Rational Gauss quadrature

Adhemar Bultheel (KU Leuven)

Abstract

Gauss quadrature formulas are well known. They have nodes that are the zeros of orthogonal polynomials which can all be found in the interval of orthogonality and the corresponding Christoffel numbers are positive. The n-point Gauss quadrature rules are exact in the space of polynomials up to degree $2n-1$, which is in a sense optimal.

Rational Gauss quadrature formulas try to achieve the same kind of properties but the polynomials are replaced by rational functions with a set of pre-fixed poles.

Like the polynomials, the spaces $\mathcal{L}_n$, $n = 0, 1, \ldots$ of rational functions of type (n/n) are nested. So, given a sequence $\{\alpha_j\}$ of complex numbers anywhere in the extended complex plane, but not in the interval of integration, a function $f \in \mathcal{L}_n$ has a numerator that is an arbitrary polynomial at most of degree n and the denominator has the form $\prod_{j=1}^{n}(1 - z/\alpha_j)$. Hence by choosing all the poles $\alpha_j = \infty$, $\mathcal{L}_n$ is the space of polynomials of degree $\leq n$, which is thus a special case.

We shall consider the properties and computation of these quadrature formulas. Almost all the properties of the polynomial case will get analogs for the rational case: location of the nodes, positivity of the weights and maximal degree of exactness.

Moreover we shall consider the case where we give up some of the freedom in choosing the nodes in an optimal way as the zeros of orthogonal rational functions. That will allow us to fix some of the nodes at particular places: the so called rational Gauss-Radau and Gauss-Lobatto formulas and their generalizations.