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▶Non Linear Integral Equations of Prandtl's Type

We are interested in the numerical solution of integral equations of the form

$$-\frac{\varepsilon}{\pi}\int_{-1}^{1}\frac{v(t)}{(t-x)^2}dt+\gamma(x,v(x))=f(x)\,,\quad |x|<1\,,\qquad (1)$$

where $0<\varepsilon\leq 1$ and the unknown function v satisfies the boundary conditions

$$v(\pm 1)=0\,.$$

In a two-dimensional crack problem v is the crack opening displacement defined by the density of the distributed dislocations $k(x)$ as

$$v(x)=-\int_{-1}^{x}k(t)dt\,.$$

If we suppose that the nondimensional half crack length is equal to 1, the parameter ε in (1) corresponds to the inverse of the normalized crack length, measured in terms of a physical length parameter which is small relative to the physical crack length. The stress field at a crack tip has a square-root singularity with respect to the distance measured from the crack tip. This requires that the dislocation density $k(x)$ is similarly singular, and it turns out that the Cauchy singular integral remains bounded at the crack tip. Thus, again we suppose that

$$v(x) = \varphi(x)u(x), \qquad \varphi(x) = \sqrt{1-x^2}\,.$$

Then,

$$-\frac{\varepsilon}{\pi}\int_{-1}^{1} \frac{\varphi(t)u(t)}{(t-x)^2}dt + \gamma(x, \varphi(x)u(x)) = f(x)\,, \quad |x| < 1\,.$$

By physical reasons the functions $f(x)$ and $\gamma(x, v)$ are both nonnegative. This happens since f and γ represent the applied tensile tractions that pull the crack surfaces apart and the stiffness of the reinforcing fibres that resist crack opening, respectively.
We are particularly interested in the class of problems for which $\gamma(x, v)$ is a monotone function with respect to v , i.e.

$$[v_1 - v_2][\gamma(x; v_1) - \gamma(x; v_2)] \geq 0, \qquad |x| \leq 1,\ v_1, v_2 \in \mathbb{R}.$$

For example, the case

$$\gamma(x, v) = \Gamma(x)\sqrt{|v|}\,\mathrm{sgn}\, v, \qquad |x| \leq 1,\ v \in \mathbb{R},$$

where $\Gamma(x) > 0$, $|x| \leq 1$, occurs in the analysis of a relatively long crack in unidirectionally reinforced ceramics.

▶The Operators V and F

We consider equation (1) in the pair $X = L^{2,\frac{1}{2}}_{\varphi} \longrightarrow X^* = L^{2,-\frac{1}{2}}_{\varphi}$ where $L^{2,-\frac{1}{2}}_{\varphi} := \left(L^{2,\frac{1}{2}}_{\varphi}\right)^*$ is the dual space of $L^{2,\frac{1}{2}}_{\varphi}$ with the dual product

$$\langle u, v \rangle_{\varphi} = \sum_{n=0}^{\infty} \langle u, p_n^{\varphi} \rangle_{\varphi} \langle v, p_n^{\varphi} \rangle_{\varphi}, \quad u \in X^*, \quad v \in X.$$

We write equation (1) in operator form

$$A(u) := \varepsilon V u + F(u) = f \,, \tag{2}$$

where the Hypersingular Operator $V : L_{\varphi}^{2,\frac{1}{2}} \longrightarrow L_{\varphi}^{2,-\frac{1}{2}}$

$$(Vu)(x) = -\frac{d}{dx}\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(t)u(t)}{t-x}dt, \quad |x| < 1,$$

is an isometrical isomorphism

$$Vu = \sum_{n=0}^{\infty}(n+1)\langle u, p_n^{\varphi}\rangle p_n^{\varphi}\,. \tag{3}$$

For $u, v \in \mathbf{L}_{\varphi}^{2,s}$, we consider the inner product

$$\langle u, v\rangle\varphi, s = \sum_{n=0}^{\infty}(1+n)^{2s}\langle u, p_n^{\varphi}\rangle\langle v, p_n^{\varphi}\rangle\,.$$

We recall that

$$\left|\langle u, v\rangle_{\varphi,s}\right| \leq ||u||_{\varphi,s-t}||v||_{\varphi,s+t}, \quad u \in \mathbf{L}_{\varphi}^{2,s-t}, \, v \in \mathbf{L}_{\varphi}^{2,s+t}.$$

Moreover, for the operator $V : \mathbf{L}_{\varphi}^{2,s+\frac{1}{2}} \longrightarrow \mathbf{L}_{\varphi}^{2,s-\frac{1}{2}}$ defined by (3), we have

$$\langle Vu, u\rangle_{\varphi,s} = ||u||^2_{\varphi,s+\frac{1}{2}}, \quad u \in \mathbf{L}_{\varphi}^{2,s+\frac{1}{2}}.$$

Now we focalize our attention on the operator $F : \mathbf{X} \rightarrow \mathbf{X}^*$ defined by

$$\Big(F(u)\Big)(x) = \gamma(x, \varphi(x)u(x)).$$

With respect to the function $\gamma : [-1, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ we can make different assumptions, for example

(A) $(v_1 - v_2)[\gamma(x, v_1) - \gamma(x, v_2)] \geq 0\,,\ x \in [-1, 1]\,,\ v_1, v_2 \in \mathbb{R}\,,$

and

(B) $|\gamma(x, v_1) - \gamma(x, v_2)| \leq \lambda(x)\,|v_1 - v_2|^{\alpha}\,,\ x \in [-1, 1]$, $v_1, v_2 \in \mathbb{R}\,,$ for some $0 < \alpha \leq 1$, where

$$c_\alpha := \left\{ \begin{array}{ccc} \displaystyle\int_{-1}^{1} [\lambda(x)]^{\frac{2}{1-\alpha}} [\varphi(x)]^{\frac{1+\alpha}{1-\alpha}} \, dx & : & 0 < \alpha < 1 \\ \sup \{ \lambda(x)\varphi(x) : -1 \leq x \leq 1 \} & : & \alpha = 1 \end{array} \right\} < \infty$$

and $\gamma(.,0) \in \mathbf{L}^2_\varphi$.

In any case we assume that $t \longrightarrow \gamma(x,t)$ is continuous on $\mathbb{R}$ for almost all $x \in [-1,1]$ and that $x \longrightarrow \gamma(x,t)$ is measurable for all $t \in \mathbb{R}$.

▶The Solvability of the Hypersingular Integral Equation

In order to show the solvability of (2) we need some Definitions and some Lemmas.

An operator $A : \mathbf{X} \longrightarrow \mathbf{X}^*$ is called
−**hemicontinuous** if the function $s \longrightarrow \langle A(u+sv), w\rangle$ is continuous on $[0,1]$ for any fixed $u, v, w \in \mathbf{X}$;
−**strictly monotone** if $\langle A(u) - A(v), u - v\rangle > 0$ for all $u, v \in \mathbf{X}$ with $u \neq v$;
−**strongly monotone** if there exists a constant $m > 0$ such that $\langle A(u) - A(v), u - v\rangle \geq m\,||u - v||^2_{\mathrm{X}}$ for all $u, v \in X$;
−**coercive** if there exists a function $\rho : [0, \infty) \longrightarrow \mathbb{R}$ satisfying $\lim_{s\to\infty} \rho(s) = \infty$ and$\langle A(u), u\rangle \geq \rho\,(||u||_{\mathrm{X}})\,||u||_{\mathrm{X}}$ for all $u \in \mathbf{X}$.

Lemma 1. *If (A) is fulfilled and if* $\boldsymbol{F}$ *maps* $\mathbf{X}$ *into* $\mathbf{X}^*$, *then the operator* $\boldsymbol{A}:\mathbf{X}\longrightarrow\mathbf{X}^*$ *in (2) is strongly monotone with* $\boldsymbol{m}=\varepsilon$ *for each* $\varepsilon>\mathbf{0}$.

Lemma 2. *If (B) is fulfilled, then the operator* $\boldsymbol{F}$ *maps* $\mathbf{L}^{\mathbf{2}}_{\varphi}$ *into* $\mathbf{L}^{\mathbf{2}}_{\varphi}$, *where* $\boldsymbol{F}:\mathbf{L}^{\mathbf{2}}_{\varphi}\longrightarrow\mathbf{L}^{\mathbf{2}}_{\varphi}$ *is Hölder continuous with exponent* $\boldsymbol{\alpha}$ **.**

Lemma 3. [Zeidler] *If the assumptions (A) and (B) are fulfilled, then the operator* $\boldsymbol{A}$ *is also coercive and equation (2) has a unique solution in* $\mathbf{X}$ *for each* $\boldsymbol{f}\in\mathbf{X}^*$ *and* $\varepsilon>\mathbf{0}$ **.**

Theorem 1. [C.,Criscuolo,Junghanns] *Let the assumptions (A) and (B) be fulfilled. Moreover, let* $\boldsymbol{f}\in\mathbf{L}^{\mathbf{2},\boldsymbol{s}}_{\varphi}$ *for some* $\boldsymbol{s}>\mathbf{0}$ **.** *If* $\boldsymbol{u}\in\mathbf{L}^{\mathbf{2},\mathbf{1}}_{\varphi}$ *implies* $\boldsymbol{F}(\boldsymbol{u})\in\mathbf{L}^{\mathbf{2},\boldsymbol{s}}_{\varphi}$, *then the unique solution* $\boldsymbol{u}^*\in\mathbf{X}$ *of equation (2) belongs to* $\mathbf{L}^{\mathbf{2},\boldsymbol{s}+\mathbf{1}}_{\varphi}$ **.**

▶A Collocation Method

Denote by

$$x_{nk}^{\varphi} = \cos\frac{k\pi}{n+1}, \quad k = 1, \ldots, n,$$

the zeros of the n-th orthonormal polynomial $p_n^{\varphi} = \sqrt{2}U_n$.

Let $\mathbf{X}_n$ denote the space of all algebraic polynomials of degree less than n and let L_n^{φ} be the Lagrange interpolation operator onto $\mathbf{X}_n$ with respect to the nodes x_{nk}^{φ}, $k = 1, \ldots, n$. We recall that L_n^{φ} is defined by

$$L_n^{\varphi}(f;x) = \sum_{k=1^n} f(x_{nk}^{\varphi})\ell_{nk}^{\varphi}(x), \quad \ell_{nk}^{\varphi}(x) = \prod_{r=1, r\neq k}^{n} \frac{x - x_{nr}^{\varphi}}{x_{nk}^{\varphi} - x_{nr}^{\varphi}}.$$

Again, we look for an approximate solution $u_n \in \mathbf{X}_n$ to the solution of equation (2) by solving the collocation equations

$$A_n(u_n) := \varepsilon V u_n + F_n(u_n) = L_n^{\varphi} f\,, \quad u_n \in \mathbf{X}_n\,, \tag{4}$$

where

$$F_n(u_n) := L_n^{\varphi} F(u_n)\,.$$

The following Theorem on the convergence of the collocation method holds true for all $0 < \alpha \leq 1$.

Theorem 2. *Consider equation (2) for a function* $f : (-1, 1) \longrightarrow \mathbb{C}$. *Assume that the conditions (A) and (B) are satisfied. Then the equations (4) have a unique solution* $u_n^* \in \mathbf{X}_n$. *If the solution* $u^* \in \mathbf{X}$ *of (2) belongs to* $\mathbf{L}_\varphi^{2,s+1}$ *for some* $s > \frac{1}{2}$, *then the solutions* u_n^* *converge in* $\mathbf{X}$ *to* u^*,,

$$||u_n^* - u^*||_{\varphi,\frac{1}{2}} \leq \text{const}\, n^{-s} ||u^*||_{\varphi,s+1}$$

Moreover, if we assume that there is an $r \geq \frac{1}{2}$ *such that* $\langle F_n(u_n) - F_n(v_n), u_n - v_n \rangle_{\varphi,r} \geq 0$, $\quad u_n, v_n \in \mathbf{X}_n$, $n \geq n_0$, *and such that* $s \geq r - \frac{1}{2}$. *Then*

$$||u_n^* - u^*||_{\varphi,r+\frac{1}{2}} \leq \text{const}\, n^{r-\frac{1}{2}-s} ||u^*||_{\varphi,s+1} ,$$

and the constant does not depend on n, ε, *and* u^*.

To construct an effective method to solve the approximate the nonlinear system of equations (4), we need to assume $\alpha = 1$. In this case we have

Lemma 4. *If condition (B) with $\alpha = 1$ is fulfilled, then the operator $A : \mathbf{X} \longrightarrow \mathbf{X}^*$ as well as the operator $A_n : \mathbf{X}_n \subset \mathbf{X} \longrightarrow \mathbf{X}_n \subset \mathbf{X}^*$ are Lipschitz continuous with constant $c_1 + \varepsilon$.*

For some fixed $t > 0$ we consider the following equations that are equivalent to (4).

$$u_n = u_n - tV^{-1}\left[\varepsilon V u_n + F_n(u_n) - L_n^{\varphi} f\right] =: B_n(u_n)\,. \quad (5)$$

Furthermore, if we choose $t \in (0, t_\varepsilon)$ with $t_\varepsilon = 2\varepsilon/(c_1^2 + \varepsilon^2)$ then one can prove that the operator $B_n : \mathbf{X}_n \subset \mathbf{X} \longrightarrow \mathbf{X}_n \subset \mathbf{X}$ is a k_ε-contractive mapping with $k_\varepsilon = \sqrt{(1 - t\varepsilon)^2 + t^2 c_1^2} < 1$, i.e.

$$||B_n(u_n^1) - B_n(u_n^2)||_{\varphi, \frac{1}{2}} \leq k_\varepsilon ||u_n^1 - u_n^2||_{\varphi, \frac{1}{2}}.$$

By this, under the assumptions (A) and (B) with $\alpha = 1$ the collocation equations (4) can be solved by applying the method of successive approximation to the fixed-point equation (5).
The smallest possible k_ε for given ε and c_1 is equal to

$$k_\varepsilon^* = \frac{c_1}{\sqrt{\varepsilon^2 + c_1^2}} \qquad \Leftrightarrow \qquad t = t_\varepsilon^* = \frac{\varepsilon}{\varepsilon^2 + c_1^2}.$$

▶A Fast Algorithm

As in the linear case, we seek the solution of (4) in the form

$$u_n(x) = \sum_{k=1}^{n} \xi_{nk} \ell_{nk}^{\varphi}(x) \,,$$

then (4) can be written as

$$\varepsilon\, \mathbf{V}_n \mathbf{\Lambda}_n \xi_n + \mathbf{F}_n(\xi_n) = \eta_n \,, \quad \xi_n = [\xi_{nk}]_{k=1}^{n} \,, \quad \eta_n = [f(x_{nk}^{\varphi})]_{k=1}^{n} \,,$$

$$\mathbf{V}_n = \mathbf{U}_n^T \mathbf{D}_n \mathbf{U}_n \,, \quad \mathbf{U}_n = [p_j^{\varphi}(x_{nk}^{\varphi})]_{j=0,k=1}^{n-1,\,n} \,,$$
$$\mathbf{D}_n = \mathrm{diag}[1, \ldots, n] \,, \quad \mathbf{\Lambda}_n = \mathrm{diag}[\lambda_{n1}^{\varphi}, \ldots, \lambda_{nn}^{\varphi}] \,,$$
$$\mathbf{F}_n(\xi_n) = [\gamma(x_{nk}^{\varphi}, \varphi(x_{nk}^{\varphi})\xi_{nk})]_{k=1}^{n} \,.$$

Recalling that, $\mathbf{U}_n \mathbf{\Lambda}_n \mathbf{U}_n^T = \mathbf{I}_n =: [\delta_{jk}]_{j,k=1}^n$, thus, the fixed point iteration takes the form

$$\xi_n^{(m+1)} = (1-t\varepsilon)\xi_n^{(m)} - t\mathbf{\Lambda}_n^{-1}\mathbf{V}_n^{-1}\left[\mathbf{F}_n(\xi_n^{(m)}) - \eta_n\right], \quad m = 0, 1, \ldots, \tag{6}$$

and $\mathbf{\Lambda}_n^{-1}\mathbf{V}_n^{-1}$ can be written as

$$\mathbf{\Lambda}_n^{-1}\mathbf{V}_n^{-1} = \mathbf{\Lambda}_n^{-1}\mathbf{U}_n^{-1}\mathbf{D}_n^{-1}(\mathbf{U}_n^T)^{-1} = \mathbf{U}_n^T\mathbf{D}_n^{-1}\mathbf{U}_n\mathbf{\Lambda}_n .$$

The matrix $\mathbf{U}_n$ can be written as

$$\mathbf{U}_n = \widetilde{\mathbf{U}}_n\widetilde{\mathbf{D}}_n^{-1}, \quad \widetilde{\mathbf{U}}_n = \sqrt{\frac{2}{\pi}}\left[\sin\frac{jk\pi}{n+1}\right]_{j,k=1}^n, \; \widetilde{\mathbf{D}}_n = \mathrm{diag}\left[\sin\frac{k\pi}{n+1}\right]_{k=1}^n,$$

The matrix $\widetilde{\mathbf{U}}_n$ can be applied to a vector with $O(n\log n)$ computational complexity.

Another way to solve (4) is given by the following collocation-iteration scheme

$$\widetilde{u}_m = L^{\varphi}_{n_m}\left[\widetilde{u}_{m-1} - tV^{-1}\left(\varepsilon V\widetilde{u}_{m-1} + F(\widetilde{u}_{m-1}) - f\right)\right], \quad m = 1, 2, \ldots,$$

where $1 < n_0 < n_1 < n_2 < \cdots$ and $\widetilde{u}_0 \equiv 0$. This is equivalent to

$$\widetilde{u}_m = \widetilde{u}_{m-1} - tV^{-1}\left[\varepsilon V\widetilde{u}_{m-1} + F_{n_m}(\widetilde{u}_{m-1}) - L^{\varphi}_{n_m} f\right] =: T_m(\widetilde{u}_{m-1}) \tag{7}$$

with $T_m u = B_{n_m}(P_{n_m} u)$, and $P_n : \mathbf{L}^{2,\frac{1}{2}}_{\varphi} \longrightarrow \mathbf{L}^{2,\frac{1}{2}}_{\varphi}$ denotes the projection

$$P_n u = \sum_{k=0}^{n-1} \langle u, p^{\varphi}_k \rangle \, p^{\varphi}_k \,.$$

Also in this case, one can prove that the operator $T_m : \mathbf{L}^{2,\frac{1}{2}}_{\varphi} \longrightarrow \mathbf{L}^{2,\frac{1}{2}}_{\varphi}$ is a k_{ε}-contractive operator.

It is possible to show that this projection-iteration method converges under the same assumptions of Theorem 2.

In pratice, we can write the collocation-iteration (7) in the form

$$\widetilde{\boldsymbol{\xi}}_m = (\mathbf{1}-t\,\boldsymbol{\varepsilon})\mathbf{E}_m\widetilde{\boldsymbol{\xi}}_{m-1} - t\boldsymbol{\Lambda}_{n_m}^{-1}\mathbf{V}_{n_m}^{-1}\left[\mathbf{F}_{n_m}(\mathbf{E}_m\widetilde{\boldsymbol{\xi}}_{m-1}) - \boldsymbol{\eta}_{n_m}\right]\,, \quad \widetilde{\boldsymbol{\xi}}_0 = \mathbf{0}\,, \tag{8}$$

where

$$\mathbf{E}_m = \mathbf{U}_{n_m}^T \mathbf{P}_{n_m n_{m-1}} \mathbf{U}_{n_{m-1}} \boldsymbol{\Lambda}_{n_{m-1}}$$

and

$$\mathbf{P}_{nm} = [\delta_{jk}]_{j=1,k=1}^{n\;\;,\;\;m}\,.$$

Since the fast transformations $\mathbf{U}_n$ can be realized most effectively for particular n, for example $n = 2^r - 1$, $r \in \mathbb{N}$ we can combine (8) with the fixed point iteration (6) in this way:

1. Choose a finite sequence $n_1 < n_2 < \cdots < n_{M+1}$ of natural numbers and a natural number K.

2. For $m = 1, \ldots, M$ do K iterations of the form (6) on level $n = n_m$ and use (8) to get a good initial approximation $u_{n_{m+1}}^{(0)}$ for (6) on level n_{m+1}.

3. Apply (6) with the initial approximation $u_{n_{M+1}}^{(0)}$ till the desired accuracy is achieved.

▶Numerical Tests

We solve the hypersingular integral equation (2), choosing different functions $\gamma(x,v)$, by the collocation method together with the fixed point iteration method with $u_n^{(0)} \equiv 0$, and with the combination of the fixed point and the projection iteration method.
In case of the fixed point iteration method the iteration is stopped if the $\mathbf{L}_{\varphi}^{2,\frac{1}{2}}$-norm of the difference of two consecutive iterations is smaller than $toll$, that means

$$||u_n^{(N)} - u_n^{(N-1)}||_{\varphi,\frac{1}{2}} < toll \,, \tag{9}$$

where $u_n^{(m)} = \sum_{k=1}^{n} \xi_{nk}^{(m)} \ell_{nk}^{\varphi}$. For the combination of the fixed point and the projection iteration method we choose the sequence

$$n_1 = 7 < \cdots < n_j = 2^{j+2} - 1 < \cdots < n = n_{M+1} = 2^{M+3} - 1$$

and the number K of iterations realized on the levels $n_1, \ldots, n_M$. The number of iterations needed on the last level n_{M+1} to get the same accuracy (9) is denoted by N_K.

Test 1.a

$$\gamma(x, v) = (1 - x^2)^{-1/2} v$$

. Then, the hypersingular integral equation is linear. $f(x) = f_a(x) = x|x| - \frac{\varepsilon x}{\pi}\left(\frac{2 - 3\,x^2}{\sqrt{1 - x^2}} \ln \frac{1 + \sqrt{1 - x^2}}{1 - \sqrt{1 - x^2}} - 6\right)$ In this case the solution is given by $u^*(x) = x|x|$ (independent of ε). We have, for $n = 2k - 1$,

$$a^*_{2k-1} := \langle u^*, p^{\varphi}_{2k-1} \rangle = \frac{4\sqrt{2}\,(-1)^k (n+1)}{\sqrt{\pi}(n^2-4)n(n+4)}, \quad k = 1, 2, \ldots ,$$

such that $u^* \in \mathbf{L}^{2,2.5-\delta}_{\varphi}$ for all $\delta > 0$. Thus, the convergence rate predicted by Theorem 2 for $t = 0.5$ is

$$||u^*_n - u^*||_{\varphi,1} = O\left(n^{\delta-1.5}\right), \quad \delta > 0 \text{ arbitrarily small},$$

which is confirmed by the numerical results presented in the following table, in which the values

$$d_s = ||u^{(N)}_n - P_n u^*||_{\varphi,s} = \sqrt{\sum_{k=0}^{n-1} (k+1)^{2s} \left[\langle u^{(N)}_n, p^{\varphi}_k \rangle_{\varphi} - a^*_k \right]^2}$$

are presented for $s = 0.5$ and $s = 1$. To compute these values we use the relation $\left[\langle u_n^{(N)}, p_k^{\varphi}\rangle_{\varphi}\right]_{k=0}^{n-1} = \mathbf{U}_n \mathbf{\Lambda}_n \xi_n^{(N)}$.

n	N	N_1	N_3	$d_{1/2}$	d_1	$n^{2.0}d_{1/2}$	$n^{1.5}d_1$
31	16			1.13e-03	5.43e-03	1.090	0.938
63	16			2.93e-04	2.00e-03	1.164	0.999
127	16			7.46e-05	7.21e-04	1.204	1.032
255	16			1.88e-05	2.58e-04	1.225	1.049
511	16			4.73e-06	9.15e-05	1.235	1.057
1023	16	13	12	1.19e-06	3.24e-05	1.241	1.062
2047	16	12	11	2.97e-07	1.15e-05	1.243	1.064
4095	16	11	10	7.42e-08	4.06e-06	1.245	1.065
8191	16	10	10	1.86e-08	1.44e-06	1.246	1.066
16383	16	9	9	4.64e-09	5.08e-07	1.246	1.066
32767	16	9	8	1.16e-09	1.80e-07	1.246	1.066
65535	16	8	7	2.92e-10	6.41e-08	1.256	1.076

Test 1.a: $\varepsilon = 1.0$, $t = 0.75$, $toll = 10^{-12}$

Here the value t is equal to $1.5\, t_\varepsilon^*$ with $t_\varepsilon^* = \frac{\varepsilon}{\varepsilon^2 + c_1^2}$, and $c_1 = 1$. For $t = t_\varepsilon^*$, $N = 31$ iteration steps are needed in (6) to get the same accuracy in (9).

The next table presents the respective results for $\varepsilon = 0.2$. We observe the same convergence rates and that the numbers in the last two columns do not depend on ε as predicted by the second part of Theorem 2 . Here t is about $9\, t_\varepsilon^*$. For $t = t_\varepsilon^*$ $N = 410$ iterations are needed in (6) to fulfil (9).

n	N	N_3	N_5	N_{10}	$d_{1/2}$	d_1	$n^{2.0}d_{1/2}$	$n^{1.5}d_1$
31	40				1.06e-03	5.17e-03	1.018	0.892
63	42				2.82e-04	1.94e-03	1.119	0.972
127	43				7.30e-05	7.11e-04	1.178	1.017
255	44				1.86e-05	2.56e-04	1.210	1.041
511	44				4.70e-06	9.12e-05	1.228	1.053
1023	44	37	36	36	1.18e-06	3.24e-05	1.237	1.060
2047	44	35	33	33	2.96e-07	1.15e-05	1.241	1.063
4095	44	32	30	30	7.42e-08	4.06e-06	1.244	1.065
8191	44	30	27	26	1.86e-08	1.44e-06	1.245	1.065
16383	44	27	24	23	4.64e-09	5.08e-07	1.246	1.066
32767	44	24	20	20	1.16e-09	1.80e-07	1.246	1.066
65535	44	21	17	16	2.92e-10	6.41e-08	1.256	1.076

Test 1.a: $\varepsilon = 0.2\,,\ t = 1.7\,,\ toll = 10^{-12}$

Test 1.b $f(x) = f_b(x) = x|x|$
In this case the solution u^* is unknown. For this reason we compare the approximate solutions $u_n^{(N)}$ with $u_{65535}^{(N)}$. We have $f_b \in \mathbf{L}_{\varphi}^{2,2.5-\delta}$ for all $\delta > 0$. Thus, by Theorem 2

$$||u_n^* - u^*||_{\varphi,1} = O\left(n^{\delta-2.5}\right)\ , \quad \delta > 0 \text{ arbitrarily small.}$$

The numerical results presented in the tables confirm these theoretical estimate. Here we observe the values of the norms
$D_s = \|u_n^{(N)} - u_{65535}^{(N)}\|_{\varphi,s}$ for $s = 0.5$ and $s = 1$.

n	N	N_1	N_3	$D_{1/2}$	D_1	$n^{3.0}D_{1/2}$	$n^{2.5}D_1$
31	14			2.90e-05	1.43e-04	0.865	0.763
63	14			3.83e-06	2.68e-05	0.958	0.845
127	14			4.93e-07	4.90e-06	1.009	0.890
255	14			6.25e-08	8.80e-07	1.036	0.914
511	14			7.87e-09	1.57e-07	1.050	0.926
1023	14	10	8	9.87e-10	2.78e-08	1.057	0.932
2047	14	9	6	1.24e-10	4.93e-09	1.060	0.935
4095	14	8	5	1.55e-11	8.71e-10	1.061	0.935
8191	14	7	3	2.00e-12	1.60e-10	1.100	0.974

Test 1.b: $\varepsilon = 1.0\,,\ t = 0.75\,,\ toll = 10^{-12}$

n	N	N_3	N_5	N_{10}	$D_{1/2}$	D_1	$n^{3.0}D_{1/2}$	$n^{2.5}D_1$
31	37				1.24e-04	6.17e-04	3.697	3.303
63	38				1.76e-05	1.24e-04	4.405	3.919
127	39				2.36e-06	2.36e-05	4.829	4.281
255	39				3.05e-07	4.31e-06	5.064	4.479
511	39				3.89e-08	7.76e-07	5.189	4.582
1023	39	29	26	25	4.91e-09	1.38e-07	5.253	4.635
2047	39	26	22	20	6.16e-10	2.46e-08	5.286	4.662
4095	39	23	17	15	7.72e-11	4.35e-09	5.300	4.673
8191	39	20	13	10	9.77e-12	7.80e-10	5.368	4.738

Test 1.b: $\varepsilon = 0.2\,,\ t = 1.7\,,\ toll = 10^{-12}$

Test 2. We solve the nonlinear hypersingular integral equation with $\gamma(x,v) = |v| \arctan(v)$ and $f(x)$ equal to

$$x^2\sqrt{1-x^2}\arctan\left(x|x|\sqrt{1-x^2}\right) - \frac{\varepsilon x}{\pi}\left(\frac{2-3\,x^2}{\sqrt{1-x^2}}\ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} - 6\right)$$

by the collocation method together with the fixed point iteration method with $u_n^{(0)} \equiv 0$, and with the combination of the fixed point and the projection iteration method.
Condition (B) is fulfilled with $\alpha = 1$ and $c_1 = \pi/2$. The solution is given by $u^*(x) = x|x|$ (independent of ε). In the following tables (where $\varepsilon = 1.0$ and $\varepsilon = 0.2$) we can observe the convergence rate which is predicted by Theorem 2.

n	N	N_3	$d_{1/2}$	d_1	$n^{2.0}d_{1/2}$	$n^{1.5}d_1$
31	12		1.16e-03	5.52e-03	1.116	0.953
63	12		2.97e-04	2.01e-03	1.178	1.007
127	12		7.51e-05	7.24e-04	1.212	1.036
255	12		1.89e-05	2.58e-04	1.229	1.051
511	12		4.74e-06	9.16e-05	1.237	1.058
1023	12	8	1.19e-06	3.25e-05	1.242	1.062
2047	12	8	2.97e-07	1.15e-05	1.244	1.064
4095	12	7	7.43e-08	4.06e-06	1.245	1.065
8191	12	6	1.86e-08	1.44e-06	1.246	1.066
16383	12	6	4.64e-09	5.08e-07	1.246	1.066
32767	12	5	1.16e-09	1.80e-07	1.246	1.066
65535	12	5	2.92e-10	6.41e-08	1.256	1.076

Test 2: $\varepsilon = 1.0\,,\ t = 0.9\,,\ toll = 10^{-12}$

n	N	N_3	N_5	N_{10}	$d_{1/2}$	d_1	$n^{2.0}d_{1/2}$	$n^{1.5}d_1$
31	22				1.16e-03	5.51e-03	1.113	0.951
63	22				2.97e-04	2.01e-03	1.178	1.007
127	22				7.51e-05	7.24e-04	1.211	1.036
255	22				1.89e-05	2.58e-04	1.229	1.051
511	22				4.74e-06	9.16e-05	1.237	1.058
1023	22	15	15	15	1.19e-06	3.25e-05	1.242	1.062
2047	22	14	14	14	2.97e-07	1.15e-05	1.244	1.064
4095	22	12	12	12	7.43e-08	4.06e-06	1.245	1.065
8191	22	11	11	11	1.86e-08	1.44e-06	1.246	1.066
16383	22	10	10	10	4.64e-09	5.08e-07	1.246	1.066
32767	22	9	9	9	1.16e-09	1.80e-07	1.246	1.066
65535	22	8	7	7	2.92e-10	6.41e-08	1.256	1.076

Test 2: $\boldsymbol{\varepsilon = 0.2}$ **,** $t = 3.4$, $toll = 10^{-12}$

▶Newton Methods

To solve the collocation equations

$$A_n(u_n) := \varepsilon V u_n + F_n(u_n) = L_n^{\varphi} f\,, \quad u_n \in \mathbf{X}_n\,, \tag{10}$$

where

$$F_n(u_n) := L_n^{\varphi} F(u_n)\,,$$

we now apply the Newton method. We make the following assumptions:

(i) $\gamma(x,g)$ possesses a partial derivative $\gamma_g(x,g)$ continuous with respect to $g \in \mathbb{R}$ for all $x \in [-1,1]$,

$$\gamma_g(x,g) \geq 0 \quad \text{for all } x \in [-1,1],\ g \in \mathbb{R}, \quad \text{and}$$

$$L := \sup_{(x,g)\in[-1,1]\times R} (\varphi(x)\gamma_g(x,g)) < \infty.$$

Moreover, $\gamma(x, 0) \in L^2_\varphi$.

(ii) $|\gamma_g(x, \varphi u_1) - \gamma_g(x, \varphi u_2)| \leq \lambda(x)|\varphi(x)|^\eta |u_1 - u_2|^\eta$, where $0 < \eta \leq 1$ and $\lambda(x)$ is such that

$$L_0 = \sup_{x \in [-1,1]} [\varphi(x)\lambda(x)] < \infty.$$

We define

$$\left(F_g(u^{(i)})u\right)(x) = \gamma_g(x, \varphi(x)u^{(i)}(x))\varphi(x)u(x) \quad \text{and}$$

$$F'_n(u_n^{(i)})u_n := L_n^\varphi F_g(u_n^{(i)})u_n.$$

The Newton method is described by the following sequence of equations

$$L_n^\varphi[\varepsilon V \triangle u_n^{(i)} + F_g(u_n^{(i)})\triangle u_n^{(i)}] = L_n^\varphi[f - \varepsilon V u_n^{(i)} - F(u_n^{(i)})], \qquad (11)$$

with $u_n^{(i+1)} = u_n^{(i)} + \triangle u_n^{(i)}$, $i = 0, 1, 2, \ldots,$ and $u_n^{(i+1)}, u_n^{(i)}, \triangle u_n^{(i)} \in X_n$. Thus,

$$u_n^{(i+1)} = \sum_{j=0}^{n-1} c_{nj}^{(i+1)} p_j^{\varphi}(x), \qquad u_n^{(i)} = \sum_{j=0}^{n-1} c_{nj}^{(i)} p_j^{\varphi}(x), \qquad \triangle u_n^{(i)} = \sum_{j=0}^{n-1} \triangle c_{nj}^{(i)} p_j^{\varphi}(x).$$

The equations (11) are equivalent to

$$\varepsilon V \triangle u_n^{(i)} + F_n'(u_n^{(i)}) \triangle u_n^{(i)} = L_n^{\varphi}[f - \varepsilon V u_n^{(i)} - F(u_n^{(i)})]. \qquad (12)$$

If we define the following operator

$$A_n u_n := \varepsilon V u_n + F_n(u_n) - L_n^{\varphi} f,$$

and its Frechet–derivative at $u_n^{(i)} \in X_n$

$$A_n^{'}(u_n^{(i)})u_n := \varepsilon V u_n + F_n'(u_n^{(i)})u_n, \qquad u_n \in X_n,$$

we can rewrite equations (12) in the equivalent operator form

$$A_n^{'}(u_n^{(i)})(\triangle u_n^{(i)}) + A_n(u_n^{(i)}) = 0, \qquad u_n^{(i+1)} = u_n^{(i)} + \triangle u_n^{(i)}, \qquad i = 0, 1, 2, \ldots \tag{13}$$

Due to the invariance property, we have that equations (13) are equivalent to

$$\sum_{j=0}^{n-1} \triangle c_{nj}^{(i)} \left[\varepsilon (j+1) p_j^{\varphi}(x_{nk}^{\varphi}) + \gamma_g(x_{nk}^{\varphi}, \varphi(x_{nk}^{\varphi}) \sum_{s=0}^{n-1} c_{ns}^{(i)} p_s^{\varphi}(x_{nk}^{\varphi}))\varphi(x_{nk}^{\varphi}) p_j^{\varphi} \right]$$

$$= -\varepsilon \sum_{s=0}^{n-1} (s+1) c_{ns}^{(i)} p_s^{\varphi}(x_{nk}^{\varphi}) - \gamma(x_{nk}^{\varphi}, \varphi(x_{nk}^{\varphi}) \sum_{t=0}^{n-1} c_{nt}^{(i)} p_t^{\varphi}(x_{nk}^{\varphi})) + f(x_{nk}^{\varphi}),$$

$$c_{nj}^{(i+1)} = c_{nj}^{(i)} + \triangle c_{nj}^{(i)}.$$

For each fixed n, we start from an initial approximation $u_n^{(0)}$. Thus, starting from $[c_{nj}^{(0)}]_{j=0}^{n-1}$ we compute $[\triangle c_{nj}^{(i)}]_{j=0}^{n-1}$ and then $[c_{nj}^{(i+1)}]_{j=0}^{n-1} = [c_{nj}^{(i)}]_{j=0}^{n-1} + [\triangle c_{nj}^{(i)}]_{j=0}^{n-1}$. When $\|u_n^{(i^*)} - u_n^{(i^*-1)}\|_{\varphi, \frac{1}{2}}$ is less or equal to a suitable accuracy, we take the polynomial $u_n^{(i^*)}(x) = \sum_{j=0}^{n-1} c_{nj}^{(i^*)} p_j^{\varphi}(x)$ as approximation of the solution u_n^* of (**??**). The modified version of (13) is

$$A_n'(u_n^{(0)})(\triangle u_n^{(i)}) + A_n(u_n^{(i)}) = 0, \qquad u_n^{(i+1)}(x) = u_n^{(i)} + \triangle u_n^{(i)}, \quad i = 0, 1, 2, \ldots, \tag{14}$$

which is more advantageous from a computational point of view, even if the choice of initial approximation requires more attention. The trouble of the modified Newton method consists in choosing the initial approximation. To solve this problem we propose the following method:

1. For a fixed n_0, we apply the Newton method to $A_{n_0}(u_{n_0}) = 0$ with respect to a chosen $u_{n_0}^{(0)}$ and compute $u_{n_0}^{(i^*)}$.

2. For $n_1 > n_0$ we apply the modified Newton method with the initial guess $u_{n_0}^{(i^*)}$, i.e.
$$A'_{n_1}(u_{n_0}^{(i^*)})(\triangle u_{n_1}^{(i)}) + A_{n_1}(u_{n_1}^{(i)}) = 0,$$
and compute $u_{n_1}^{(i^*)}$.

3. Then, we iterate.

In particular, in our numerical test we shall apply the Newton method with

$n_0 = 2$ and $u_{n_0}^{(0)} = 0$. Then, for all $n > n_0$, we shall apply the modified Newton method in the form

$$A'_n(u_{n-1}^{(i^*)})(\triangle u_n^{(i)}) + A_n(u_n^{(i)}) = 0. \tag{15}$$

For all n, we call the steps to compute $u_n^{(i^*)}$ (without considering the previous steps to reach $u_{n-1}^{(i^*)} \equiv u_n^{(0)}$) *local steps on level* n. On the other hand, the sum of the numbers of the *local steps* on all levels we call the *total number of steps*. We observe that the the numbers of *local steps* decrease when n increases.

Ii the numerical test we present here, on the level n, we compute the approximate solution $u_n^{(i^*)}$ such that $\|u_n^{(i^*)} - u_n^{(i^*-1)}\|_{\varphi,\frac{1}{2}} \leq toll$.

We remark that, for the Newton method (13), the iterative steps decrease when ε increases and, for the method (15), the number of *local*

steps decrease when n increases. The tables show the number of iterative steps necessary to obtain the accuracy $toll$ by using the Newton method (13) and the numbers of the *local steps* ($l.s.$) as well as the *total number of steps* ($t.s.$) for the method (15). **Numerical Test**
We solve the equation with

$$\gamma(x, \varphi(x)u(x)) = \frac{1}{2}\arctan(\varphi(x)u(x)),$$

and

$$f(x) = \frac{\varepsilon}{\pi}\left[6x + \frac{3x^3 - 2x}{\sqrt{1-x^2}}\log\left(\frac{2 - x^2 + 2\sqrt{1-x^2}}{x^2}\right)\right] + \frac{1}{2}\arctan(x|x|\sqrt{1-x^2}).$$

Then, the exact solution is $u(x) = x|x|$. Moreover, $u \in L_{\varphi}^{2,\frac{5}{2}-\sigma}$ and $f \in L_{\varphi}^{2,\frac{3}{2}-\sigma}$, $\sigma > 0$. Since the exact solution is known, we compute $u_n^{(i^*)}$ in some points belonging to $[0,1]$ in order to evaluate

$$norm := \left\| u_n^{(i^*)} - u^* \right\|_{\varphi,\frac{1}{2}},$$

and

$$num := \varepsilon \cdot n^s \cdot norm.$$

The following Tables show the numerical results for this test equation.

Table 1: $toll = 10^{-3}$ $(s = 1)$

Newton method (13)					Method (15)			
ε	n	$s.$	$norm$	num	$l.s.$	$t.s.$	$norm$	num
0.01	8	5	.105D-01	.842D-03	4	35	.105D-01	.842D-03
0.1		4	.117D-01	.939D-02	4	33	.117D-01	.939D-02
0.25		4	.124D-01	.248D-01	3	32	.124D-01	.248D-01
0.5		4	.128D-01	.513D-01	3	29	.128D-01	.513D-01
1.		4	.131D-01	.104D+00	3	28	.131D-01	.104D+00
0.01	16	5	.286D-02	.459D-03	3	61	.286D-02	.459D-03
0.1		4	.321D-02	.514D-02	3	58	.321D-02	.514D-02
0.25		4	.335D-02	.134D-01	3	57	.335D-02	.134D-01
0.5		4	.342D-02	.274D-01	3	53	.342D-02	.274D-01
1.		4	.347D-02	.556D-01	3	52	.347D-02	.556D-01
0.01	32	5	.769D-03	.246D-03	2	107	.769D-03	.246D-03
0.1		4	.864D-03	.276D-02	2	103	.864D-03	.276D-02
0.25		4	.891D-03	.713D-02	2	101	.891D-03	.713D-02
0.5		4	.904D-03	.144D-01	2	97	.904D-03	.144D-01
1.		4	.911D-03	.291D-01	2	94	.911D-03	.291D-01

Table 2: $toll = 10^{-3} \quad (s = 1)$

Newton method (13)					**Method (15)**			
ε	n	$s.$	$norm$	num	$l.s.$	$t.s.$	$norm$	num
0.01	64	5	.203D-03	.130D-03	2	175	.203D-03	.130D-03
0.1		4	.227D-03	.145D-02	2	170	.227D-03	.145D-02
0.25		4	.231D-03	.370D-02	2	167	.231D-03	.370D-02
0.5		4	.233D-03	.747D-02	2	161	.233D-03	.747D-02
1.		4	.234D-03	.150D-01	2	158	.234D-03	.150D-01
0.01	128	5	.536D-04	.687D-04	2	303	.536D-04	.687D-04
0.1		4	.585D-04	.749D-03	2	298	.585D-04	.749D-03
0.25		4	.592D-04	.189D-02	2	295	.592D-04	.189D-02
0.5		4	.595D-04	.380D-02	2	289	.595D-04	.380D-02
1.		4	.596D-04	.763D-02	2	286	.596D-04	.763D-02
0.01	256	5	.140D-04	.358D-04	2	559	.140D-04	.358D-04
0.1		4	.148D-04	.381D-03	2	554	.148D-04	.381D-03
0.25		4	.149D-04	.959D-03	2	551	.149D-04	.959D-03
0.5		4	.150D-04	.192D-02	2	545	.150D-04	.192D-02
1.		4	.150D-04	.385D-02	2	542	.150D-04	.385D-02

Table 3: $x = 0.25$, exact value $u(x) = 0.0625$

Newton method (13)				**Method (15)**		
ε	n	$s.$	$u_n^{(i^*)}(x)$ $(toll = 10^{-03})$	$l.s.$	$t.s.$	$u_n^{(i^*)}(x)$ $(toll = 10^{-03})$
0.01	8	5	.579275D-01	4	35	.579275D-01
	16	4	.618371D-01	3	61	.618371D-01
	32		.623908D-01	2	107	.623908D-01
	64		.624727D-01	2	175	.624727D-01
	128		.625017D-01	2	303	.625017D-01
	256		.624997D-01	2	559	.624997D-01
0.1	8	4	.568227D-01	4	33	.568227D-01
	16		.615515D-01	3	58	.615515D-01
	32		.624700D-01	2	103	.623808D-01
	64		.624700D-01	2	170	.624700D-01
	128		.625014D-01	2	298	.625014D-01
	256		.624996D-01	2	554	.624996D-01

Table 4: $x = 0.25,$ exact value $u(x) = 0.0625$

Newton method (13)				**Method (15)**		
ε	n	$s.$	$u_n^{(i^*)}(x)$ $(toll = 10^{-03})$	$l.s.$	$t.s.$	$u_n^{(i^*)}(x)$ $(toll = 10^{-03})$
0.25	8	4	.562021D-01	3	32	.562021D-01
	16		.614105D-01	3	57	.614105D-01
	32		.623668D-01	2	101	.623668D-01
	64		.624679D-01	2	167	.624679D-01
	128		.625011D-01	2	295	.625011D-01
	256		.624996D-01	2	551	.624996D-01
0.5	8	4	.558301D-01	3	29	.558301D-01
	16		.613307D-01	3	53	.613307D-01
	32		.623578D-08	2	97	.623578D-08
	64		.624666D-01	2	161	.624666D-01
	128		.625010D-01	2	289	.625010D-01
	256		.624996D-01	2	545	.624996D-01

Table 5: $x = 0.25$, exact value $u(x) = 0.0625$

Newton method (13)				**Method (15)**		
ε	n	$s.$	$u_n^{(i^*)}(x)$ $(toll = 10^{-03})$	$l.s.$	$t.s.$	$u_n^{(i^*)}(x)$ $(toll = 10^{-03})$
1.	8	4	.555788D-01	3	28	.555788D-01
	16		.612786D-01	3	52	.612786D-01
	32		.623515D-01	2	94	.623515D-01
	64		.624657D-01	2	158	.624657D-01
	128		.625008D-01	2	286	.625008D-01
	256		.624996D-01	2	542	.624996D-01

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