$$g(x)v(x) - \frac{1}{\pi}\int_{-1}^{1}\frac{v'(t)}{t-x}dt + \frac{1}{\pi}\int_{-1}^{1}h(x,t)v(t)dt = f(x), \qquad (1)$$

$$v(-1) = v(1) = 0. \qquad (2)$$

For a function $v \in L^p(-1,1)$ possessing a generalized derivative $v' \in L^p(-1,1)$,we have

$$\frac{d}{dx}\int_{-1}^{1}\frac{v(t)}{t-x}dt = \int_{-1}^{1}\frac{v'(t)}{t-x}dt - \frac{v(-1)}{1+x} + \frac{v(1)}{1-x}, \quad x \in (-1,1),$$

Eq. (1) together with (2) can be written in the form

$$g(x)v(x) - \frac{1}{\pi}\int_{-1}^{1}\frac{v(t)}{(t-x)^2}dt + \frac{1}{\pi}\int_{-1}^{1}h(x,t)v(t)dt = f(x), -1 < x < 1 \quad (3)$$

where the hypersingular integral operator has to be understood in the sense of

$$\int_{-1}^{1}\frac{v(t)}{(t-x)^2}dt = \frac{d}{dx}\int_{-1}^{1}\frac{v(t)}{t-x}dt \quad (4)$$

Recall that for physical reason we look for a solution in the form

$$v(x) = \varphi(x)u(x), \qquad \varphi(x) = \sqrt{1-x^2} \quad (5)$$

$$g(x)\varphi(x)u(x) - \frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(x)u(x)}{(t-x)^2}dt + \frac{1}{\pi}\int_{-1}^{1}h(x,t)\varphi(x)u(x)dt = f(x), \quad (6)$$

where the hypersingular integral operator has to be understood in the sense of

$$\int_{-1}^{1} \frac{\varphi(x)u(x)}{(t-x)^2} dt = \frac{d}{dx} \int_{-1}^{1} \frac{\varphi(x)u(x)}{t-x} dt \tag{7}$$

Multiplication Operator

$$\Gamma(x) = g(x)\varphi(x), \qquad (M_\Gamma)u(x) = \Gamma(x)u(x) \tag{8}$$

Cauchy Singular Integral Operator

$$(Su)(x) = -\frac{1}{\pi} \int_{-1}^{1} \frac{u(t)}{t-x} \varphi(t) dt \tag{9}$$

Hypersingular Integral Operator

$$V = DS, \qquad D = \frac{d}{dx}, \tag{10}$$

Kernel Integral Operator

$$(Hu)(x) = \frac{1}{\pi}\int_{-1}^{1} h(x,t)u(t)\varphi(t)dt, \tag{11}$$

$$(M_\Gamma + V + H)u = f \tag{12}$$

▶Collocation method

Let

$$v^{\gamma,\delta}(x) = (1-x)^{\gamma}(1+x)^{\delta}, \quad \gamma, \delta > -1$$

be a Jacobi weight and let $p_n^{\gamma,\delta}$ refer as the normalized Jacobi polynomial (with positive leading coefficient) of degree n with respect to the Jacobi weight $v^{\gamma,\delta}$. Moreover, let $x_{nk}^{\gamma,\delta}$ with $-1 < x_{nn}^{\gamma,\delta} < \ldots < x_{n1}^{\gamma,\delta} < 1$ be the zeros of $p_n^{\gamma,\delta}$ and denote by $L_n^{\gamma,\delta}$ the Lagrange interpolation operator

$$L_n^{\gamma,\delta} f = \sum_{k=1}^{n} f(x_{nk}^{\gamma,\delta}) l_{nk}^{\gamma,\delta}, \quad l_{nk}^{\gamma,\delta}(x) = \prod_{j=1, j\neq k}^{n} \frac{x - x_{nj}^{\gamma,\delta}}{x_{nk}^{\gamma,\delta} - x_{nj}^{\gamma,\delta}}.$$

Since we look for a solution of our hypersingular integral equation of the form $u(x)\varphi(x) = u(x)\sqrt{1-x^2}$, this suggest us to choose $\gamma = \delta = \frac{1}{2}$, then p_n^{φ} is the n-th orthonormal Tchebychev polynomial of the second kind

$p_n^\varphi = \sqrt{2} U_n$ where $U_n(x) = \frac{\sin[(n+1)\xi]}{\sin \xi}$ and the zeros have the following expressions

$$x_{nk}^\varphi = \cos \frac{k\pi}{n+1}, \quad k = 1, \ldots, n \, .$$

The collocation method consists looking for an approximate solution $u_n \in \; P_{n-1}$ of equation (12) and solving the linear system

$$[(M_\Gamma + V + H)u_n](x_{nk}^\varphi) = f(x_{nk}^\varphi) \, , \quad k = 1, ..., n,$$

or (that is the same) to solve the equation

$$L_n^\varphi (M_\Gamma + V + H) u_n = L_n^\varphi f \, .$$

Now, we remember that for each $s \geq 0$, the hypersingular integral operator

$V = DS \in \mathcal{L}(L^{2,s+1}_{\varphi}, L^{2,s}_{\varphi})$ is a continuous isomorphism and

$$Vu = DSu = \sum_{n=0}^{\infty}(n+1)\langle u, p_n^{\varphi}\rangle_{\varphi} p_n^{\varphi}. \tag{13}$$

By this, we deduce that the operator V transforms polynomials into polynomials i.e.

$$L_n^{\varphi} V u_n = V u_n$$

Then, the previous equation is equivalent to

$$[V + L_n^{\varphi}(M_{\Gamma} + H)]u_n = L_n^{\varphi} f. \tag{14}$$

Theorem 1. *Assume that*

- $s > 1/2$;
- $f \in L^{2,s}_{\varphi}$;
- *For* $f \equiv 0$ *equation (12) has only the trivial solution* $u \equiv 0$ *in* $L^{2,1}_{\varphi}$;
- $\Gamma \in C^{r}_{\varphi}$ *for some integer* r, $0 \leq s \leq r$;
- $h(.,t) \in L^{2,s}_{\varphi}$ *uniformly w.r.t.* $t \in [-1,1]$.

Then, for all sufficiently large n, *the approximate equation (14) is uniquely solvable, and the solution* u^*_n *converges to the unique solution* u^* *of* *hypersingular integral equation (12) in the norm of* $L^{2,s+1}_{\varphi}$. *Moreover, for* $0 \le t \le s$,

$$||u_n^* - u^*||_{\varphi,t+1} \le const\; n^{t-s} ||u^*||_{\varphi,s+1}.$$

- **Quadrature method (discrete collocation method)**

We consider the operator H and we approximate this with the following gaussian quadrature rule with respect to the variable t

$$(Hu)(x) = \frac{1}{\pi} \int_{-1}^{1} h(x,t)u(t)\varphi(t)\, dt \approx$$

$$\approx (H_n u)(x) = \sum_{k=1}^{n} \lambda_{nk}^{\varphi} h(x, x_{nk}^{\varphi}) u(x_{nk}^{\varphi}),$$

where

$$x_{nk}^{\varphi} = \cos \frac{k\pi}{n+1},$$

$$\lambda_{nk}^{\varphi} = \frac{1-(x_{nk}^{\varphi})^2}{n+1} = \frac{1}{n+1}\sin^2\frac{k\pi}{n+1}, \quad k = 1,\dots,n.$$

The quadrature or discrete collocation method consists in solving the equation

$$V + L_n^{\varphi}(M_\Gamma + H_n)u_n = L_n^{\varphi} f.$$

The solution of this equation again belongs to $\mathbb{P}_{n-1}$. Since, for such u_n, we have

$$(H_n u_n)(x) = \sum_{k=1}^{n} \lambda_{nk}^{\varphi} h(x, x_{nk}^{\varphi}) u_n(x_{nk}^{\varphi})$$

$$= \frac{1}{\pi}\int_{-1}^{1} u_n(t) L_{nt}^{\varphi}[h(x,t)]\varphi(t)\,dt =: (\widehat{H}_n u_n)(x), \qquad (15)$$

the approximate equation is equivalent to

$$V + L_n^{\varphi}(M_{\Gamma} + \widehat{H}_n)u_n = L_n^{\varphi} f \,.$$

The following Lemma is crucial for proving the convergence of both the discrete collocation method and the fast algorithm we now present.

Lemma 1. *Assume* $h(x,.) \in L_{\varphi}^{2,s}$ *for some* $s > 1/2$ *uniformly w.r.t.* $x \in [-1,1]$. *Then, for* $0 \le t \le s$ *and* $u \in L_{\varphi}^2$,

$$||L_m^{\gamma,\delta}(\widehat{H}_n - H)u||_{\gamma,\delta,t} \le \mathrm{const} m^t n^{-s} ||u||_{\varphi} \,.$$

▶A Fast Algorithm

Now we show as we can construct a fast algorithm to solve numerically the simple hypersingular integral equation

$$\gamma_0 u(x) - \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{(t-x)^2}\varphi(t)dt + \frac{1}{\pi}\int_{-1}^{1} h(x,t)u(t)\varphi(t)\,dt = f(x),$$

or in the operator form

$$M_{\gamma_0} + V + H = f. \tag{16}$$

This technique can be used also for the more general equation

$$\gamma_0 u(x) - \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{(t-x)^2}\varphi(t)dt +$$

$$+\frac{1}{\pi}\int_{-1}^{1}\Big[h(x,t)-\gamma_1\ln|x-t|\Big]u(t)\varphi(t)\,dt=f(x).$$

We approximate the operator $M_{\gamma_0}+V+H$ by using the quadrature method.

$$(M_{\gamma_0}+V+L_n^{\varphi}\widehat{H}_n)u_n=L_n^{\varphi}f,$$

where we recall that

$$(\widehat{H}_n u)(x):=\frac{1}{\pi}\int_{-1}^{1}u(t)L_{nt}^{\varphi}[h(x,t)]\varphi(t)\,dt$$

We again remark that each solution u_n belongs to $\mathbb{P}_{n-1}$, such that

$$(\widehat{H}_n u_n)(x)=\sum_{k=1}^{n}\lambda_{nk}^{\varphi}h(x,x_{nk}^{\varphi})u_n(x_{nk}^{\varphi})\,.$$

We recall that,since $\alpha = \beta = \frac{1}{2}$, the n-th orthonormal Tchebychev polynomial of the second kind $p_n^{\varphi} = \sqrt{2} U_n$ where $U_n(x) = \frac{\sin[(n+1)\xi]}{\sin \xi}$ and the zeros and the weights have the following expressions

$$x_{nk}^{\varphi} = \cos \frac{k\pi}{n+1},$$

$$\lambda_{nk}^{\varphi} = \frac{1 - (x_{nk}^{\varphi})^2}{n+1} = \frac{1}{n+1} \sin^2 \frac{k\pi}{n+1}, \quad k = 1, \ldots, n.$$

We use the following representation for the fundamental Lagrange polynomials (recalling that $l_{nk}^{\varphi}(x_{nj}^{\varphi}) = \delta_{jk}, \quad j,k = 1,...,n$ and δ_{jk} is the Kronecher symbol)

$$l_{nk}^{\varphi}(x) = \lambda_{nk}^{\varphi} \sum_{j=0}^{n-1} p_j^{\varphi}(x_{nk}^{\varphi}) p_j^{\varphi}(x)$$

$$l_{nk}^{\varphi}(x) = \sum_{j=0}^{n-1} c_j^{\varphi} p_j^{\varphi}(x)$$

$$c_j^{\varphi} = \int_{-1}^{1} l_{nk}^{\varphi}(x) p_j^{\varphi}(x) \varphi(x) dx = \sum_{r=1}^{n} \lambda_{nr}^{\varphi} l_{nk}^{\varphi}(x_{nr}^{\varphi}) p_j^{\varphi}(x_{nr}^{\varphi}) = \lambda_{nk}^{\varphi} p_j^{\varphi}(x_{nk}^{\varphi})$$

We seek the approximate solution u_n in the form

$$u_n(x) = \sum_{k=1}^{n} \xi_{nk} l_{nk}^{\varphi}(x), \quad u_n(x_{nk}^{\varphi}) = \xi_{nk}$$

where ξ_{nk} are n unknown. Therefore we need to solve the linear system and to do this we choose as collocation points the zeros of the n-th Tchebychev polynomial $x_{nr}^{\varphi}, \quad r = 1, ..., n$.

$$(M_{\gamma_0} + V + \widehat{H}_n) u_n(x_{nr}^{\varphi}) = f(x_{nr}^{\varphi}), \quad r = 1, ..., n$$

Recalling that

$$(\widehat{H}_n u_n)(x) = \sum_{k=1}^{n} \lambda_{nk}^{\varphi} h(x, x_{nk}^{\varphi}) u_n(x_{nk}^{\varphi}),$$

and using Lemma 9

$$(V l_{nk}^{\varphi})(x) = \lambda_{nk}^{\varphi} \sum_{j=0}^{n-1} p_j^{\varphi}(x_{nk}^{\varphi})(j+1) p_j^{\varphi}(x),$$

we obtain

$$(M_{\gamma_0} u_n)(x_{nr}^{\varphi}) = \gamma_0 \sum_{k=1}^{n} \xi_{nk} l_{nk}^{\varphi}(x_{nr}^{\varphi}) = \gamma_0 \xi_{nr}$$

$$(Vu_n)(x_{nr}^{\varphi}) = \sum_{k=1}^{n} \xi_{nk} (V l_{nk}^{\varphi})(x_{nr}^{\varphi}) =$$

$$= \sum_{k=1}^{n} \xi_{nk} \lambda_{nk}^{\varphi} \sum_{j=0}^{n-1} p_j^{\varphi}(x_{nk}^{\varphi})(j+1) p_j^{\varphi}(x_{nr}^{\varphi})$$

$$(\widehat{H}_n u_n)(x_{nr}^{\varphi}) = \sum_{k=1}^{n} \lambda_{nk}^{\varphi} \xi_{nk} h(x_{nr}^{\varphi}, x_{nk}^{\varphi}).$$

Denoting by

$$\xi_n = [\xi_{nk}]_{k=1}^{n}, \quad \eta_n = [f(x_{nj}^{\varphi})]_{j=1}^{n},$$

$$\mathbf{I}_n = [\delta_{jk}]_{j,k=1}^{n}, \quad \mathbf{D}_n = \text{diag}[1, \ldots, n], \ \Lambda_n = \text{diag}[\lambda_{n1}^{\varphi}, \ldots, \lambda_{nn}^{\varphi}]$$

$$\mathbf{U}_n = [p_j^{\varphi}(x_{nk}^{\varphi})]_{j=0,k=1}^{n-1,\ n},$$

$$\mathbf{V}_n = \mathbf{U}_n^T \mathbf{D}_n \mathbf{U}_n, \ \mathbf{H}_n = [h(x_{nj}^{\varphi}, x_{nk}^{\varphi})]_{j,k=1}^{n},$$

then, we can write the system in the vectorial form

$$(\gamma_0 \mathbf{I}_n + \mathbf{V}_n \mathbf{\Lambda}_n + \mathbf{H}_n \mathbf{\Lambda}_n)\, \xi_n = \eta_n$$

From $\delta_{jk} = \langle p_k^\varphi, p_j^\varphi \rangle_\varphi = \sum_{l=1}^{n} \lambda_{nl}^\varphi p_k^\varphi(x_{nl}^\varphi) p_j^\varphi(x_{nl}^\varphi)$ it follows

$$\mathbf{I}_n = \mathbf{U}_n \mathbf{\Lambda}_n \mathbf{U}_n^T .$$

We assume that the vector η_n of the values of the function f at the collocation points x_{nj}^φ, $j = 1, \ldots, n$, as well as the values $h(x_{nj}^\varphi, x_{nk}^\varphi)$, $j, k = 1, \ldots, n$, are given.

Choose an integer $0 < m < n$ and write

$$u_n = \sum_{k=0}^{m-1} \alpha_k p_k^{\varphi} + \sum_{k=m}^{n-1} \alpha_k p_k^{\varphi} = \mathcal{P}_m u_n + \mathcal{Q}_m u_n \,,$$

where

$$\mathcal{P}_m u = \sum_{k=0}^{m-1} \langle u, p_k^{\varphi} \rangle_{\varphi} p_k^{\varphi} \quad \text{and} \quad \mathcal{Q}_m = I - \mathcal{P}_m \,.$$

Set $\alpha_k = \langle v_n^*, p_k^{\varphi} \rangle_{\varphi}$, $k = m, \ldots, n-1$, where $v_n^* = \sum_{k=0}^{n-1} \beta_{nk}^* p_k^{\varphi}$ is the solution of

$$(M_{\gamma_0} + V) v_n = L_n^{\varphi} f \,. \tag{17}$$

At first we note that for the case of $h \equiv 0$, Theorem 1 shows that this equation (17) is uniquely solvable for all sufficiently large n, if for $f \equiv 0$

equation has only the trivial solution $u \equiv 0$ in $L^{2,1}_{\varphi}$.

For $\beta_n = [\beta_{nk}]_{k=0}^{n-1}$ we have

$$\left[(M_{\gamma_0} v_n)(x^{\varphi}_{nj})\right]_{j=1}^{n} = \left[\gamma_0 \sum_{k=0}^{n-1} \beta_{nk} p^{\varphi}_k(x^{\varphi}_{nj})\right]_{j=1}^{n} = \gamma_0 \mathbf{U}_n^T \beta_n \,,$$

and

$$\left[(V v_n)(x^{\varphi}_{nj})\right]_{j=1}^{n} = \left[\sum_{k=0}^{n-1} \beta_{nk}(k+1) p^{\varphi}_k(x^{\varphi}_{nj})\right]_{j=1}^{n} = \mathbf{U}_n^T \mathbf{D}_n \beta_n \,.$$

So we can write equation (17) in the form

$$\mathbf{U}_n^T(\gamma_0 \mathbf{I}_n + \mathbf{D}_n)\beta_n = \eta_n$$

Recalling that $\mathbf{I}_n = \mathrm{U}_n \mathbf{\Lambda}_n \mathrm{U}_n^T$, we can write also (17) in this way

$$(\gamma_0 \mathbf{I}_n + \mathbf{D}_n)\boldsymbol{\beta}_n = \mathrm{U}_n \mathbf{\Lambda}_n \boldsymbol{\eta}_n \,.$$

Since the transform

$$\mathrm{U}_n \mathbf{\Lambda}_n = \frac{\sqrt{2}}{n+1} \left[\sin \frac{jk\pi}{n+1}\right]_{j,k=1}^{n} \mathrm{diag} \left[\sin \frac{k\pi}{n+1}\right]_{k=1}^{n}$$

can be applied to a vector with $O(n \ln n)$ computational complexity, we can compute $\boldsymbol{\beta}_n$ (and so $\alpha_m, \ldots, \alpha_{n-1}$) with

$O(n \ln n)$-complexity

taking into account the simple structure of the matrix on the left hand side of $(\gamma_0 \mathbf{I}_n + \mathbf{D}_n)\boldsymbol{\beta}_n = \mathrm{U}_n \mathbf{\Lambda}_n \boldsymbol{\eta}_n \,.$

The second step of our algorithm consists in setting $\mathcal{P}_m u_n = w_m^*$, where w_m^* is the solution of

$$\left(M_{\gamma_0} + V + L_m^{\varphi} \widehat{H}_m\right) w_m = L_m^{\varphi} \left(f - (M_{\gamma_0} + V) \mathcal{Q}_m v_n^*\right) . \quad (18)$$

This equation is equivalent to

$$\left[\mathbf{U}_m^T (\gamma_0 \mathbf{I}_m + \mathbf{D}_m) \mathbf{U}_m + \mathbf{H}_m\right] \mathbf{\Lambda}_m \boldsymbol{\omega}_m = \widetilde{\eta}_m ,$$

with $\boldsymbol{\omega}_m = [w_m(x_{mk}^{\varphi})]_{k=1}^m$ and

$$\widetilde{\eta}_m = [f(x_{mj}^{\varphi}) - ((M_{\gamma_0} + V) \mathcal{Q}_m v_n^*)(x_{mj}^{\varphi})]_{j=1}^m .$$

The matrix $\mathbf{U}_m$ can be generated with $O(m^2)$-complexity using the three term recurrence relation of the orthogonal polynomials $p_j^{\varphi}(x)$. Thus this equation can be solved with $O(m^3)$-complexity.

The values $f(x^{\varphi}_{mj})$ are already been given if we choose m in such a way that $\frac{n+1}{m+1}$ is an integer, which implies $x^{\varphi}_{mj} \in \{x^{\varphi}_{nk} : k = 1, \ldots, n\}$ for $j = 1, \ldots, m$.

So, it remains to compute $\mathbf{U}_n^T(\gamma_0 \mathbf{I}_n + \mathbf{D}_n)\widetilde{\beta}_n$, where

$$\widetilde{\beta}_n = [0, \ldots, 0, \beta_{nm}, \ldots, \beta_{n,n-1}]^T .$$

This can be done with $O(n \ln n)$ operations taking into account that

$$\mathbf{U}_n^T = \sqrt{2}\,\mathrm{diag}\left[\sin^{-1}\frac{k\pi}{n+1}\right]_{k=1}^{n}\left[\sin\frac{jk\pi}{n+1}\right]_{k,j=1}^{n}$$

can again be handled as fast discrete sine transform.

The determination of the Fourier coefficients α_{nk}, $k = 0, \ldots, m-1$, needs $O(m \ln m)$ operations, since $[\alpha_{nk}]_{k=0}^{m-1} = \mathbf{U}_m \Lambda_m \omega_m$.

The computation of the Fourier coefficients of $u_n = w_m^* + \mathcal{Q}_m v_n^*$, can be done with $O(m^3 + n \ln n)$ numerical complexity.

Now, we make the following assumptions. For some $s > 1/2$

1) For $f \equiv 0$ equation (21) possesses in $L_\varphi^{s,1}$ only the trivial solution $u \equiv 0$. The same is assumed for the equation $(M_{\gamma_0} + V)u = 0$.

2) $h(.,t) \in L_\varphi^{2,s+\delta}$ uniformly w.r.t. $t \in [-1,1]$ and

3) $h(x,.) \in L_\varphi^{2,s+\delta}$ uniformly w.r.t. $x \in [-1,1]$ for some $\delta \geq 0$.

4) The right hand side f of equation (21) belongs to $L_\varphi^{2,s}$.

Then, we can summarize our results with the

Theorem 2. [C.,Criscuolo,Junghanns] *Let* $s > 1/2$, *and* m, n, $0 < m < n$, *be integers such that* $\frac{n+1}{m+1}$ *is an integer and* $c_1 n \le m^3 \le c_2 n$ *with some positive constants* c_1 *and* c_2. *Then, if the assumptions 1),2),3),4) are verified, for all sufficiently large* m, *equations (17) and (18) are uniquely solvable and* $\widetilde{u}_n^* = w_m^* + \mathcal{Q}_m v_n^*$ *converges in the norm of* $L_\varphi^{2,t+1}$, $0 \le t < s$, *to the unique solution* $u^* \in \mathbf{L}_\varphi^{2,s+1}$ *of equation (16), where, for* $\max\left\{\frac{1}{2}, s - \frac{\delta}{2}\right\} < t \le s$,

$$||\widetilde{u}_n^* - u^*||_{\varphi,t+1} \le \mathrm{const} n^{t-s} ||u^*||_{\varphi,s+1} .$$

Moreover, the solution of (17) and (18) needs $O(n \ln n)$ *operations.*

▶Numerical Examples

We apply the fast algorithm to the following hypersingular integral equations
1.

$$u(x) - \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{(t-x)^2}\varphi(t)dt + \frac{1}{\pi}\int_{-1}^{1} (|x|+|t|)u(t)\varphi(t)\,dt =$$

$$= 2 + \frac{|x|}{2} + \frac{2}{3\pi}; \quad u(x) \equiv 1, \quad h(.,t), h(x,.), f \in L^2_{\varphi,\frac{3}{2}-\epsilon}, \quad \epsilon > 0.$$

The assumptions of Theorem 2 are satisfied, for example, for $s = 0.8$ and $\delta = 0.6$. Therefore,we can expect theoretically the convergence rate

$$||u_n^* - u^*||_{\varphi,t+1} \leq \mathrm{const}\, n^{t-0.8} ||u^*||_{\varphi,1.8}, \quad 0.5 < t \leq 0.8.$$

Example 1.		
n	m	$\lvert\lvert u_n^* - u^* \rvert\rvert_{\varphi,1.51}$
8	2	0.123D-00
27	3	0.801D-01
64	4	0.562D-01
125	5	0.519D-01
216	6	0.386D-01
343	7	0.393D-01
399	15	0.205D-01

2.

$$u(x) - \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{(t-x)^2}\varphi(t)dt + \frac{1}{\pi}\int_{-1}^{1}[t\left(x^2|x| + t|t|\right)]u(t)\varphi(t)\,dt =$$

$$= x\left[\left(1 + \frac{4x}{15\pi}\right)|x| + \frac{6}{\pi} + \frac{3x^2-2}{\pi\sqrt{1-x^2}}\ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right],$$

$$u(x) = x|x|, \quad h(.,t), h(x,.) \in L^2_{\varphi,\frac{7}{2}-\epsilon}, f \in L^2_{\varphi,\frac{3}{2}-\epsilon} \quad \epsilon > 0.$$

The assumptions of Theorem 2 are satisfied, for example, for $s = 1.5 - \epsilon$ and $\delta = 2$. Therefore,we can expect theoretically the convergence rate, for $s < 1.5$,

$$||u_n^* - u^*||_{\varphi,t+1} \leq \mathrm{const} n^{t-s}||u^*||_{\varphi,s+1}, \quad 0.5 < t \leq s.$$

Example 2.			
n	m	$\|\|u_n^* - \mathcal{P}_n u^*\|\|_{\varphi,1.51}$	$\|\|u_n^* - \mathcal{P}_n u^*\|\|_{\varphi,1.85}$
8	2	0.552D-01	0.107D-00
27	3	0.186D-01	0.520D-01
64	4	0.838D-02	0.283D-01
125	5	0.447D-02	0.212D-01
216	6	0.226D-02	0.125D-01
343	7	0.168D-02	0.112D-01
399	15	0.141D-02	0.102D-01

Finally, in the following two tables one can see the results obtained by means of the quadrature method.
Example 2 is more convenient than Example 1 for applying the fast algorithm, since already for small m in comparison with n the errors for the quadrature method and the fast algorithm are essentially the same. The reason for this is that in Example 2 the kernel $h(x, t)$ is really smoother

than the right hand side $f(x)$.

Example 1.	
n	$\|\|u_n^* - P_n u^*\|\|_{\varphi,1.51}$
8	0.980D-03
27	0.199D-03
64	0.184D-04
125	0.980D-05
216	0.165D-05
343	0.131D-05
399	0.972D-06

Example 2.		
n	$\|\|u_n^* - \mathcal{P}_n u^*\|\|_{\varphi,1.51}$	$\|\|u_n^* - \mathcal{P}_n u^*\|\|_{\varphi,1.85}$
8	0.522D-01	0.107D-00
27	0.178D-01	0.511D-01
64	0.695D-02	0.271D-01
125	0.436D-02	0.211D-01
216	0.211D-02	0.124D-01
343	0.164D-02	0.112D-01
399	0.141D-02	0.102D-01