On the Numerical Solution of Integro-Differential Equations of Prandtl's Type

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OUTLINE OF THE COURSE

- **Introduction on Integro-Differential Equations of Prandtl's Type**
- **Mapping Properties of Hypersingular Operators**
- **Collocation and Quadrature Methods for Linear Equations**
- **Fast Algorithms for Linear Equations**
- **Collocation Method and Iterative Schemes for Nonlinear Equations**
- **Fast Algorithms for Nonlinear Equations**

$$g(x)v(x) - \frac{1}{\pi}\int_{-1}^{1}\frac{v'(t)}{t-x}dt + \frac{1}{\pi}\int_{-1}^{1} h(x,t)v(t)dt = f(x), \qquad (1)$$

$$v(-1) = v(1) = 0. \qquad (2)$$

For a function $v \in L^p(-1,1)$ possessing a generalized derivative $v' \in L^p(-1,1)$,we have

$$\frac{d}{dx}\int_{-1}^{1}\frac{v(t)}{t-x}dt = \int_{-1}^{1}\frac{v'(t)}{t-x}dt - \frac{v(-1)}{1+x} + \frac{v(1)}{1-x}, \quad x \in (-1,1),$$

Eq. (??) together with (??) can be written in the form

$$g(x)v(x)-\frac{1}{\pi}\int_{-1}^{1}\frac{v(t)}{(t-x)^2}dt+\frac{1}{\pi}\int_{-1}^{1}h(x,t)v(t)dt=f(x),-1<x<1 \tag{3}$$

where the hypersingular integral operator has to be understood in the sense of

$$\int_{-1}^{1}\frac{v(t)}{(t-x)^2}dt=\frac{d}{dx}\int_{-1}^{1}\frac{v(t)}{t-x}dt \tag{4}$$

Most of the physical problems we can model with such equations, suggest that the solution of (??)-(??) or (??)-(??) has an endpoint behavior of the form $\sqrt{1-x^2}$. Thus, it is convenient to represent v as the product

$$v(x)=\varphi(x)u(x), \qquad \varphi(x)=\sqrt{1-x^2} \tag{5}$$

▶ MAPPING PROPERTIES

Multiplication Operator

$$\Gamma(x) = g(x)\varphi(x), \qquad (M_\Gamma)u(x) = \Gamma(x)u(x) \tag{6}$$

Cauchy Singular Integral Operator

For real numbers a and b with $a - ib = e^{i\pi\alpha}$, $0 < \alpha < 1$, $\beta = 1 - \alpha$, define the Jacobi weight function $v^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ and the singular integral operator of Cauchy type

$$(Au)(x) = av^{\alpha,\beta}(x)u(x) + \frac{b}{\pi}\int_{-1}^{1}\frac{u(t)}{t-x}v^{\alpha,\beta}(t)dt \tag{7}$$

If $a = 0$ and $b = -1$ (i.e. $\alpha = \beta = \frac{1}{2}$)

$$(Su)(x) = -\frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{t-x}\varphi(t)dt \qquad (8)$$

Hypersingular Integral Operator

$$(DAu)(x) = a\,\frac{d}{dx}[v^{\alpha,\beta}(x)u(x)] + \frac{b}{\pi}\int_{-1}^{1} \frac{u(t)}{(t-x)^2}\,v^{\alpha,\beta}(t)\,dt \qquad (9)$$

$$V = DS, \qquad D = \frac{d}{dx}, \quad (\alpha = \beta = \frac{1}{2}) \qquad (10)$$

Kernel Integral Operator

$$(Hu)(x) = \frac{1}{\pi}\int_{-1}^{1} h(x,t)v(t)dt, \tag{11}$$

We assume that the function h is continuous on $[-1,1]^2$.

At first, we consider the hypersingular integral equation , written in operator form:

$$(M_\Gamma + V + H)u = f \tag{12}$$

Let

$$v^{\gamma,\delta}(x) = (1-x)^\gamma(1+x)^\delta, \quad \gamma, \delta > -1$$

be a Jacobi weight and $L^2_{\gamma,\delta}$, $\gamma, \delta > -1$ denote the weighted space of square integrable functions on the interval $[-1, 1]$ endowed with the scalar product

$$\langle u, v \rangle_{\gamma,\delta} = \frac{1}{\pi}\int_{-1}^{1} u(x)\overline{v(x)}v^{\gamma,\delta}(x)dx,$$

and the norm

$$\|u\|_{\gamma,\delta} = \sqrt{\langle u, u \rangle_{\gamma,\delta}}.$$

Let $p_n^{\gamma,\delta}$ refer as the normalized Jacobi polynomial (with positive leading coefficient) of degree n with respect to the Jacobi weight $v^{\gamma,\delta}$.

For real numbers $s \geq 0$ define the weighted Sobolev space $L^{2,s}_{\gamma,\delta}$ by

$$L^{2,s}_{\gamma,\delta} = \left\{ u \in L^2_{\gamma,\delta} \;:\; \sum_{n=0}^{\infty} (1+n)^{2s} \left| \langle u, p_n^{\gamma,\delta} \rangle_{\gamma,\delta} \right|^2 < \infty \right\},$$

with the norm

$$\|u\|_{\gamma,\delta,s} = \left[\sum_{n=0}^{\infty} (1+n)^{2s} \left| \langle u, p_n^{\gamma,\delta} \rangle_{\gamma,\delta} \right|^2 \right]^{1/2}.$$

In the following we summarize some results concerning the properties of weighted Sobolev spaces, of interpolation operators with respect to the zeros of the orthogonal polynomials $p_n^{\gamma,\delta}$, the multiplication operator M_Γ defined by (??), the hypersingular integral operator V defined by (??) and the kernel operator H defined by (??).

By $\mathcal{L}(\mathrm{X}, \mathrm{Y})$ we will denote the Banach space of all bounded linear operators between the Banach spaces X and Y.

Lemma 1. [Berthold,Hoppe and Silbermann] *For* $0 \le s < t$ *the space* $L^{2,t}_{\gamma,\delta}$ *is compactly imbedded in* $L^{2,s}_{\gamma,\delta}$.

Lemma 2. [Junghanns] *If the operator* B *belongs to* $\mathcal{L}(\mathrm{L}^{2,\mathrm{s}_1}_{\alpha_1,\beta_1}, \mathrm{L}^{2,\mathrm{s}_2}_{\alpha_2,\beta_2})$ *and* $\mathcal{L}(L^{2,t_1}_{\alpha_1,\beta_1}, L^{2,t_2}_{\alpha_2,\beta_2})$ *then* $B \in \mathcal{L}(L^{2,s(\tau)}_{\alpha_1,\beta_1}, L^{2,t(\tau)}_{\alpha_2,\beta_2})$, **where** $s(\tau) = (1-\tau)s_1$ *and* $t(\tau) = (1-\tau)s_2 + \tau\, t_2\,,\ 0 \le \tau \le 1.$

Lemma 3. [Berthold,Hoppe and Silbermann] *Let* $r \ge 0$ *be an integer. Then* $u \in L^{2,r}_{\gamma,\delta}$ *if and only if* $u^{(k)}\varphi^k$ *belongs to* $L^2_{\gamma,\delta}$ *for all* $k = 0, \ldots, r$. *Moreover, the norms* $||u||_{\gamma,\delta,r}$ *and* $||u||_{\gamma,\delta,r,\varphi} = \sum_{k=0}^{r} ||u^{(k)}\varphi^k||_{\gamma,\delta}$ **are equivalent.**

Let $x_{nk}^{\gamma,\delta}$ with $-1 < x_{nn}^{\gamma,\delta} < \ldots < x_{n1}^{\gamma,\delta}$ be the zeros of $p_n^{\gamma,\delta}$ and denote by $L_n^{\gamma,\delta}$ the Lagrange interpolation operator

$$L_n^{\gamma,\delta} f = \sum_{k=1}^{n} f(x_{nk}^{\gamma,\delta}) l_{nk}^{\gamma,\delta}, \quad l_{nk}^{\gamma,\delta}(x) = \prod_{j=1, j\neq k}^{n} \frac{x - x_{nj}^{\gamma,\delta}}{x_{nk}^{\gamma,\delta} - x_{nj}^{\gamma,\delta}}.$$

Lemma 4. [C.,Mastroianni] *For $s > 1/2$ we have*

(a) $\lim_{n\to\infty} ||f - L_n^{\gamma,\delta} f||_{\gamma,\delta,s} = 0$ *for all* $f \in \mathrm{L}_{\gamma,\delta}^{2,s}$,

(b) $||f - L_n^{\gamma,\delta} f||_{\gamma,\delta,t} \leq const\ n^{t-s} ||f||_{\gamma,\delta,s}$ *if* $0 \leq t \leq s$.

By C^r_φ, $r \geq 0$ **an integer, we denote the space of all** r **times differentiable functions** $u : (-1,1) \to C$ **satisfying the conditions** $u^{(k)}\varphi^k \in \mathrm{C}[-1,1]$ **for** $k = 0, 1, \ldots, r$. **Let** $||u||_{\mathrm{C}^r_\varphi} = \sum_{k=0}^r ||u^{(k)}\varphi^k||_\infty$.

Lemma 5. [Junghanns] *Let* $r \geq 0$ *be an integer and* $\Gamma \in C^r_\varphi$. *Then the multiplication operator* M_Γ *belongs to* $\mathcal{L}(L^{2,r}_{\gamma,\delta}, L^{2,r}_{\gamma,\delta})$ *and* $||M_\Gamma||_{L^{2,r}_{\gamma,\delta} \to L^{2,r}_{\gamma,\delta}} \leq const ||\Gamma||_{C^r_\varphi}$.

Lemma 6. *Taking into account Lemma 1, under the assumptions of Lemma 5, if* $M_\Gamma \in \mathcal{L}(L^2_{\gamma,\delta}, L^2_{\gamma,\delta})$, *the condition* $\Gamma \in C^r_\varphi$ *implies* $M_\Gamma \in \mathcal{L}(L^{2,r}_{\gamma,\delta}, L^{2,r}_{\gamma,\delta})$ *for* $0 \leq s \leq r$.

We use the notations
$L^{2,s}_{\varphi} = L^{2,s}_{1/2,1/2}$, $\langle ., . \rangle_{\varphi} = \langle ., . \rangle_{1/2,1/2}$, and $p^{\varphi}_n = p^{1/2,1/2}_n$, and

$$L^{2,s,0}_{\gamma,\delta} = \left\{ f \in L^{2,s}_{\gamma,\delta} : \langle f, p^{\gamma,\delta}_0 \rangle_{\gamma,\delta} = 0 \right\}.$$

Lemma 7. [Berthold,Hoppe and Silbermann] *For all $s \geq 0$, the Cauchy singular integral operator A belongs to $\mathcal{L}(L^{2,s}_{\alpha,\beta}, L^{2,s}_{-\alpha,-\beta})$. Moreover, $A : L^{2,s}_{\alpha,\beta} \rightarrow L^{2,s,0}_{-\alpha,-\beta}$ is a bijection, and the inverse operator is given by*

$$A^{-1} = \widehat{A}, \quad (\widehat{A} f)(t) := a v^{-\alpha,-\beta} - \frac{b}{\pi} \int_{-1}^{1} \frac{f(x)}{x - t} v^{-\alpha,-\beta}(x)\, dx. \tag{13}$$

Lemma 8. [Proosdorf, Silbermann] *For the Cauchy singular integral operator A defined in (??)(we recall that $\alpha, \beta > 0$) we have the* *relation*

$$Ap_n^{\alpha,\beta} = p_{n+1}^{-\alpha,-\beta}, \quad n = 0, 1, 2, \ldots$$

Lemma 9. *For all $s \geq 0$ and $\gamma, \delta > -1$, the operator D of generalized differentiation is a continuous isomorphism from $L_{\gamma,\delta}^{2,s+1,0}$ onto $L_{1+\gamma,1+\delta}^{2,s}$. Moreover, For each $s \geq 0$, the finite part integral operator DA is a continuous isomorphism between the spaces $L_{\alpha,\beta}^{2,s+1}$ and $L_{\beta,\alpha}^{2,s}$. Finally, for $u \in L_{\alpha,\beta}^{2,s+1}$,*

$$DAu = \sum_{n=0}^{\infty} (n+1) \langle u, p_n^{\alpha,\beta} \rangle_{\alpha,\beta} p_n^{\beta,\alpha}. \tag{14}$$

In our case $a = 0$, $b = -1$ i.e. $\alpha = \beta = 1/2$, it follows that

$V \in \mathcal{L}(L^{2,s+1}_{\varphi}, L^{2,s}_{\varphi})$ and

$$Vu = DSu = \sum_{n=0}^{\infty} (n+1) \langle u, p_n^{\varphi} \rangle_{\varphi} p_n^{\varphi}. \tag{15}$$

Remark. We remember that to prove the previous Lemmas we need also these two relations for the orthonormal polynomials:

$$v^{\gamma,\delta}(x) p_n^{\gamma,\delta}(x) = -[n(n+\gamma+\delta+1)]^{-1/2} \frac{d}{dx} \left[v^{1+\gamma,1+\delta}(x) p_{n-1}^{1+\gamma,1+\delta}(x) \right],$$

$$\frac{d}{dx} p_n^{\gamma,\delta}(x) = \sqrt{n(n+\gamma+\delta+1)}\, p_{n-1}^{1+\gamma,1+\delta}(x), \quad n = 1, 2, \ldots$$

and that $\alpha + \beta = 1$.

$$(Hu)(x) = \frac{1}{\pi}\int_{-1}^{1} h(x,t)u(t)v^{\alpha,\beta}(t)\,dt$$

We assume that the function h is continuous on $[-1,1]^2$.

Lemma 10. [Berthold, Hoppe, Silbermann] *If $h(.,t) \in L^{2,s}_{\gamma,\delta}$ uniformly w.r.t. $t \in [-1,1]$ then $H \in \mathcal{L}(L^2_{\alpha,\beta}, L^{2,s}_{\gamma,\delta})$.*

$$(Hu)(x) = a\int_{-1}^{x} h(x,t)u(t)v^{\alpha,\beta}(t)\,dt +$$

$$-\frac{b}{\pi}\int_{-1}^{1} h(x,t)\ln|x-t|u(t)v^{\alpha,\beta}(t)\,dt\,;$$

$$(Hu)(x) = \frac{1}{\pi}\int_{-1}^{1} h(x,t)|x-t|^{-\eta}u(t)v^{\alpha,\beta}(t)\,dt\,.$$

▶Collocation method

We investigate the simple hypersingular integral equation

$$Bu := (M_\Gamma + V + H)u = f. \tag{16}$$

The collocation method consists in looking for an approximate solution $u_n \in\ P_{n-1}$ of equation by solving the equation

$$L_n^\varphi(M_\Gamma + V + H)u_n = L_n^\varphi f\,.$$

In view of relation (14) this equation is equivalent to

$$B_n u_n := [V + L_n^\varphi(M_\Gamma + H)]u_n = L_n^\varphi f. \tag{17}$$

We can observe, that in view of Lemma 9 on the operator V, each solution $u_n \in L^{2,s+1}_{\varphi}$ of equation (16) is a polynomial (Invariance Property), therefore we can consider equation (16) in the pair of spaces $(L^{2,s+1}_{\varphi}, L^{2,s}_{\varphi})$.

In order to prove the convergence of the collocation method we need some assumptions.

1) For $f \equiv 0$ equation has only the trivial solution $u \equiv 0$ in $L^{2,1}_{\varphi}$.

2) $\Gamma \in C^r_{\varphi}$ for some integer $r \geq 0$.

3) $h(.,t) \in L^{2,\widetilde{s}}_{\varphi}$ uniformly w.r.t. $t \in [-1,1]$.

The first step is to show the invertibility of the operator B.

Recalling Lemma 5 and 6 on the mapping properties of the multiplication operator M_Γ, Lemma 10 on the mapping property of the kernel operator H and Lemma 1 about the imbedding property of the Sobolev spaces, we can conclude that the operators M_Γ and H are compact in the pair of spaces $(L^{2,s+1}_\varphi, L^{2,s}_\varphi)$, as well as in the pair of spaces $(L^{2,1}_\varphi, L^2_\varphi)$.

Furthermore, by Lemma 9 on the operator V and Lemma 1 about the imbedding property of the Sobolev spaces, we can conclude that the operator $B : L^{2,t+1}_\varphi \longrightarrow L^{2,t}_\varphi$ is invertible for $0 \le t \le s$ and equation possesses a unique solution $u^* \in L^{2,s+1}_\varphi$.

Now we are able to prove the convergence of the collocation method.

Theorem 1. *Assume that*

- $s > 1/2$;

- $f \in L^{2,s}_{\varphi}$;

- *For* $f \equiv 0$ *equation has only the trivial solution* $u \equiv 0$ *in* $L^{2,1}_{\varphi}$;

- $\Gamma \in C^{r}_{\varphi}$ *for some integer* r, $0 \leq s \leq r$;

- $h(.,t) \in L^{2,s}_{\varphi}$ *uniformly w.r.t.* $t \in [-1,1]$.

Then, for all sufficiently large n, the approximate equation (16) is uniquely solvable, and the solution u_n^* converges to the unique solution u^* of hypersingular integral equation (15) in the norm of $L_\varphi^{2,s+1}$. Moreover, for $0 \leq t \leq s$,

$$||u_n^* - u^*||_{\varphi,t+1} \leq const\ n^{t-s}||u^*||_{\varphi,s+1}.$$

Proof. By Lemma 6 we can deduce the compactness of the multiplication operator $M_\Gamma : L_\varphi^{2,t+1} \longrightarrow L_\varphi^{2,t}$ for $0 \leq t \leq s$. Since $H : L_\varphi^{2,s+1} \longrightarrow L_\varphi^{2,s}$ is compact, it follows

$$\lim_{n\to\infty} ||B_n - B||_{L_\varphi^{2,s+1}\to L_\varphi^{2,s}} = \lim_{n\to\infty} ||(I - L_n^\varphi)(M_\Gamma + H)||_{L_\varphi^{2,s+1}\to L_\varphi^{2,s}} = 0$$

taking into account assertion (a) of Lemma 4.

By assertion (b) of Lemma 4, recalling that $r \geq 1$, we can deduce

$$||(M_\Gamma - L_n^\varphi M_\Gamma)u||_\varphi \leq \mathrm{const}\ n^{-1}||u||_{\varphi,1} \quad for \quad u \in \mathbf{L}_\varphi^{2,1},$$

and again by assertion (b) of Lemma 4 and Lemma 10,

$$||(H - L_n^\varphi H)u||_\varphi \leq \mathrm{const}\ n^{-s}||u||_{\varphi,s} \quad for \quad u \in \mathbf{L}_\varphi^{2,\mathrm{s}},$$

and therefore

$$||B_n - B||_{\mathbf{L}_\varphi^{2,1} \to \mathbf{L}_\varphi^2} \leq const n^{-\widetilde{s}},$$

where $\widetilde{s} = \min\{s, 1\}$.
This inequality, together with $\lim_{n\to\infty} ||B_n - B||_{L_\varphi^{2,s+1} \to L_\varphi^{2,s}} = 0$, and Lemma 1 imply the existence and uniform boundedness of

$B_n^{-1} \in \mathcal{L}(L_\varphi^{2,t}, \mathrm{L}_\varphi^{2,\mathrm{t}+1})$ for $0 \leq t \leq s$ and for all sufficiently large n. Consequently, since

$$u_n^* - u^* = B_n^{-1}\left[L_n^\varphi f - f + \left(I - L_n^\varphi\right)(M_\Gamma + H)u^*\right],$$

we have, with the help of assertion (b) of Lemma 4

$$||u_n^* - u^*||_{\varphi,t+1} \leq \mathrm{const}\; n^{t-s}\left(||f||_{\varphi,s} + ||(M_\Gamma + H)u^*||_{\varphi,s}\right),$$

and this prove the Theorem. $\square$

▶ Quadrature method

We consider the gaussian quadrature rule with respect to the Jacobi weight $v^{\gamma,\delta}$, i.e.

$$\frac{1}{\pi}\int_{-1}^{1} u(t)v^{\gamma,\delta}(t)\,dt \approx Q_n^{\gamma,\delta}(u) := \sum_{k=1}^{n} \lambda_{nk}^{\gamma,\delta} u(x_{nk}^{\gamma,\delta})$$

with

$$\lambda_{nk}^{\gamma,\delta} = \frac{1}{\pi}\int_{-1}^{1} l_{nk}^{\gamma,\delta}(t)v^{\gamma,\delta}(t)\,dt.$$

If we choose $v^{\gamma,\delta} = \varphi$ we can approximate the operator H by

$$(H_n u)(x) = \sum_{k=1}^{n} \lambda_{nk}^{\varphi} h(x, x_{nk}^{\varphi}) u(x_{nk}^{\varphi}).$$

The quadrature or discrete collocation method consists in solving the equation

$$V + L_n^\varphi(M_\Gamma + H_n)u_n = L_n^\varphi f.$$

The solution of this equation again belongs to $\mathbb{P}_{n-1}$. Since, for such u_n, we have

$$(Hu_n)(x) = Q_n^\varphi\left(u_n L_n^\varphi[h(x,.)]\right) =$$

$$= \frac{1}{\pi}\int_{-1}^{1} u_n(t) L_{nt}^\varphi[h(x,t)]\varphi(t)\,dt =: (\widehat{H}_n u_n)(x)\,, \tag{18}$$

the approximate equation is equivalent to

$$\widetilde{B}_n u_n := V + L_n^\varphi(M_\Gamma + \widehat{H}_n)u_n = L_n^\varphi f\,.$$

Lemma 11. *Assume $h(x,.) \in L^{2,s}_{\varphi}$ for some $s > 1/2$ uniformly w.r.t. $x \in [-1,1]$. Then, for $0 \le t \le s$ and $u \in L^2_{\varphi}$,*

$$||L^{\gamma,\delta}_m(\widehat{H}_n - H)u||_{\gamma,\delta,t} \le \mathrm{const}\, m^t n^{-s} ||u||_{\varphi}\,.$$

With this Lemma, we can prove the same result as that in Theorem 1.

We can generalize all these results in the more general hypersingular integral equation

$$M_\Gamma + DA + H = f$$

where DA is defined in (14) and the kernel operator can have logarithmic or algebraic singularity.

▶Weighted Uniform Convergence

To prove similar results in suitable weighted uniform norm, we need to investigate a regularized version of the hypersingular integral equation

$$(M_\Gamma + V + H)u = f.$$

Our first aim is to describe the inverse operator of $V : L^{2,s+1}_\varphi \longrightarrow L^{2,s}_\varphi$ and to study the mapping properties of such operator in appropriate pairs of weighted spaces of continuous functions.
We recall Lemma 9.
For each $s \geq 0$, the hypersingular integral operator V is a continuous isomorphism between $L^{2,s+1}_\varphi$ and $L^{2,s}_\varphi$ and

$$Vu = DSu = \sum_{n=0}^{\infty}(n+1)\langle u, p^\varphi_n\rangle_\varphi p^\varphi_n.$$

In view of this Lemma the operator $V^{-1}: L^{2,s}_{\varphi} \longrightarrow L^{2,s+1}_{\varphi}$ is uniquely determined by

$$V^{-1} p_n^{\varphi} = \frac{1}{n+1} p_n^{\varphi}, \quad n = 0, 1, 2, \ldots\ldots$$

and by continuity since the set of polynomials is dense in $L^{2,s}_{\gamma,\delta}$. Moreover, we can show that

Lemma 12. [C.,Criscuolo,Junghanns,Luther] *The inverse operator $V^{-1}: L^{2,s}_{\varphi} \longrightarrow L^{2,s+1}_{\varphi}$ of the hypersingular integral operator V can be written in the form:*

$$V^{-1} = -S^{-1} W S \tag{19}$$

with

$$(Wu)(x) = \frac{1}{\pi}\int_{-1}^{1}\varphi^{-1}(t)\ln|x-t|u(t)dt,$$

$$(S^{-1}u)(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{u(t)}{t-x}\varphi^{-1}(t)dt$$

We can generalize this result recalling Lemma 9 in the general case and that $\alpha+\beta=1$.

Lemma 13. [C.,Criscuolo,Junghanns,Luther] *The inverse operator $(DA)^{-1}: L^{2,s}_{\beta,\alpha} \to L^{2,s+1}_{\alpha,\beta}$ of the hypersingular integral operator DA can be written in the form:*

$$(DA)^{-1} = A^{-1}WB \tag{20}$$

where the continuous operators

$$B : L^{2,s}_{\beta,\alpha} \longrightarrow L^{2,s}_{-\beta,-\alpha}, \quad W : L^{2,s}_{-\beta,-\alpha} \longrightarrow L^{2,s+1}_{\beta-1,\alpha-1} = L^{2,s+1}_{-\alpha,-\beta},$$

and

$$A^{-1} : L^{2,s+1}_{-\alpha,-\beta} \longrightarrow L^{2,s+1}_{\alpha,\beta}$$

are defined by

$$(Bu)(x) = av^{\beta,\alpha}(x)u(x) - \frac{b}{\pi}\int_{-1}^{1} \frac{u(t)}{t-x} v^{\beta,\alpha}(t)dt,$$

$$(Wu)(x) = a\int_{-1}^{x} v^{-\beta,-\alpha}(t)u(t)dt - \frac{b}{\pi}\int_{-1}^{1} u(t)v^{-\beta,-\alpha}(t) \ln|x-t|u(t)dt$$

and A^{-1} is the inverse of the Cauchy singular integral operator defined in (15).

Equation

$$(V + M_\Gamma + H)u = f,$$

can be write in the regularized equivalent form

$$(I + V^{-1}(M_\Gamma + H))u = V^{-1}f.$$

Therefore, the Collocation Method can be represented with this operator equation

$$(I + V^{-1}L_n(M_\Gamma + H))u_n = V^{-1}L_n f,$$

and also the Quadrature Method can be represented with this operator equation

$$(I + V^{-1}L_n(M_\Gamma + H_n))u_n = V^{-1}L_n f.$$

The equations here described are studied in suitable pairs of weighted Besov spaces.