Polynomial approximation
via de la Vallée Poussin means

Lecture 3: {  
• Generalized airfoil equation (Part 1)  
• Polynomial wavelets (Part 2)  

Woula Themistoclakis

CNR - National Research Council of Italy  
Institute for Computational Applications “Mauro Picone”, Naples, Italy.
GENERALIZED AIRFOIL EQUATION

\[ -\frac{1}{\pi} \int_{-1}^{1} \frac{f(y)}{y-x} \sqrt{\frac{1-y}{1+y}} \, dy + \frac{\nu}{\pi} \int_{-1}^{1} \log |x-y| f(y) \sqrt{\frac{1-y}{1+y}} \, dy = g(x), \quad |x| < 1 \]

where the first integral is in the Cauchy principal value sense, \( \nu \) is a complex number, \( g \) is a known function and \( f \) is the sought solution.

\[ D f(x) + \nu K f(x) = g(x), \quad |x| < 1 \]

\[ \text{Operator form} \]

\[ \text{Cauchy singular integral operator:} \]

\[ D f(x) = -\frac{1}{\pi} \int_{-1}^{1} \frac{f(y)}{y-x} \nu^{\frac{1}{2},-\frac{1}{2}}(y) \, dy \]

\[ \text{Perturbation operator:} \]

\[ K f(x) = \frac{1}{\pi} \int_{-1}^{1} \log |x-y| f(y) \nu^{\frac{1}{2},-\frac{1}{2}}(y) \, dy \]
For $u(x) = (1 - x)^\gamma (1 + x)^\delta$ with $\gamma, \delta \geq 0$, we consider

- **Weighted spaces of locally continuous functions:**

$$C^0_u := \left\{ f \in C^0_{loc} : \lim_{x \to 1} (fu)(x) = 0 \quad \text{if } \gamma > 0 \quad \text{and} \quad \lim_{x \to -1} (fu)(x) = 0 \quad \text{if } \delta > 0 \right\}$$

equipped with the norm $\|f\|_{C^0_u} := \|fu\|_\infty$.

- **Hölder–Zygmund subspaces:**

$$Z_r(u) := \left\{ f \in C^0_u : \|f\|_{Z_r(u)} < \infty \right\}$$

equipped with the norm

$$\|f\|_{Z_r(u)} := \|fu\|_\infty + \sup_{t > 0} \frac{w^k(f, t)_{u, \infty}}{t^r} \sim \|fu\|_\infty + \sup_{k > 0} (k + 1)^r E_k(f)_{u, \infty}$$
Mapping properties of $Df(x) = -\frac{1}{\pi} \int_{-1}^{1} \frac{f(y)}{y-x} v_{\frac{1}{2},-\frac{1}{2}}(y)$.

**TH. 1:** For all $r > 0$, the map $D : Z_r(v_{\frac{1}{2},0}) \rightarrow Z_r(v_{0,\frac{1}{2}})$ is linear, bounded, with bounded inverse given by $\hat{D}f(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(y)}{y-x} v_{-\frac{1}{2},\frac{1}{2}}(y)dy$. Moreover

$$\sup_{t>0} \frac{\omega_k^k(Df, t)_{v_{0,\frac{1}{2}}, \infty}}{t^r} \sim \sup_{t>0} \frac{\omega_k^k(f, t)_{v_{\frac{1}{2},0}, \infty}}{t^r}, \quad k > r > 0$$

**Note:** More generally, in the first lecture we studied

$$D^{\alpha,-\alpha}f(x) := \cos \pi \alpha f(x)v^{\alpha,-\alpha}(x) - \frac{\sin \pi \alpha}{\pi} \int_{-1}^{1} \frac{f(y)}{y-x} v^{\alpha,-\alpha}(y)dy,$$

establishing TH.1 for the map $D^{\alpha,-\alpha} : Z_r(v_{\alpha,0}) \rightarrow Z_r(v_{0,\alpha})$. 

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Mapping properties of $Kf(x) = \frac{1}{\pi} \int_{-1}^{1} \log |x - y| f(y) v_{\frac{1}{2}, -\frac{1}{2}}(y) dy$

**TH. 2:** For all $r > 0$, the map $K : Z_r(v_{\frac{1}{2}, 0}) \rightarrow Z_{r+1}$ is bounded and

$$\|Kf\|_{\infty} \leq C \|fv_{\frac{1}{2}, 0}\|_{\infty}, \quad \sup_{t > 0} \frac{\omega_{\varphi}^{k+1}(Kf, t)}{tr+1} \leq C \sup_{t > 0} \frac{\omega_{\varphi}^{k}(f, t)}{tr} v_{\frac{1}{2}, 0, \infty}$$

hold for all $k > r$, $C > 0$ being independent of $f \in Z_r(v_{\frac{1}{2}, 0})$.

**Note that:**

- The identity $(Kf)' = Df$ and TH.1 can be used in order to prove the second inequality of TH.2.

- Since $Z_{r+1}$ is compactly embedded into $Z_s$ for all $s < r + 1$, by TH.2 we also get that the map $K : Z_r(v_{\frac{1}{2}, 0}) \rightarrow Z_r$ is compact.
Solvability of \((D + \nu K)f = g\):

By the previous theorems we can apply the Fredholm alternative theorem to the regularized equation \((I + \nu \hat{D}K)f = \hat{D}g\), obtaining the following:

**Corollary:** Assume \(\ker\{D + \nu K\} = \{0\}\). Then for any \(g \in Z_r(v^{0, \frac{1}{2}})\) the generalized airfoil equation has a unique and stable solution \(f \in Z_r(v^{1, 0})\).

**Note:** (D. Berthold, W. Hoppe, B. Silbermann) \(\ker\{D + \nu K\} = \{0\}, \forall \nu \in \mathbb{R}\)

**Polynomial projection methods** attempt to find a polynomial approximation of \(f\), namely \(f_n\), solving the approximate equation \((D + \nu P_nK)f_n = P_n g\), where \(P_n\) is the polynomial projection defining the method.

**Condition to require:**

\[
\lim_{n \to \infty} \|K - P_nK\|_{Z_r(v^{0, \frac{1}{2}})} \to Z_r(v^{1, 0}) = 0
\]
Projections:

\[ L_n : f \rightarrow L_n(v^{-\frac{1}{2}}, \frac{1}{2}, f) \in \mathbb{P}_{n-1} \]  
\[ \tilde{V}_{n,m} : f \rightarrow \tilde{V}_{n,m}(v^{-\frac{1}{2}}, \frac{1}{2}, f) \in S_{n,m}(v^{-\frac{1}{2}}, \frac{1}{2}) \]

Both these projections satisfy the required condition, since we have

\[ \|K - L_nK\|_{Z_r(v^{\frac{1}{2}}, 0) \rightarrow Z_r(v^{0}, \frac{1}{2})} \leq C n^{-1} \log n \]
\[ \|K - \tilde{V}_{n,m}K\|_{Z_r(v^{\frac{1}{2}}, 0) \rightarrow Z_r(v^{0}, \frac{1}{2})} \leq C n^{-1}, \quad m = \theta n, \quad 0 < \theta < 1 \]

**TH. 3:** If \( D + \nu K : Z_r(v^{\frac{1}{2}}, 0) \rightarrow Z_r(v^{0}, \frac{1}{2}) \) has bounded inverse, then the same holds for \( D + \nu \mathcal{P}_n K : Z_r(v^{\frac{1}{2}}, 0) \rightarrow Z_r(v^{0}, \frac{1}{2}) \), where either \( \mathcal{P}_n = L_n \) or \( \mathcal{P}_n = \tilde{V}_{n,m} \) with \( m = \theta n, \quad 0 < \theta < 1 \). Moreover:

\[ \sup_n \|(D + \nu \mathcal{P}_n K)^{-1}\| < \infty, \quad \lim_n \kappa(D + \nu \mathcal{P}_n K) = \kappa(D + \nu K) \]

where \( \kappa(A) := \|A\| \|A^{-1}\| \).
Airfoil equation: \((D + \nu K)f = g, \quad g \in Z_r(v^{0,\frac{1}{2}})\)

Approximate equation: \((D + \nu P_n K)f_n = P_n g, \quad P_n = L_n \text{ or } P_n = \tilde{V}_{n,m}\)

**Solvability of the approximate equation:**

There exists a unique stable solution \(f\) of the airfoil equation \(\implies\) There exists a unique stable solution \(f_n\) of the approximate equation

**Error estimates** depend on \(P_n\) and can be deduced from

\[f - f_n = (I + \nu \hat{D} P_n K)^{-1} [\hat{D} D f - \hat{D} P_n D f]\]

taking into account that

\[
\|[\hat{D} F - \hat{D} P_n F]v^{\frac{1}{2},0}\|_\infty \leq \frac{C}{n^r} \|F\|_{Z_r(v^{0,\frac{1}{2}})} \begin{cases} 
1 & \text{if } P_n = \tilde{V}_{n,m}(v^{-\frac{1}{2},\frac{1}{2}}), \ m = \theta n \\
\log n & \text{if } P_n = L_n(v^{-\frac{1}{2},\frac{1}{2}})
\end{cases}
\]
Theorem 4: The solution $f_n$ of the approximate equation corresponding to $P_n = L_n$ or $P_n = \tilde{V}_{n,m}$ with $m = \theta n$, $0 < \theta < 1$, satisfies the following error estimates, where $C > 0$ denotes a constant independent of $f$ and $n$.

Lagrange case:

\[
\|f - f_n\|_{Z_s(v^{1/2,0})} \leq C \frac{\|g\|_{Z_r(v^{0,1})}}{n^{r-s}} \log n, \quad 0 < s \leq r
\]

\[
\|(f - f_n)v^{1/2,0}\|_{\infty} \leq C \frac{\|g\|_{Z_r(v^{0,1})}}{n^r} \log n,
\]

De la V.P. case:

\[
\|f - f_n\|_{Z_s(v^{1/2,0})} \leq C \frac{\|g\|_{Z_r(v^{0,1})}}{n^{r-s}}, \quad 0 < s \leq r
\]

\[
\|(f - f_n)v^{1/2,0}\|_{\infty} \leq C \frac{\|g\|_{Z_r(v^{0,1})}}{n^r}
\]
Theorem 5 The operator $D$ maps the space $S_{n,m}(v^{1/2},-1/2)$ into the space $S_{n,m}(v^{-1/2},1/2)$. This correspondence is bijective and its inverse is $D^{-1} = \hat{D} : S_{n,m}(v^{-1/2},1/2) \rightarrow S_{n,m}(v^{1/2},-1/2)$.

Proof. $Dp_k(v^{1/2},-1/2) = p_k(v^{-1/2},1/2) \implies Dq_k(v^{1/2},-1/2) = q_k(v^{-1/2},1/2)$. □

Notes on

$Df_n + \nu \tilde{V}_{n,m}(v^{-1/2},1/2, Kf_n) = \tilde{V}_{n,m}(v^{-1/2},1/2, g)$

- Its solution $f_n \in S_{n,m}(v^{1/2},-1/2)$.
- It is equivalent to: $\tilde{V}_{n,m}(v^{-1/2},1/2, Df_n + \nu Kf_n) = \tilde{V}_{n,m}(v^{-1/2},1/2, g)$
COMPUTATION OF THE APPROXIMATE SOLUTIONS

Notations: \[ w := v^{-\frac{1}{2}, \frac{1}{2}} \] and \[ < f, g >_w := \int_{-1}^{1} f(x)g(x)w(x)dx \]

De la Vallée Poussin case: \[ Df_n + \nu \tilde{V}_{n,m}(w, Kf_n) = \tilde{V}_{n,m}(w, g) \]

We compute \( f_n = \sum_{k=0}^{n-1} a_k q_k(w^{-1}) \in S_{n,m}(w^{-1}) \) by requiring that

\[ < Df_n + \nu \tilde{V}_{n,m}(w, Kf_n), q_h(w) >_w = < \tilde{V}_{n,m}(w, g), q_h(w) >_w \]

\[ < q_h(w), q_h(w) >_w = 0 \quad h = 0, \ldots, n-1 \]

Lagrange case: \[ Df_n + \nu L_n(w, f_n) = L_n(w, g) \]

We compute \( f_n = \sum_{k=0}^{n-1} b_k p_k(w^{-1}) \in \mathbb{P}_{n-1} \) by requiring that

\[ < Df_n + \nu L_n(w, Kf_n), p_h(w) >_w = < L_n(w, g), p_h(w) >_w \]

\[ h = 0, \ldots, n-1 \]
Linear system by de la V.P. projection method

For $h = 0, \ldots, n - 1$, set $w := v^{-\frac{1}{2}, \frac{1}{2}}$, we have

$$
\frac{< Df_n + \nu \tilde{V}_{n,m}(w, Kf_n), q_h(w) >_w}{< q_h(w), q_h(w) >_w} = \frac{< \tilde{V}_{n,m}(w, g), q_h(w) >_w}{< q_h(w), q_h(w) >_w}
$$

which, by $f_n = \sum_{k=0}^{n-1} a_k q_k(w^{-1})$ and $Dq_k(w^{-1}) = q_k(w)$, gives

$$
\sum_{k=0}^{n-1} a_k \left[ \delta_{h,k} + \nu \frac{< \tilde{V}_{n,m}(w, K q_k(w^{-1})), q_h(w) >_w}{< q_h(w), q_h(w) >_w} \right] = \frac{< \tilde{V}_{n,m}(w, g), q_h(w) >_w}{< q_h(w), q_h(w) >_w}
$$

But $\tilde{V}_{n,m}(w, f) = \sum_{h=0}^{n-1} \left[ \sum_{j=1}^{n} \lambda_{n,j} p_h(w, x_{n,j}) f(x_{n,j}) \right] q_h(w)$, hence

$$
\sum_{k=0}^{n-1} a_k \left[ \delta_{h,k} + \nu \sum_{j=1}^{n} \lambda_{n,j} K q_k(w^{-1})(x_{n,j}) p_h(w, x_{n,j}) \right] = \sum_{j=1}^{n} \lambda_{n,j} g(x_{n,j}) p_h(w, x_{n,j})
$$
By de la V.P. interpolation: \( f_n(x) = \sum_{k=0}^{n-1} a_k \ q_k(w^{-1}, x) \)

\[
\sum_{k=0}^{n-1} a_k \left[ \delta_{h,k} + \nu \sum_{j=1}^{n} \lambda_{n,j} K q_k(w^{-1})(x_{n,j}) p_h(w, x_{n,j}) \right] = \sum_{j=1}^{n} \lambda_{n,j} g(x_{n,j}) p_h(w, x_{n,j})
\]

\[ h = 0, \ldots, n - 1 \]

By Lagrange interpolation: \( f_n(x) = \sum_{k=0}^{n-1} b_k \ p_k(w^{-1}, x) \)

\[
\sum_{k=0}^{n-1} b_k \left[ \delta_{h,k} + \nu \sum_{j=1}^{n} \lambda_{n,j} K p_k(w^{-1})(x_{n,j}) p_h(w, x_{n,j}) \right] = \sum_{j=1}^{n} \lambda_{n,j} g(x_{n,j}) p_h(w, x_{n,j})
\]

\[ h = 0, \ldots, n - 1 \]

where \( w := v^{-\frac{1}{2}, \frac{1}{2}}, x_{n,j} \) and \( \lambda_{n,j} \) correspond to \( w \) and \( K p_k(w^{-1}, x_{n,j}) = \int_{-1}^{1} \log |x_{n,j} - y| p_k(w^{-1}, y) w^{-1}(y) dy \), as well as \( K q_k(w^{-1}, x_{n,j}) \) can be computed without any integration.
Theorem 6 [D.Berthold, W.Hoppe and B.Silbermann] The operator

\[ Kf(x) = \frac{1}{\pi} \int_{-1}^{1} \log|x - y|f(y)v^{1/2,-1/2}(y)dy \]

acts on polynomials according to the rule:

\[
Kp_0(v^{1/2,-1/2})(x) = (x - \log 2)/\sqrt{\pi},
\]

\[
Kp_k(v^{1/2,-1/2})(x) = \frac{1}{2} \left[ \frac{p_{k+1}(v^{-1/2,1/2}, x)}{k+1} - \frac{p_k(v^{-1/2,1/2}, x)}{k(k+1)} - \frac{p_{k-1}(v^{-1/2,1/2}, x)}{k} \right]
\]

A similar result holds for \( Kq_k(v^{1/2,-1/2})(x_{n,j}) \) too, recalling the definition

\[
q_k(w) := \begin{cases} 
  p_k(w) & \text{if } 0 \leq k \leq n - m \\
  \frac{m+n-k}{2m}p_k(w) - \frac{m-n+k}{2m}p_{2n-k}(w) & \text{if } n - m < k < n 
\end{cases}
\]

and using \( p_{2n-k}(v^{-1/2,1/2}, x_{n,j}) = -p_k(v^{-1/2,1/2}, x_{n,j}) \).
Theorem 7 For all $n \in \mathbb{N}$ and any $k, j = 1, \ldots, n$, we have

$$K q_0(v^{\frac{1}{2}, -\frac{1}{2}})(x_{n,j}) = (x_{n,j} - \log 2)/\sqrt{\pi}$$

$$K q_k(v^{\frac{1}{2}, -\frac{1}{2}})(x_{n,j}) = \alpha_k p_{k+1}(v^{-\frac{1}{2}, \frac{1}{2}}, x_{n,j}) - \beta_k p_k(v^{-\frac{1}{2}, \frac{1}{2}}, x_{n,j}) - \gamma_k p_{k-1}(v^{-\frac{1}{2}, \frac{1}{2}}, x_{n,j})$$

where for $k = 1, \ldots, n - m$, it is

$$\alpha_k := \frac{1}{2(k + 1)}, \quad \beta_k := \frac{1}{2k(k + 1)}, \quad \gamma_k := \frac{1}{2k}$$

while in the case $k = n - m + 1, \ldots, n$, we have

$$\alpha_k := \frac{1}{4m} \left[ \frac{n + m - k}{k + 1} - \frac{m - n + k}{2n - k} \right], \quad \gamma_k := \frac{1}{4m} \left[ \frac{n + m - k}{k} - \frac{m - n + k}{2n - k + 1} \right]$$

$$\beta_k := \frac{1}{4m} \left[ \frac{n + m - k}{k(k + 1)} + \frac{m - n + k}{(2n - k)(2n - k + 1)} \right]$$
▶ Matrix system from de la V.P. interpolation:
\[ M_n = I_n + \nu A_n \]
\[
A_n = \begin{pmatrix}
-\beta_0 & -\gamma_1 & 0 \\
\alpha_0 & -\beta_1 & -\gamma_2 \\
& \ddots & \ddots & \ddots \\
0 & \alpha_{n-2} & -\beta_{n-1}
\end{pmatrix}
\]
with \[ \begin{cases} 
\beta_0 := \frac{1}{2} - \ln 2, \\
\alpha_0 := \frac{1}{2}
\end{cases} \]

▶ Matrix system from Lagrange interpolation:
\[ M_n = I_n + \nu B_n \]
\[
B_n = \begin{pmatrix}
-\frac{1}{2} + \ln 2 & -\frac{1}{2} & 0 \\
\frac{1}{2} & -\frac{1}{4} & -\frac{1}{2} \\
& \ddots & \ddots & \ddots \\
0 & & & -\frac{1}{2(n-1)}
\end{pmatrix}
\]
\[
& & & \frac{1}{2(n-1)} & -\frac{1}{2n(n-1)}
\]
SOME REFERENCES (Part 1: Generalized airfoil equation):

POLYNOMIAL WAVELETS: some historical remarks

Polynomial wavelets based on de la V. P. interpolation

In order to have a **multiresolution structure**, we take the integers $n > m$ as functions of the *resolution level* $j \in \mathbb{N}$, i.e. we assume

$$n := n_j \quad \text{and} \quad m := m_j.$$  

**The choice** of $n_j$ and $m_j$ is different in dependence on which Chebyshev weight $w$ we consider. More precisely we set

$$n_j := 2 \cdot 3^j, \quad m_j := 3^j \quad \text{if} \quad w(x) = \frac{1}{\sqrt{1 - x^2}},$$

$$n_j := 2^{j+2} - 1, \quad m_j := 2^j - 1 \quad \text{if} \quad w(x) = \sqrt{1 - x^2},$$

$$n_j := \frac{3^{j+1} - 1}{2}, \quad m_j := \frac{3^j - 1}{2} \quad \text{if} \quad w(x) = \sqrt{1 \pm x}.$$  

**Reason for this choice:** The zeros of $p_{n_j}(w)$ are also zeros of $p_{n_j+1}(w)$.  

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**SIMPLIFIED NOTATIONS:** For all resolution level \( j \in \mathbb{N} \), we set 
\[
x_{j,k} := x_{n_j,k}(w), \quad \lambda_{j,k} := \lambda_{n_j,k}(w)
\]
and define:

- **Scaling functions:** 
  \[
  \Phi_{j,k}(x) := \lambda_{j,k} H_{n_j,m_j}(w, x, x_{j,k}), \quad k = 1, \ldots, n_j
  \]

- **Sample spaces:** 
  \[
  S_j := S_{n_j,m_j}(w) := \text{span} \{ \Phi_{j,k} : k = 1, \ldots, n_j \}
  \]

- **De la V.P. projection:** 
  \[
  V_j f(x) := \tilde{V}_{n_j,m_j}(w, f, x) = \sum_{k=1}^{n_j} f(x_{j,k}) \Phi_{j,k}(x)
  \]

**Properties:** The choices of \( n_j \) and \( m_j \) guarantee that:

- The interpolation knots of level \( j \) are also knots of level \( j + 1 \), i.e. we have the partition 
  \[
  \{x_{j+1,k}\}_{k=1,\ldots,n_{j+1}} = \{x_{j,k}\}_{k=1,\ldots,n_j} \cup \{y_{j,k}\}_{k=1,\ldots,(n_{j+1}-n_j)}
  \]
- We have a nested sequence of polynomial spaces \( S_j \subset S_{j+1} \)
Figure 1: Scaling functions $\Phi_{n_j,k}^{m_j}$ of level $j = 3$
Figure 2: Scaling functions $\Phi_{n_j,k}^{mj}$ associated with $w(x) = \sqrt{1 - x^2}$ and $x_{j,k} = 0$ for increasing resolution levels $j = 1, 2, 3$.
**WAVELET SPACES** $W_j$ are defined as the orthogonal complement of $S_j$ in $S_{j+1}$, i.e. $S_{j+1} = S_j \oplus W_j$ and $S_j \perp W_j$.

**WAVELET FUNCTIONS** $\Psi_{j,k}$ provide local bases in the spaces $W_j$. Generally they are orthogonal or interpolating. In our case, they are uniquely determined by the conditions:

- $< \Psi_{j,h}, \Phi_{j,k} >_w = 0$, $h = 1, \ldots, n_{j+1} - n_j$, $k = 1, \ldots, n_j$,
- $\Psi_{j,h}(y_{j,k}) = \delta_{h,k}$, $h, k = 1, \ldots, n_{j+1} - n_j$.

where $\{y_{j,k}\}_k$ are those zeros of $p_{n_{j+1}}(w)$ which are not zeros of $p_{n_j}(w)$.

The previous requirements allow us to compute uniquely the unknown coefficients in the expansion $\Psi_{j,h}(x) = \sum_{k=1}^{n_{j+1}} \Psi_{j,h}(x_{j+1,k}) \Phi_{j+1,k}(x)$.
Computation of the wavelet functions

Since we partitioned \( \{x_{j+1,k}\}_k = \{x_{j,k}\}_k \cup \{y_{j,k}\}_k \), set

\[
\Phi_j(x, y) := \lambda_j(x) H_{n_j, m_j}(w, x, y), \quad \lambda_j(x) = \left[ \sum_{k=0}^{n_j-1} p_k^2(w, x) \right]^{-1}
\]

we write

\[
\Psi_j, h(x) = \sum_{k=1}^{n_j+1} \Psi_j, h(x_{j+1,k}) \Phi_{j+1,k}(x)
\]

as follows

\[
\Psi_j, h(x) = \sum_{k=1}^{n_j+1-n_j} \Psi_j, h(y_{j,k}) \Phi_{j+1}(y_{j,k}, x) + \sum_{k=1}^{n_j} \Psi_j, h(x_{j,k}) \Phi_{j+1}(x_{j,k}, x)
\]

where we required

\[
\Psi_j, h(y_{j,k}) = \delta_{h,k}
\]

and

\[
< \Psi_j, h, \Phi_{j,k} >_w = 0 \implies \Psi_j, h(x_{j,k}) = \frac{-\lambda_{j+1}(y_{j,h})}{\lambda_{j+1}(x_{j,k})} \Phi_{j,k}(y_{j,h})
\]
In fact, easy computations give

\[ \langle \Phi_{j,k}, \Phi_{j+1,r} \rangle_w = \lambda_{j+1}(x_{j+1,r})\Phi_{j,k}(x_{j+1,r}) \]

Then by

\[
\Psi_{j,h}(x) = \Phi_{j+1}(y_{j,h}, x) + \sum_{s=1}^{n_j} \Psi_{j,h}(x_{j,s})\Phi_{j+1}(x_{j,s}, x),
\]

we deduce

\[
0 = \langle \Phi_{j,k}, \Psi_{j,h} \rangle_w
\]

\[
= \lambda_{j+1}(y_{j,h})\Phi_{j,k}(y_{j,h}) + \sum_{s=1}^{n_j} \Psi_{j,h}(x_{j,s})\lambda_{j+1}(x_{j,s})\Phi_{j,k}(x_{j,s})
\]

\[
= \lambda_{j+1}(y_{j,h})\Phi_{j,k}(y_{j,h}) + \Psi_{j,h}(x_{j,k})\lambda_{j+1}(x_{j,k})
\]

i.e.

\[
\Psi_{j,h}(x_{j,k}) = -\frac{\lambda_{j+1}(y_{j,h})}{\lambda_{j+1}(x_{j,k})}\Phi_{j,k}(y_{j,h}).
\]

\[\Box\]
Figure 3: Wavelet functions of level $j = 3$

- $w(x) = (1-x^2)^{-1/2}$
- $w(x) = (1-x^2)^{1/2}$
- $w(x) = (1-x)^{1/2}(1+x)^{-1/2}$
- $w(x) = (1-x)^{-1/2}(1+x)^{1/2}$
WAVELET DECOMPOSITION AND RECONSTRUCTION

\[ S_J = S_{J-1} \oplus W_{J-1} \]

\[ = S_{J-2} \oplus W_{J-2} \oplus W_{J-1} = \ldots = S_{J-s} \oplus W_{J-s} \oplus \ldots \oplus W_{J-1} \]

Consequently, any \( f_J \in S_J \) can be uniquely decomposed:

\[ f_J = f_{J-1} + g_{J-1} \]

\[ = f_{J-2} + g_{J-2} + g_{J-1} = \ldots = f_{J-s} + g_{J-s} + \ldots + g_{J-1} \]

where for \( j = J - 1, J - 2, \ldots: \)

\[
\begin{align*}
  f_j(x) &= \sum_{k=1}^{n_j} a_{j,k} \Phi_{j,k}(x) \in S_j \quad \text{lower degree approximations} \\
  g_j(x) &= \sum_{k=1}^{n_{j+1}-n_j} b_{j,k} \Psi_{j,k}(x) \in W_j \quad \text{details we lost}
\end{align*}
\]
TWO SCALE RELATIONS:

\[
\Psi_{j,h}(x) = \Phi_{j+1}(y_j,h,x) - \sum_{k=1}^{n_j} \frac{\lambda_{j+1}(y_j,h)}{\lambda_{j+1}(x_j,k)} \Phi_j(y_j,h) \Phi_{j+1}(x_j,k,x) \\
\quad j \in \mathbb{N}, \ h = 1, \ldots, n_{j+1} - n_j
\]

\[
\Phi_{j,h}(x) = \Phi_{j+1}(x_j,h,x) + \sum_{k=1}^{n_{j+1} - n_j} \Phi_{j,h}(y_{j,k}) \Phi_{j+1}(y_{j,k},x) \\
\quad j \in \mathbb{N}, \ h = 1, \ldots, n_j
\]

where, for each \( j \in \mathbb{N} \), we set

\[
\Phi_j(x, y) := \lambda_j(x) H_{n_j,m_j}(w, x, y), \quad \lambda_j(x) = \left[ \sum_{k=0}^{n_j-1} p_k^2(w, x) \right]^{-1}
\]
Matrix formulation:

\[
\begin{pmatrix}
\Phi_j \\
\Psi_j
\end{pmatrix} = \begin{pmatrix}
I & A_j \\
B_j & I
\end{pmatrix} \begin{pmatrix}
\Phi'_{j+1} \\
\Phi''_{j+1}
\end{pmatrix}
\]

where the matrices \( A_j \) and \( B_j \) are defined by

\[
\begin{cases}
(A_j)_{h,k} := \Phi_{j,h}(y_{j,k}), & h = 1, \ldots, n_j, & k = 1, \ldots, n_{j+1} - n_j, \\
(B_j)_{h,k} := \Psi_{j,h}(x_{j,k}), & h = 1, \ldots, n_{j+1} - n_j, & k = 1, \ldots, n_j,
\end{cases}
\]

\( I \) is the identity matrix and we set

\[
\Phi_j(x) := \left( \Phi_{j,1}(x), \ldots, \Phi_{j,n_j}(x) \right)^T,
\]

\[
\Psi_j(x) := \left( \Psi_{j,1}(x), \ldots, \Psi_{j,n_{j+1} - n_j}(x) \right)^T,
\]

\[
\Phi'_{j+1}(x) := \left( \Phi_{j+1}(x_{j,1}, x), \ldots, \Phi_{j+1}(x_{j,n_j}, x) \right)^T,
\]

\[
\Phi''_{j+1}(x) := \left( \Phi_{j+1}(y_{j,1}, x), \ldots, \Phi_{j+1}(y_{j,n_{j+1} - n_j}, x) \right)^T,
\]
THEOREM: Under the previous notations, we have

\[
\begin{pmatrix}
\Phi'_{j+1} \\
\Phi''_{j+1}
\end{pmatrix} = \begin{pmatrix}
G_j^{-1} & -G_j^{-1}A_j \\
-B_jG_j^{-1} & I + B_jG_j^{-1}A_j
\end{pmatrix}
\begin{pmatrix}
\Phi_j \\
\Psi_j
\end{pmatrix}
\]

where \(G_j^{-1}\) is the inverse matrix of \(G_j\) defined by

\[
(G_j)_{h,k} := \frac{1}{\lambda_{j+1}(x_{j,k})} < \Phi_j, h, \Phi_j, k >_w, \quad h, k = 1, \ldots, n_j.
\]

Proof. It is based on the identity \(G_j = I - A_jB_j\), which follows from

\[
\Phi_j, h(x) = \Phi_{j+1}(x_{j,h}, x) + \sum_{k=1}^{n_j+1-n_j} \Phi_j, h(y_{j,k})\Phi_{j+1}(y_{j,k}, x)
\]

taking into account that \(< \Phi_j, k, \Phi_j+1, r >_w = \lambda_{j+1}(x_{j+1,r})\Phi_j, k(x_{j+1,r}). \square
NOTATIONS: For all resolution level \( j \), assume \( f_{j+1} = f_j + g_j \) with

\[
f_j(x) = \sum_{k=1}^{n_j} a_{j,k} \Phi_{j,k}(x) \in S_j, \quad g_j(x) = \sum_{k=1}^{n_j+1-n_j} b_{j,k} \Psi_{j,k}(x) \in W_j
\]

and recalling that \( \{x_{j+1,k}\}_k = \{x_{j,k}\}_k \cup \{y_{j,k}\}_k \), set

\[
f_{j+1}(x) = \sum_{k=1}^{n_j+1} a_{j+1,k} \Phi_{j+1}(x_{j+1,k}, x)
\]

\[
= \sum_{k=1}^{n_j} a'_{j+1,k} \Phi_{j+1}(x_{j,k}, x) + \sum_{k=1}^{n_j+1-n_j} a''_{j+1,k} \Phi_{j+1}(y_{j,k}, x)
\]

Basis coefficients:

\[
a_j := (a_{j,1}, \ldots, a_{j,n_j}), \quad b_j := (b_{j,1}, \ldots, b_{j,n_j+1-n_j}),
\]

\[
a'_{j+1} := (a'_{j+1,1}, \ldots, a'_{j+1,n_j}), \quad a''_{j+1} := (a''_{j+1,1}, \ldots, a''_{j+1,n_j+1-n_j})
\]
RECONSTRUCTION FORMULA: 

\[
\begin{pmatrix} a'_{j+1}, a''_{j+1} \end{pmatrix} = (a_j, b_j) \begin{pmatrix} I & A_j \\ B_j & I \end{pmatrix}
\]

where

\[
\begin{cases}
(A_j)_{h,k} := \Phi_{j,h}(y_{j,k}) = \lambda_j(x_{j,h}) \sum_{k=0}^{n-1} p_k(w, x_{j,h}) q_k(w, y_{j,k}) \\
(B_j)_{h,k} := \Psi_{j,h}(x_{j,k}) = -\frac{\lambda_{j+1}(y_{j,h})}{\lambda_{j+1}(x_{j,k})} \Phi_{j,k}(y_{j,h})
\end{cases}
\]

Reconstruction scheme

\[
\begin{array}{cccccc}
a_1 & \rightarrow & a_2 & \rightarrow & a_3 & \cdots & a_{J-1} & \rightarrow & a_J \\
& & b_1 & \rightarrow & b_2 & \cdots & b_{J-1} & \\
\end{array}
\]
DECOMPOSITION FORMULA:

\[
(a_j, b_j) = (a_j', a_j'') \begin{pmatrix}
G_j^{-1} & -G_j^{-1} A_j \\
-B_j G_j^{-1} & I + B_j G_j^{-1} A_j
\end{pmatrix}
\]

where

\[
\begin{aligned}
(A_j)_{h,k} &:= \Phi_j,h(y_{j,k}), \\
(B_j)_{h,k} &:= \Psi_j,h(x_{j,k}), \\
(G_j)_{h,k} &:= \frac{\langle \Phi_j,h, \Phi_j,k \rangle}{\lambda_{j+1}(x_{j,h})}
\end{aligned}
\]

← High computational cost!

Decomposition scheme
THEOREM: The elements of the matrix $G_j^{-1}$ are given by

\[ (G_j^{-1})_{r,s} := \lambda_{j+1}(x_{j,r}) \sum_{h=0}^{n_j-1} \nu_{j,h} p_h(w, x_{j,r}) p_h(w, x_{j,s}), \quad r, s = 1, \ldots, n_j, \]

where $\nu_{j,h} := \begin{cases} 1 & \text{if } 0 \leq h \leq n_j - m_j, \\ \frac{2m_j^2}{m_j^2 + (n_j - h)^2} & \text{if } n_j - m_j < h < n_j. \end{cases}$

Proof. Recalling $\Phi_{j,h}(x) = \lambda_{j,h} \sum_{k=0}^{n_j-1} p_k(w, x_{j,h}) q_k(w, x)$, we get

\[ (G_j)_{r,s} := \frac{\langle \Phi_{j,r}, \Phi_{j,s} \rangle}{\lambda_{j+1}(x_{j,s})} = \frac{\lambda_{j,r} \lambda_{j,s}}{\lambda_{j+1}(x_{j,s})} \sum_{h=0}^{n_j-1} \frac{1}{\nu_{j,h}} p_h(w, x_{j,r}) p_h(w, x_{j,s}) \]

Thus $G_j = \Delta_j C_j^T M_j C_j D_j$ holds, where $\Delta_j, M_j, D_j$ are diagonal and $(C_j)_{r,s} := \sqrt{\lambda_{j,s}} p_{r-1}(w, x_{j,s}), \ r, s = 1, \ldots, n_j$, is orthogonal. □
Decomposition formulas:

For \( k = 1, \ldots, n_j \)

\[
a_{j,k} = \sum_{s=1}^{n_j} a'_{j+1,s} \lambda_{j+1}(x_{j,s}) \left[ \sum_{r=0}^{n_j-1} \nu_{j,r} p_{r}(w, x_{j,k}) p_{r}(w, x_{j,s}) \right] \\
+ \sum_{s=1}^{n_{j+1}-n_j} a''_{j+1,s} \lambda_{j+1}(y_{j,s}) \left[ \sum_{r=0}^{n_j-1} \nu_{j,r} p_{r}(w, y_{j,k}) q_{r}(w, y_{j,s}) \right]
\]

For \( k = 1, \ldots, n_{j+1} - n_j \)

\[
b_{j,k} = a''_{j+1,k} - \sum_{s=1}^{n_j} a'_{j+1,s} \lambda_{j+1}(x_{j,s}) \left[ \sum_{r=0}^{n_j-1} \nu_{j,r} q_{r}(w, y_{j,k}) p_{r}(w, x_{j,s}) \right] \\
- \sum_{s=1}^{n_{j+1}-n_j} a''_{j+1,s} \lambda_{j+1}(y_{j,s}) \left[ \sum_{r=0}^{n_j-1} \nu_{j,r} q_{r}(w, y_{j,k}) q_{r}(w, y_{j,s}) \right]
\]
Reconstruction formulas:

For \( k = 1, \ldots, n_j \)

\[
a'_{j+1,k} = a_{j,k} - \frac{\lambda_j(x_{j,k})}{\lambda_{j+1}(x_{j,k})} \sum_{s=1}^{n_j+1-n_j} b_{j,s} \lambda_{j+1}(y_{j,s}) \left[ \sum_{r=0}^{n_j-1} p_r(w, x_{j,k}) q_r(w, y_{j,s}) \right]
\]

For \( k = 1, \ldots, n_{j+1} - n_j \)

\[
a''_{j+1,k} = b_{j,k} + \sum_{s=1}^{n_j} a_{j,s} \lambda_j(x_{j,s}) \left[ \sum_{r=0}^{n_j-1} q_r(w, y_{j,k}) p_r(w, x_{j,s}) \right]
\]

where we recall that

\[
q_k(w) := \begin{cases} 
p_k(w) & \text{if } 0 \leq k \leq n - m \\
\frac{m + n - k}{2m} p_k(w) - \frac{m - n + k}{2m} p_{2n-k}(w) & \text{if } n - m < k < n
\end{cases}
\]
**Decomposition Algorithm:**

1. Compute 
   \[ \alpha_r = \sum_{s=1}^{n_j} \frac{\lambda_{j+1}(x_{j,s})a'_{j+1,s}}{n_{j+1} - n_j} p_r(w, x_{j,s}), \quad r = 0, \ldots, n_j - 1 \]

2. Compute 
   \[ \beta_r = \sum_{s=1}^{n_j} \frac{\lambda_{j+1}(y_{j,s})a''_{j+1,s}}{n_{j+1} - n_j} q_r(w, y_{j,s}), \quad r = 0, \ldots, n_j - 1 \]

3. Compute 
   \[ a_{j,k} = \sum_{r=0}^{n_j-1} \nu_{j,r}(\alpha_r + \beta_r)p_r(w, x_{j,k}), \quad k = 1, \ldots, n_j \]

4. Compute 
   \[ b_{j,k} = \sum_{r=0}^{n_j-1} \nu_{j,r}(\alpha_r + \beta_r)q_r(w, y_{j,k}), \quad k = 1, \ldots, n_{j+1} - n_j \]
Reconstruction Algorithm:

1. Compute $\alpha_r = \sum_{s=1}^{n_j} \lambda_j(x_{j,s}) a_{j,s} p_r(w, x_{j,s}), \quad r = 0, \ldots, n_j - 1$
2. Compute $\beta_r = \sum_{s=1}^{n_{j+1}-n_j} \lambda_{j+1}(y_{j,s}) b_{j,s} q_r(w, y_{j,s}), \quad r = 0, \ldots, n_j - 1$
3. Compute $a'_{j+1,k} = a_{j,k} - \frac{\lambda_j(x_{j,k})}{\lambda_{j+1}(x_{j,k})} \sum_{r=0}^{n_j-1} \beta_r p_r(w, x_{j,k}), \quad k = 1, \ldots, n_j$
4. Compute $a''_{j+1,k} = b_{j,k} + \sum_{r=0}^{n_j-1} \alpha_r q_r(w, y_{j,k}), \quad k = 1, \ldots, n_{j+1} - n_j$
SOME REFERENCES (Part 2: Polynomial wavelets):