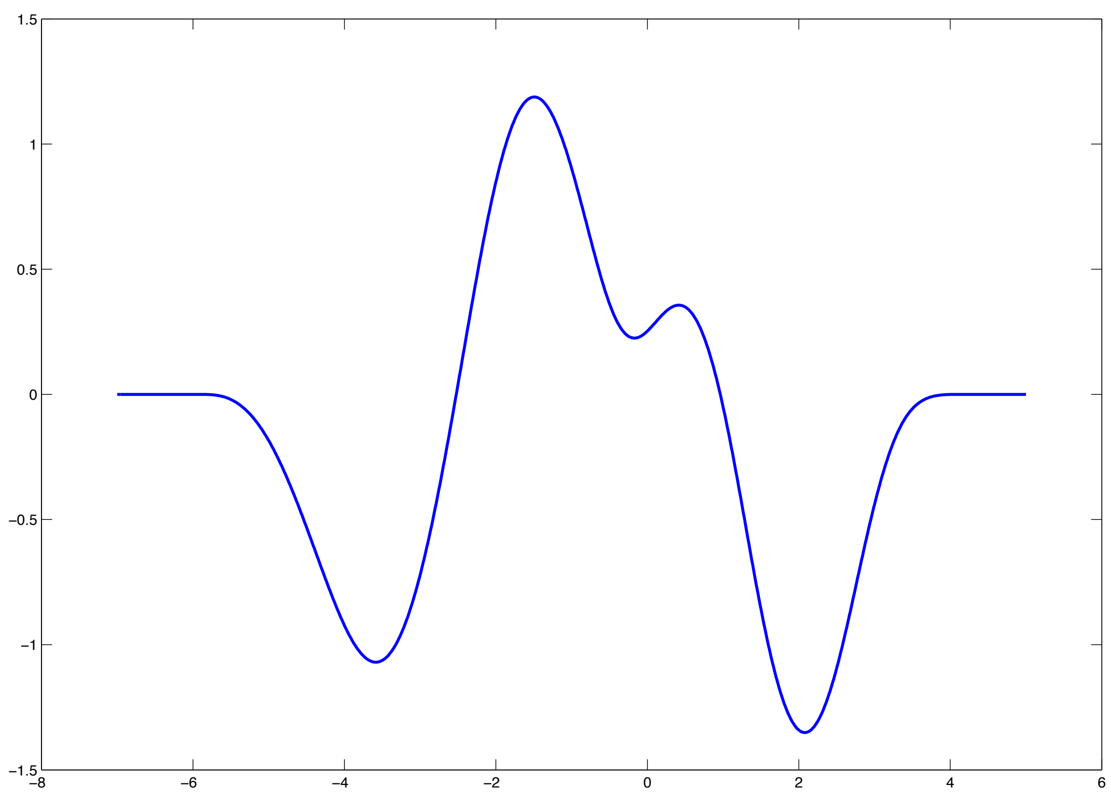
Theorem 2 (Plonka, Wischerhoff '13)

Let $-\infty < T_1 < T_2 < \ldots < T_{N+m} < \infty$ be a knot sequence and let $c_j^m \in \mathbb{R}, j = 1, \ldots, N$.
Let $h > 0$ satisfy $|hT_j| < \pi$ for all $j = 1, \ldots, N + m$.

Then the spline function f can be exactly recovered from $N+m$ Fourier samples $\widehat{f}(\ell h), \ell = 1, \ldots, N + m$.

Idea of proof:
The $(m-1)$-th derivative of f is a step function.

Example



j	T_j	$\lvert T_j^* - T_j \rvert \approx$	c_j	$\lvert c_j^* - c_j \rvert \approx$
1	-6	$2.665 \cdot 10^{-15}$	-3.2	$1.377 \cdot 10^{-14}$
2	-5.8	$4.441 \cdot 10^{-15}$	3.1	$4.441 \cdot 10^{-15}$
3	-4	$4.441 \cdot 10^{-16}$	-0.8	$2.156 \cdot 10^{-13}$
4	-2.25	$4.441 \cdot 10^{-16}$	1.5	$8.136 \cdot 10^{-13}$
5	-0.6	$9.992 \cdot 10^{-16}$	-3	$1.792 \cdot 10^{-12}$
6	0	$2.053 \cdot 10^{-15}$		
7	1.3	$1.11 \cdot 10^{-15}$		
8	2.73	$8.882 \cdot 10^{-16}$		
9	3.5	$1.332 \cdot 10^{-15}$		
10	4.2	$8.882 \cdot 10^{-16}$		

Spline function with $N = 5$ and $m = 5$

Table: Parameter of the original function and reconstruction error $h = 0.5$.

Reconstruction of linear combinations of translates of Φ

Consider

$$f(x) = \sum_{j=1}^{N} c_j \, \Phi(x - T_j)$$

with unknown parameters $c_j \in \mathbb{R}$, $T_j \in \mathbb{R}$, and known Φ.

Reconstruction of linear combinations of translates of Φ

Consider

$$f(x) = \sum_{j=1}^{N} c_j \, \Phi(x - T_j)$$

with unknown parameters $c_j \in \mathbb{R}$, $T_j \in \mathbb{R}$, and known Φ.

Theorem 3 (Plonka, Wischerhoff '13)

Let $-\infty < T_1 < T_2 < \ldots < T_N < \infty$ be a real sequence and $c_j \in \mathbb{R}, j = 1, \ldots, N$.
Let Φ be a given function with $\widehat{\Phi}(\omega) \neq 0$ for $\omega \in (-T, T)$, and let $h > 0$ be such that $|hT_j| < \pi$ for all $j = 1, \ldots, N$.

Then $f(x)$ can be exactly recovered from $N+1$ Fourier samples $\widehat{f}(\ell h)$, $\ell = 0, \ldots, N$.

Proof: Apply Prony method to $\hat{f}(\omega) = \left(\sum_{j=1}^{N} c_j \, e^{-\mathrm{i} T_j \omega}\right) \, \hat{\Phi}(\omega)$.

Two-dimensional case (nonseparable)

Consider

$$f(x_1, x_2) = \sum_{j=1}^{N} c_j \Phi(x_1 - v_{j,1}, x_2 - v_{j,2}).$$

with unknown parameters $c_j > 0$, $\mathbf{v}_j := (v_{j,1}, v_{j,2}) \in \mathbb{R}^2$, and known function Φ.

Fourier transform gives

$$\widehat{f}(\omega_1, \omega_2) = \left(\sum_{j=1}^{N} c_j e^{-\mathrm{i}\cdot(\omega_1 v_{j,1} + \omega_2 v_{j,2})} \right) \widehat{\Phi}(\omega_1, \omega_2)$$

Hence

$$\frac{\widehat{f}(\omega_1, 0)}{\widehat{\Phi}(\omega_1, 0)} = \left(\sum_{j=1}^{N} c_j e^{-\mathrm{i}\cdot(\omega_1 v_{j,1})} \right), \qquad \frac{\widehat{f}(0, \omega_2)}{\widehat{\Phi}(0, \omega_2)} = \left(\sum_{j=1}^{N} c_j e^{-\mathrm{i}\cdot(\omega_2 v_{j,2})} \right)$$

Theorem 4 (Plonka, Wischerhoff '13)

Let Φ be given with $\widehat{\Phi}(\omega_1, \omega_2) \neq 0$ for $|\omega|_2 < T$.
and let $h > 0$ be such that $h\|\mathbf{v}_j\|_2 < \min\{\pi, T\}$ for all $j = 1, \ldots, N$.

Then, the coefficients $c_j > 0$ and the shifts
$v_j = (v_{j,1}, v_{j,2}) \in \mathbb{R}^2$ can be exactly reconstructed from $3N + 1$ Fourier samples

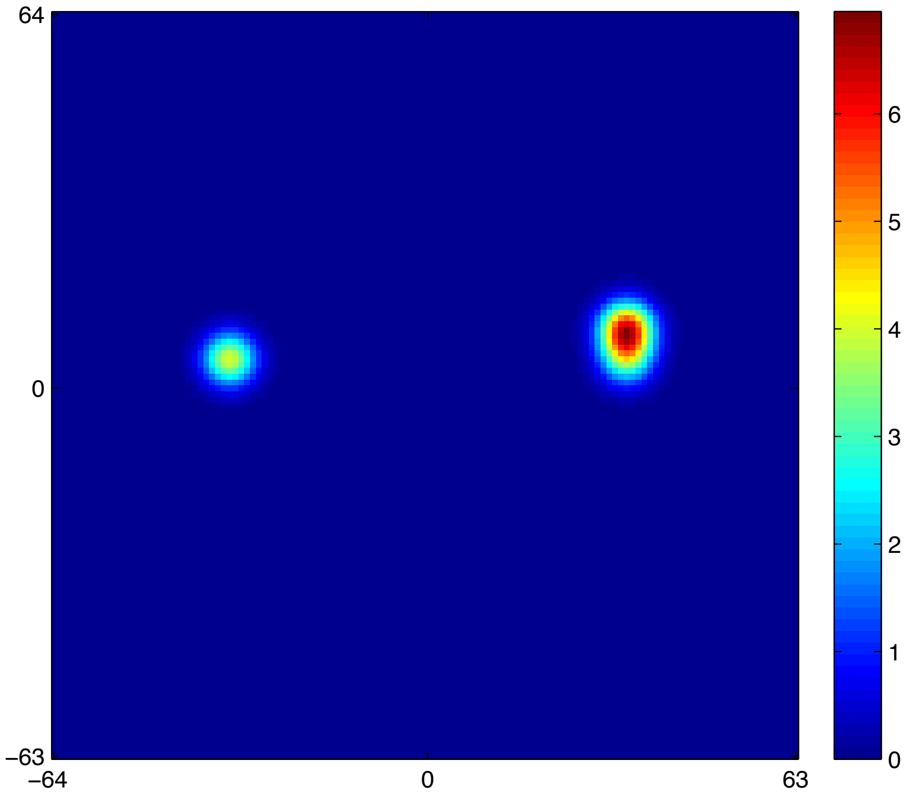
$$\{\widehat{f}(0,0), \widehat{f}(\ell h, 0), \widehat{f}(0, \ell h), \widehat{f}(\cos(\alpha\pi)\ell h, \sin(\alpha\pi)\ell h), \ell = 1, \ldots, N\},$$

where $\alpha \in (0, \frac{1}{2})$ needs to be chosen suitably.

Idea of proof:
Prony method in "x-direction": yields a set of candidates for $v_{j,1}$
Prony method in "y-direction": yields a set of candidates for $v_{j,2}$
We obtain all possible shifts $\mathbf{v}$ as a cartesian product of these sets.

Choose α such that the orthogonal projections of all possible shifts onto the third line (with slope $\tan \alpha$) are pairwise different.

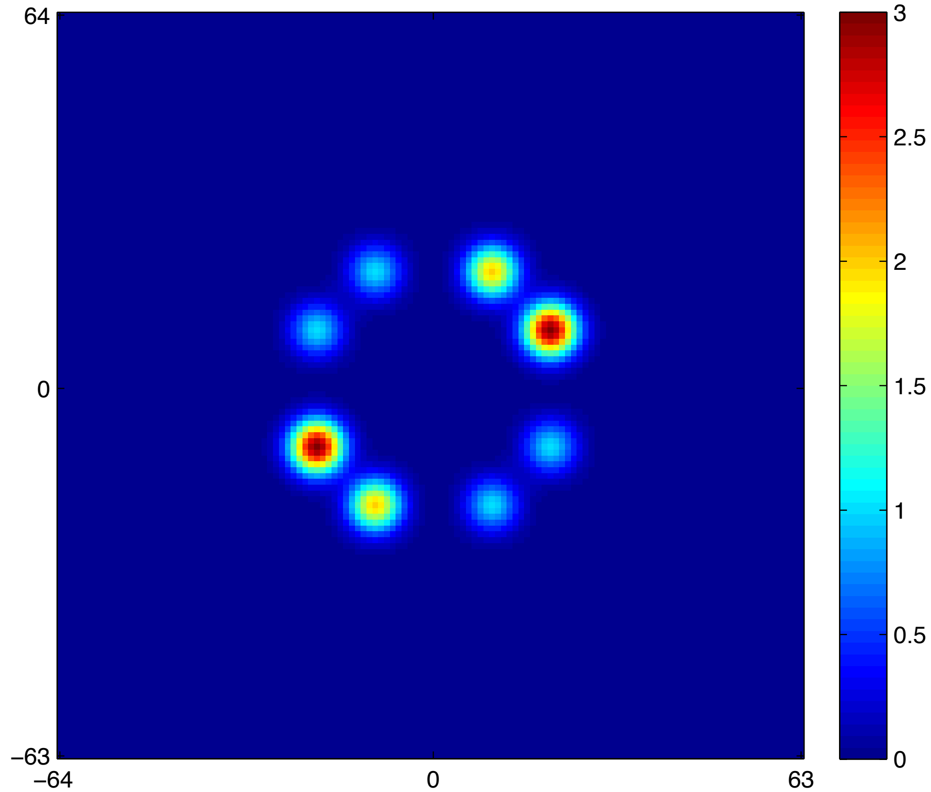
Example 1



Gaussian $\Phi(x_1, x_2) = \exp(\frac{x_1^2+x_2^2}{20})$
with four shifts

j	$v_{j,1}$	$\lvert v^*_{j,1} - v_{j,1}\rvert \approx$	$v_{j,2}$	$\lvert v^*_{j,2} - v_{j,2}\rvert \approx$	c_j	$\lvert c^*_j - c_j\rvert \approx$
1	34	$1.421 \cdot 10^{-14}$	5	$2.958 \cdot 10^{-13}$	3	$2.531 \cdot 10^{-8}$
2	-34	0	5	$2.958 \cdot 10^{-13}$	4	$4.406 \cdot 10^{-9}$
3	34	$1.421 \cdot 10^{-14}$	10	$2.603 \cdot 10^{-11}$	2	$5.795 \cdot 10^{-7}$
4	34	$1.421 \cdot 10^{-14}$	10.25	$6.908 \cdot 10^{-12}$	4	$5.586 \cdot 10^{-7}$

Example 2



Gaussian $\Phi(x_1, x_2) = \exp(\frac{x_1^2+x_2^2}{20})$ with eight shifts

j	$v_{j,1}$	$\lvert v^*_{j,1} - v_{j,1}\rvert \approx$	$v_{j,2}$	$\lvert v^*_{j,2} - v_{j,2}\rvert \approx$	c_j	$\lvert c^*_j - c_j\rvert \approx$
1	-10	$1.954 \cdot 10^{-14}$	20	$1.066 \cdot 10^{-14}$	1	$6.646 \cdot 10^{-9}$
2	10	$1.066 \cdot 10^{-14}$	20	$1.066 \cdot 10^{-14}$	2	$8.371 \cdot 10^{-9}$
3	20	$7.105 \cdot 10^{-15}$	10	$1.421 \cdot 10^{-14}$	3	$9.27 \cdot 10^{-9}$
4	20	$7.105 \cdot 10^{-15}$	-10	$3.02 \cdot 10^{-14}$	1	$1.139 \cdot 10^{-8}$
5	10	$1.066 \cdot 10^{-14}$	-20	$1.066 \cdot 10^{-14}$	1	$6.217 \cdot 10^{-9}$
6	-10	$1.954 \cdot 10^{-14}$	-20	$1.066 \cdot 10^{-14}$	2	$6.036 \cdot 10^{-9}$
7	-20	$2.842 \cdot 10^{-14}$	-10	$3.02 \cdot 10^{-14}$	3	$1.353 \cdot 10^{-8}$
8	-20	$2.842 \cdot 10^{-14}$	10	$1.421 \cdot 10^{-14}$	1	$1.198 \cdot 10^{-8}$

Classical Prony method

Function $$f(x) = \sum_{j=1}^{M} c_j \, e^{T_j x}$$

We have $M,\ f(\ell),\ \ell = 0, \ldots, 2M-1$

We want $c_j \in \mathbb{C}, \quad T_j \in [-\alpha, 0] + i[-\pi, \pi),\ j = 1, \ldots, M.$

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We want $c_j \in \mathbb{C}, \quad T_j \in [-\alpha, 0] + i[-\pi, \pi), \; j = 1, \ldots, M.$

Introduce the **Prony polynomial**

$$P(z) := \prod_{j=1}^{M} \left(z - e^{T_j}\right) = \sum_{\ell=0}^{M} p_\ell \, z^\ell$$

with unknown parameters T_j and $p_M = 1$.

$$\begin{aligned} \sum_{\ell=0}^{M} p_\ell f(\ell+m) &= \sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j e^{T_j(\ell+m)} = \sum_{j=1}^{M} c_j \, e^{T_j m} \sum_{\ell=0}^{M} p_\ell \, e^{T_j \ell} \\ &= \sum_{j=1}^{M} c_j e^{T_j m} P(e^{T_j}) = 0, \qquad m = 0, \ldots, M-1. \end{aligned}$$

Reconstruction of M-sparse polynomials
Ben-Or & Tiwari algorithm

Function $$f(x) = \sum_{j=1}^{M} c_j \, x^{d_j}$$

$$0 \leq d_1 < d_2 < \ldots < d_M = N, \; d_j \in \mathbb{N}_0, \; c_j \in \mathbb{C}.$$

We have $M, \; f(x_0^\ell), \; \ell = 0, \ldots, 2M-1,$ (x_0^ℓ pairwise different)

We want coefficients $c_j \in \mathbb{C} \setminus \{0\}$, indices d_j of "active" monomials

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We want coefficients $c_j \in \mathbb{C} \setminus \{0\}$, indices d_j of "active" monomials

Consider the **Prony polynomial**

$$P(z) := \prod_{j=1}^{M} \left(z - x_0^{d_j} \right) = \sum_{\ell=0}^{M} p_\ell z^\ell$$

with unknown zeros $x_0^{d_j}$ and $p_M = 1$.

Prony polynomial $$P(z) := \prod_{j=1}^{M} \left(z - x_0^{d_j}\right) = \sum_{\ell=0}^{M} p_\ell z^\ell$$

Then

$$\sum_{\ell=0}^{M} p_\ell f(x_0^{\ell+m}) = \sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j x_0^{d_j(\ell+m)} = \sum_{j=1}^{M} c_j x_0^{d_j m} \sum_{\ell=0}^{M} p_\ell x_0^{d_j \ell}$$
$$= \sum_{j=1}^{M} c_j x_0^{d_j m} P(x_0^{d_j}) = 0, \qquad m = 0, \ldots, M-1.$$

Hence

$$\sum_{\ell=0}^{M-1} p_\ell f(x_0^{\ell+m}) = -f(x_0^{\ell+m}), \qquad m = 0, \ldots, M-1.$$

Compute p_ℓ $\quad\Rightarrow\quad$ $P(z)$ $\quad\Rightarrow\quad$ zeros $x_0^{d_j}$ $\quad\Rightarrow\quad$ d_j $\quad\Rightarrow\quad$ c_j.

Literature

[Ben-Or, Tiwari] (1988):	Reconstruction of multivariate M-sparse polynomials
[Grigoriev, Karpinski, Singer] (1990):	Sparse polynomial interpolation over finite fields
[Dress, Grabmeir] (1991):	Interpolation of k-sparse character sums
[Lakshman, Saunders] (1995):	sparse polynomial interpolation in non-standard bases
[Kaltofen, Lee] (2003):	Early termination in sparse polynomial interpolation
[Giesbrecht, Labahn, Lee] (2008)	Sparse interpolation of multivariate polynomials

The generalized Prony method (Peter, Plonka (2013))

Let V be a normed vector space and let $\mathcal{A} : V \to V$ be a linear operator.

Let $\{e_n : n \in I\}$ be a set of eigenfunctions of $\mathcal{A}$ to **pairwise different** eigenvalues $\lambda_n \in \mathbb{C}$,

$$\mathcal{A}\, e_n = \lambda_n\, e_n.$$

Let

$$f = \sum_{j \in J} c_j\, e_j, \qquad J \subset I \text{ with } |J| = M,\ c_j \in \mathbb{C}.$$

Let $F : V \to \mathbb{C}$ be a linear functional with $F(e_n) \neq 0$ for all $n \in I$.

We have $M,\ F(\mathcal{A}^\ell f)$ for $\ell = 0, \ldots, 2M-1$

We want $J \subset I,\ c_j \in \mathbb{C}$ for $j \in J$

Prony polynomial

$$P(z) = \prod_{j \in J} (z - \lambda_j) = \sum_{\ell=0}^{M} p_\ell z^\ell$$

with unknown λ_j, i.e., with unknown J. Hence, for $m = 0, 1, \ldots$

$$\begin{aligned}
\sum_{\ell=0}^{M} p_\ell F(\mathcal{A}^{\ell+m} f) &= \sum_{\ell=0}^{M} p_\ell F\left(\sum_{j \in J} c_j \mathcal{A}^{\ell+m} e_j\right) = \sum_{\ell=0}^{M} p_\ell F\left(\sum_{j \in J} c_j \lambda_j^{\ell+m} e_j\right) \\
&= \sum_{j \in J} c_j \lambda_j^m \left(\sum_{\ell=0}^{M} p_\ell \lambda_j^\ell\right) F(e_j) \\
&= \sum_{j \in J} c_j \lambda_j^m P(\lambda_j) F(e_j) = 0.
\end{aligned}$$

Thus, if $F(\mathcal{A}^\ell f)$, $\ell = 0, \ldots, 2M-1$ is known, then we can compute the index set $J \subset I$ of the active eigenfunctions and the coefficients c_j.

Application to linear operators: shift operator

Choose the **shift operator** $\mathcal{S}_h : C(\mathbb{R}) \to C(\mathbb{R})$, $h > 0$

$$\mathcal{S}_h f(x) := f(x+h)$$

Eigenfunctions of $\mathcal{S}_h$

$$\mathcal{S}_h \, \mathrm{e}^{T_j x} = \mathrm{e}^{T_j (x+h)} = \mathrm{e}^{T_j h} \, \mathrm{e}^{T_j x}, \qquad T_j \in \mathbb{C}, \, \mathrm{Im}\, T_j \in [-\frac{\pi}{h}, \frac{\pi}{h}).$$

Prony method: For the reconstruction of

$f(x) = \sum\limits_{j=1}^{M} c_j \, \mathrm{e}^{T_j x}$ we need $F(\mathcal{S}_h^{\ell} f) = F(f(\cdot + h\ell))$, $\ell = 0, \ldots, 2M-1$.

Put $F(f) := f(x_0)$. $\qquad F(\mathcal{S}_h^{\ell} f) = f(x_0 + h\ell)$

Put $F(f) := \int_{x_0}^{x_0+h} f(x)dx$. $\qquad F(\mathcal{S}_h^{\ell} f) = \int\limits_{x_0+\ell h}^{x_0+(\ell+1)h} f(x)dx$.

Application to linear operators: dilation operator

Choose the **dilation operator** $\mathcal{D}_h : C(\mathbb{C}) \to C(\mathbb{C})$,

$$\mathcal{D}_h f(x) := f(hx)$$

Eigenfunctions of $\mathcal{D}_h$: $\quad \mathcal{D}_h\, x^{p_j} = (hx)^{p_j} = h^{p_j} x^{p_j}, \qquad p_j \in \mathbb{C},\ x \in \mathbb{R}.$
We need: $\quad h^{p_j}$ are pairwise different for all $j \in I$.

Ben-Or & Tiwari method: For reconstruction of

$$f(x) = \sum_{j=1}^{M} c_j\, x^{p_j},$$

we need $F(\mathcal{D}_h^{\ell} f) = F(f(h^{\ell} \cdot)),\ \ \ell = 0, \ldots, 2M-1.$

Put $F(f) := f(x_0)$. $\qquad F(\mathcal{D}_h^{\ell} f) = f(h^{\ell} x_0)$

Put $F(f) := \int_0^1 f(x)dx$. $\qquad F(\mathcal{D}_h^{\ell} f) = \frac{1}{h^{\ell}} \int\limits_0^{h^{\ell}} f(x)dx.$

Application to linear operators: Sturm-Liouville operator

Choose the **Sturm-Liouville operator** $\mathcal{L}_{p,q} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$,

$$\mathcal{L}_{p,q} f(x) := p(x) f''(x) + q(x) f'(x),$$

where $p(x), q(x)$ are polynomials of degree 2 and 1, respectively.

Eigenfunctions are orthogonal polynomials, where $\mathcal{L}_{p,q} Q_n = \lambda_n Q_n$.

$p(x)$	$q(x)$	λ_n	name	symbol
$(1-x^2)$	$(\beta-\alpha-(\alpha+\beta+2)x)$	$-n(n+\alpha+\beta+1)$	Jacobi	$P_n^{(\alpha,\beta)}$
$(1-x^2)$	$-(2\alpha+1)x$	$-n(n+2\alpha)$	Gegenbauer	$C_n^{(\alpha)}$
$(1-x^2)$	$-2x$	$-n(n+1)$	Legendre	P_n
$(1-x^2)$	$-x$	$-n^2$	Chebyshev 1. kind	T_n
$(1-x^2)$	$-3x$	$-n(n+2)$	Chebyshev 2. kind	U_n
1	$-2x$	$-2n$	Hermite	H_n
x	$(\alpha+1-x)$	$-n$	Laguerre	$L_n^{(\alpha)}$

Sparse sums of orthogonal polynomials

Function $$f(x) = \sum_{j=1}^{M} c_{n_j} Q_{n_j}(x).$$

We want: $c_{n_j} \in \mathbb{C} \setminus \{0\}$, indices n_j of "active" basis polynomials Q_{n_j}

Now, f can be uniquely recovered from

$$F(\mathcal{L}_{p,q}^k f) = \mathcal{L}_{p,q}^k f(x_0) = \sum_{j=1}^{M} c_{n_j} \lambda_{n_j}^k Q_{n_j}(x_0), \quad k = 0, \ldots, 2M-1.$$

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Theorem.
For each polynomial f and each $x \in \mathbb{R}$, the values $\mathcal{L}_{p,q}^k f(x)$, $k = 0, \ldots, 2M-1$, are uniquely determined by $f^{(m)}(x)$, $m = 0, \ldots, 4M-2$.

If $p(x_0) = 0$, then $\mathcal{L}_{p,q} f(x_0)$ reduces to $\mathcal{L}_{p,q} f(x_0) = q(x_0) f'(x_0)$, and the values $\mathcal{L}_{p,q}^k f(x_0)$, $k = 0, \ldots, 2M-1$, can be determined uniquely by $f^{(m)}(x_0)$, $m = 0, \ldots, 2M-1$.

Example: Sparse Laguerre expansion

Operator equation

$$x(L_n^{(\alpha)})''(x) + (\alpha + 1 - x)(L_n^{(\alpha)})'(x) = -n\, L_n^{(\alpha)}(x)$$

Sparse Laguerre expansion

$$f(x) = \sum_{j=1}^{6} c_{n_j}\, L_{n_j}^{(0)}(x) \qquad \text{(with } \alpha = 0\text{)}$$

Given values: $f(0),\ f'(0),\ \ldots,\ f^{(11)}(0)$.

j	n_j	c_{n_j}	$\tilde{n}_j$	$\tilde{c}_{n_j}$
1	142	-3	142.0000000018223	-2.99999999999987
2	125	-1	125.0000000494359	-1.00000000000034
3	91	2	90.9999998114290	2.00000000000063
4	69	-3	69.0000003316075	-3.00000000000058
5	53	-1	53.0000003445395	-0.99999999999988
6	11	2	10.9999999973030	2.00000000000004

Application to linear operators: Recovery of sparse vectors

Choose the operator $\mathbf{D} : \mathbb{C}^N \to \mathbb{C}^N$

$$\mathbf{D}\mathbf{x} := \mathrm{diag}(d_0, \ldots, d_{N-1})\,\mathbf{x}$$

with pairwise different d_j.

Eigenvectors of $\mathbf{D}$: $\quad \mathbf{D}\,\mathbf{e}_j = d_j \mathbf{e}_j \qquad j = 0, \ldots, N-1.$

We want to reconstruct

$$\mathbf{x} = \sum_{j=1}^{M} c_{n_j}\,\mathbf{e}_{n_j} \quad c_{n_j} \in \mathbb{C},\ 0 \le n_1 < \ldots < n_M \le N-1.$$

We need $F(\mathbf{D}^\ell \mathbf{x})$, $\ell = 0, \ldots 2M-1$.

Let $F(\mathbf{x}) := \mathbf{1}^{\mathrm{T}}\mathbf{x} := \sum_{j=0}^{N-1} x_j$. Then $F(\mathbf{D}^\ell \mathbf{x}) = \mathbf{1}^{\mathrm{T}}\mathbf{D}^\ell \mathbf{x} = \mathbf{a}_\ell^{\mathrm{T}}\mathbf{x}$,

where $\mathbf{a}_\ell^{\mathrm{T}} = (d_0^\ell, \ldots, d_{N-1}^\ell)$, $\ell = 0, \ldots 2M-1$.

Example: Sparse vectors

Choose

$$\mathbf{D} = \mathrm{diag}(\omega_N^0, \omega_N^1, \ldots, \omega_N^{N-1}),$$

where $\omega_N := \mathrm{e}^{-2\pi \mathrm{i}/N}$ denotes the N-th root of unity.

Then an M-sparse vector $\mathbf{x}$ can be recovered from

$$\mathbf{y} = \mathbf{F}_{2M,N}\, \mathbf{x},$$

where $\mathbf{F}_{2M,N} = (\omega_N^{k\ell})_{k,\ell=0}^{2M-1,N-1} \in \mathbb{C}^{2M\times N}$ contains the first $2M$ rows of the Fouriermatrix of order N.

Variant of the method:
For $(m_1, N) = 1$ and $0 \le m_2 \le N-1$ apply the rows of the Fourier matrix $\mathbf{F}_N$ with indices
$m_2, m_1 + m_2, 2m_1 + m_2, \ldots, (2M-1)m_1 + m_2$ to recover $\mathbf{x}$.

(Heider, Kunis, Potts, Veit '13)

Numerical issues (Generalization of [Potts,Tasche '13])

Numerically stable procedure for steps 1 and 2:

Let $\mathbf{g}_k := (F(\mathcal{A}^{k+m} f))_{m=0}^{M-1}$, $k = 0, \ldots, M-1$ and

$\mathbf{G} := (F(\mathcal{A}^{k+m} f))_{k,m=0}^{M-1} = (\mathbf{g}_0 \ldots \mathbf{g}_{M-1})$ and $\widetilde{\mathbf{G}} = (\mathbf{g}_1 \ldots \mathbf{g}_M)$.

Then (1) yields

$$\mathbf{G}\mathbf{p} = -\mathbf{g}_M,$$

Let $\mathbf{C}(P)$ be the companion matrix of the Prony polynomial $P(z)$ with eigenvalues λ_j, $j \in J$. Then

$$\mathbf{G}\,\mathbf{C}(P) = \widetilde{\mathbf{G}},$$

Hence λ_j are the eigenvalues of the matrix pencil

$$z\mathbf{G} - \widetilde{\mathbf{G}}, \qquad z \in \mathbb{C}.$$

Now apply a QR-decomposition or SVD-decomposition to $(\mathbf{g}_0 \ldots \mathbf{g}_M)$.

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\thankyou