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Polynomial approximation via de la Vallée Poussin means

Lecture 3:
- **Generalized airfoil equation (Part 1)**
- **Polynomial wavelets (Part 2)**

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GENERALIZED AIRFOIL EQUATION

$$-\frac{1}{\pi}\int_{-1}^{1}\frac{f(y)}{y-x}\sqrt{\frac{1-y}{1+y}}dy+\frac{\nu}{\pi}\int_{-1}^{1}\log|x-y|f(y)\sqrt{\frac{1-y}{1+y}}dy=g(x),\quad |x|<1$$

where the first integral is in the Cauchy principal value sense, ν is a complex number, g is a known function and f is the sought solution.

$$Df(x)+\nu Kf(x)=g(x),\quad |x|<1$$ ← **Operator form**

▶ **Cauchy singular integral operator:** $$Df(x)=-\frac{1}{\pi}\int_{-1}^{1}\frac{f(y)}{y-x}v^{\frac{1}{2},-\frac{1}{2}}(y)dy$$

▶ **Perturbation operator:** $$Kf(x)=\frac{1}{\pi}\int_{-1}^{1}\log|x-y|f(y)v^{\frac{1}{2},-\frac{1}{2}}(y)dy$$

For $u(x) = (1-x)^\gamma (1+x)^\delta$ with $\gamma, \delta \geq 0$, we consider

▶ **Weighted spaces of locally continuous functions:**

$$C_u^0 := \left\{ f \in C_{loc}^0 : \begin{array}{ll} \lim_{x\to 1}(fu)(x) = 0 & \text{if } \gamma > 0 \quad \text{and} \\ \lim_{x\to -1}(fu)(x) = 0 & \text{if } \delta > 0 \end{array} \right\}$$

equipped with the norm $\|f\|_{C_u^0} := \|fu\|_\infty$.

▶ **Hölder–Zygmund subspaces:** $Z_r(u) := \{ f \in C_u^0 : \quad \|f\|_{Z_r(u)} < \infty \}$

equipped with the norm

$$\|f\|_{Z_r(u)} := \|fu\|_\infty + \sup_{t>0} \frac{\omega_\varphi^k(f,t)_{u,\infty}}{t^r} \sim \|fu\|_\infty + \sup_{k>0} (k+1)^r E_k(f)_{u,\infty}$$

▶ **Mapping properties of** $Df(x) = -\frac{1}{\pi}\int_{-1}^{1} \frac{f(y)}{y-x}\ v^{\frac{1}{2},-\frac{1}{2}}(y).$

TH. 1: For all $r > 0$, the map $D : Z_r(v^{\frac{1}{2},0}) \to Z_r(v^{0,\frac{1}{2}})$ is linear, bounded, with bounded inverse given by $\widehat{D}f(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{f(y)}{y-x} v^{-\frac{1}{2},\frac{1}{2}}(y)dy.$ Moreover

$$\sup_{t>0} \frac{\omega_\varphi^k(Df,t)_{v^{0,\frac{1}{2}},\infty}}{t^r} \sim \sup_{t>0} \frac{\omega_\varphi^k(f,t)_{v^{\frac{1}{2},0},\infty}}{t^r}, \qquad k > r > 0$$

Note: More generally, in the first lecture we studied

$$D^{\alpha,-\alpha}f(x) := \cos\pi\alpha f(x)v^{\alpha,-\alpha}(x) - \frac{\sin\pi\alpha}{\pi}\int_{-1}^{1} \frac{f(y)}{y-x} v^{\alpha,-\alpha}(y)dy,$$

establishing TH.1 for the map $D^{\alpha,-\alpha} : Z_r(v^{\alpha,0}) \to Z_r(v^{0,\alpha})$.

▶ **Mapping properties of** $Kf(x) = \frac{1}{\pi}\int_{-1}^{1} \log|x-y| f(y) v^{\frac{1}{2},-\frac{1}{2}}(y)dy$

TH. 2: For all $r > 0$, the map $K : Z_r(v^{\frac{1}{2},0}) \to Z_{r+1}$ is bounded and

$$\|Kf\|_\infty \le C\|fv^{\frac{1}{2},0}\|_\infty, \qquad \sup_{t>0} \frac{\omega_\varphi^{k+1}(Kf,t)_\infty}{t^{r+1}} \le C \sup_{t>0} \frac{\omega_\varphi^k(f,t)_{v^{\frac{1}{2},0},\infty}}{t^r}$$

hold for all $k > r$, $C > 0$ being independent of $f \in Z_r(v^{\frac{1}{2},0})$.

Note that:

- The identity $(Kf)' = Df$ and TH.1 can be used in order to prove the second inequality of TH.2.
- Since Z_{r+1} is compactly embedded into Z_s for all $s < r+1$, by TH.2 we also get that the map $K : Z_r(v^{\frac{1}{2},0}) \to Z_r$ is compact.

▶ **Solvability of $(D + \nu K)f = g$:**

By the previous theorems we can apply the Fredholm alternative theorem to the regularized equation $(I + \nu \widehat{D}K)f = \widehat{D}g$, obtaining the following

Corollary: Assume $ker\{D + \nu K\} = \{0\}$. Then for any $g \in Z_r(v^{0,\frac{1}{2}})$ the generalized airfoil equation has a unique and stable solution $f \in Z_r(v^{\frac{1}{2},0})$.

Note:($D.Berthold,\ W.Hoppe,\ B.Silbermann$) $ker\{D+\nu K\} = \{0\}, \forall \nu \in \mathbb{R}$

▶ **Polynomial projection methods** attempt to find a polynomial approximation of f, namely f_n, solving the approximate equation $(D + \nu \mathcal{P}_n K)f_n = \mathcal{P}_n g$, where $\mathcal{P}_n$ is the polynomial projection defining the method.

Condition to require: $$\lim_{n\to\infty} \|K - \mathcal{P}_n K\|_{Z_r(v^{\frac{1}{2},0}) \to Z_r(v^{0,\frac{1}{2}})} = 0$$

Projections:

$$L_n : f \to L_n(v^{-\frac{1}{2},\frac{1}{2}}, f) \in \mathbb{P}_{n-1} \qquad \textbf{Lagrange}$$

$$\tilde{V}_{n,m} : f \to \tilde{V}_{n,m}(v^{-\frac{1}{2},\frac{1}{2}}, f) \in S_{n,m}(v^{-\frac{1}{2},\frac{1}{2}}) \qquad \textbf{de la V.P.}$$

Both these projections satisfy the required condition, since we have

$$\|K - L_n K\|_{Z_r(v^{\frac{1}{2},0}) \to Z_r(v^{0,\frac{1}{2}})} \leq C\, n^{-1} \log n$$

$$\|K - \tilde{V}_{n,m} K\|_{Z_r(v^{\frac{1}{2},0}) \to Z_r(v^{0,\frac{1}{2}})} \leq C\, n^{-1}, \qquad m = \theta n,\ 0 < \theta < 1$$

TH. 3: If $D + \nu K : Z_r(v^{\frac{1}{2},0}) \to Z_r(v^{0,\frac{1}{2}})$ has bounded inverse, then the same holds for $D + \nu \mathcal{P}_n K : Z_r(v^{\frac{1}{2},0}) \to Z_r(v^{0,\frac{1}{2}})$, where either $\mathcal{P}_n = L_n$ or $\mathcal{P}_n = \tilde{V}_{n,m}$ with $m = \theta n$, $0 < \theta < 1$. Moreover:

$$\sup_n \|(D + \nu \mathcal{P}_n K)^{-1}\| < \infty, \qquad \lim_n \kappa(D + \nu \mathcal{P}_n K) = \kappa(D + \nu K)$$

where $\kappa(A) := \|A\| \|A^{-1}\|$.

Airfoil equation: $(D+\nu K)f = g, \qquad g \in Z_r(v^{0,\frac{1}{2}})$

Approximate equation: $(D+\nu \mathcal{P}_n K)f_n = \mathcal{P}_n g, \quad \mathcal{P}_n = L_n$ or $\mathcal{P}_n = \tilde{V}_{n,m}$

- **Solvability of the approximate equation:**

There exists a unique stable solution f of the **airfoil equation**	$\Longrightarrow$	There exists a unique stable solution f_n of the **approximate equation**

- **Error estimates** depend on $\mathcal{P}_n$ and can be deduced from

$$f - f_n = (I + \nu \widehat{D}\mathcal{P}_n K)^{-1}[\widehat{D}Df - \widehat{D}\mathcal{P}_n Df]$$

taking into account that

$$\|[\widehat{D}F - \widehat{D}\mathcal{P}_n F]v^{\frac{1}{2},0}\|_\infty \le \frac{C}{n^r}\|F\|_{Z_r(v^{0,\frac{1}{2}})} \begin{cases} 1 & \text{if } \mathcal{P}_n = \tilde{V}_{n,m}(v^{-\frac{1}{2},\frac{1}{2}}),\ m = \theta n \\ \log n & \text{if } \mathcal{P}_n = L_n(v^{-\frac{1}{2},\frac{1}{2}}) \end{cases}$$

Theorem 4: The solution f_n of the approximate equation corresponding to $\mathcal{P}_n = L_n$ or $\mathcal{P}_n = \tilde{V}_{n,m}$ with $m = \theta n$, $0 < \theta < 1$, satisfies the following error estimates, where $C > 0$ denotes a constant independent of f and n.

▶ **Lagrange case**:

$$\begin{cases} \|f - f_n\|_{Z_s(v^{\frac{1}{2},0})} \leq C \dfrac{\|g\|_{Z_r(v^{0,\frac{1}{2}})}}{n^{r-s}} \log n, & 0 < s \leq r \\[2ex] \|(f - f_n)v^{\frac{1}{2},0}\|_\infty \leq C \dfrac{\|g\|_{Z_r(v^{0,\frac{1}{2}})}}{n^r} \log n, & \end{cases}$$

▶ **De la V.P. case**:

$$\begin{cases} \|f - f_n\|_{Z_s(v^{\frac{1}{2},0})} \leq C \dfrac{\|g\|_{Z_r(v^{0,\frac{1}{2}})}}{n^{r-s}}, & 0 < s \leq r \\[2ex] \|(f - f_n)v^{\frac{1}{2},0}\|_\infty \leq C \dfrac{\|g\|_{Z_r(v^{0,\frac{1}{2}})}}{n^r} & \end{cases}$$

$$Df(x) = -\frac{1}{\pi}\int_{-1}^{1}\frac{f(y)}{y-x}v^{\frac{1}{2},-\frac{1}{2}}(y)dy, \qquad \widehat{D}f(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{f(y)}{y-x}v^{-\frac{1}{2},\frac{1}{2}}(y)dy$$

Theorem 5 The operator D maps the space $S_{n,m}(v^{\frac{1}{2},-\frac{1}{2}})$ into the space $S_{n,m}(v^{-\frac{1}{2},\frac{1}{2}})$. This correspondence is bijective and its inverse is $D^{-1} = \widehat{D} : S_{n,m}(v^{-\frac{1}{2},\frac{1}{2}}) \to S_{n,m}(v^{\frac{1}{2},-\frac{1}{2}})$.

Proof. $Dp_k(v^{\frac{1}{2},-\frac{1}{2}}) = p_k(v^{-\frac{1}{2},\frac{1}{2}}) \Longrightarrow Dq_k(v^{\frac{1}{2},-\frac{1}{2}}) = q_k(v^{-\frac{1}{2},\frac{1}{2}})$. $\square$

Notes on $Df_n + \nu\tilde{V}_{n,m}(v^{-\frac{1}{2},\frac{1}{2}}, Kf_n) = \tilde{V}_{n,m}(v^{-\frac{1}{2},\frac{1}{2}}, g)$ **:**

- Its solution $f_n \in S_{n,m}(v^{\frac{1}{2},-\frac{1}{2}})$.
- It is equivalent to: $\tilde{V}_{n,m}(v^{-\frac{1}{2},\frac{1}{2}}, Df_n + \nu Kf_n) = \tilde{V}_{n,m}(v^{-\frac{1}{2},\frac{1}{2}}, g)$

COMPUTATION OF THE APPROXIMATE SOLUTIONS

Notations: $w := v^{-\frac{1}{2},\frac{1}{2}}$ and $< f, g >_w := \int_{-1}^{1} f(x)g(x)w(x)dx$

- **De la Vallée Poussin case:** $Df_n + \nu\tilde{V}_{n,m}(w, Kf_n) = \tilde{V}_{n,m}(w, g)$

We compute $f_n = \sum_{k=0}^{n-1} a_k \; q_k(w^{-1}) \in S_{n,m}(w^{-1})$ by requiring that

$$\frac{< Df_n + \nu\tilde{V}_{n,m}(w, Kf_n),\; q_h(w) >_w}{< q_h(w),\; q_h(w) >_w} = \frac{< \tilde{V}_{n,m}(w, g),\; q_h(w) >_w}{< q_h(w),\; q_h(w) >_w}$$

$$h = 0, \dots, n-1$$

- **Lagrange case:** $Df_n + \nu L_n(w, f_n) = L_n(w, g)$

We compute $f_n = \sum_{k=0}^{n-1} b_k \; p_k(w^{-1}) \in \mathbb{P}_{n-1}$ by requiring that

$$< Df_n + \nu L_n(w, Kf_n),\; p_h(w) >_w = < L_n(w, g),\; p_h(w) >_w$$

$$h = 0, \dots, n-1$$

Linear system by de la V.P. projection method

For $h = 0, \dots, n-1$, set $w := v^{-\frac{1}{2},\frac{1}{2}}$, we have

$$\frac{< Df_n + \nu \tilde{V}_{n,m}(w, Kf_n),\ q_h(w) >_w}{< q_h(w),\ q_h(w) >_w} = \frac{< \tilde{V}_{n,m}(w, g),\ q_h(w) >_w}{< q_h(w),\ q_h(w) >_w}$$

which, by $f_n = \sum_{k=0}^{n-1} a_k\ q_k(w^{-1})$ and $Dq_k(w^{-1}) = q_k(w)$, gives

$$\sum_{k=0}^{n-1} a_k \left[\delta_{h,k} + \nu \frac{< \tilde{V}_{n,m}(w, Kq_k(w^{-1})),\ q_h(w) >_w}{< q_h(w),\ q_h(w) >_w} \right] = \frac{< \tilde{V}_{n,m}(w, g),\ q_h(w) >_w}{< q_h(w),\ q_h(w) >_w}$$

But $\tilde{V}_{n,m}(w, f) = \sum_{h=0}^{n-1} \left[\sum_{j=1}^{n} \lambda_{n,j} p_h(w, x_{n,j}) f(x_{n,j}) \right] q_h(w)$, hence

$$\sum_{k=0}^{n-1} a_k \left[\delta_{h,k} + \nu \sum_{j=1}^{n} \lambda_{n,j} K q_k(w^{-1})(x_{n,j}) p_h(w, x_{n,j}) \right] = \sum_{j=1}^{n} \lambda_{n,j} g(x_{n,j}) p_h(w, x_{n,j})$$

- **By de la V.P. interpolation:** $f_n(x) = \sum_{k=0}^{n-1} a_k \; q_k(w^{-1}, x)$

$$\sum_{k=0}^{n-1} a_k \left[\delta_{h,k} + \nu \sum_{j=1}^{n} \lambda_{n,j} K q_k(w^{-1})(x_{n,j}) p_h(w, x_{n,j}) \right] = \sum_{j=1}^{n} \lambda_{n,j} g(x_{n,j}) p_h(w, x_{n,j})$$

$$h = 0, .., n-1$$

- **By Lagrange interpolation:** $f_n(x) = \sum_{k=0}^{n-1} b_k \; p_k(w^{-1}, x)$

$$\sum_{k=0}^{n-1} b_k \left[\delta_{h,k} + \nu \sum_{j=1}^{n} \lambda_{n,j} K p_k(w^{-1})(x_{n,j}) p_h(w, x_{n,j}) \right] = \sum_{j=1}^{n} \lambda_{n,j} g(x_{n,j}) p_h(w, x_{n,j})$$

$$h = 0, .., n-1$$

where $w := v^{-\frac{1}{2},\frac{1}{2}}$, $x_{n,j}$ and $\lambda_{n,j}$ correspond to w and $K p_k(w^{-1}, x_{n,j}) = \int_{-1}^{1} \log |x_{n,j} - y| p_k(w^{-1}, y) w^{-1}(y) dy$, as well as $K q_k(w^{-1}, x_{n,j})$ can be computed without any integration.

Theorem 6 [*D.Berthold, W.Hoppe and B.Silbermann*] The operator $Kf(x) = \frac{1}{\pi}\int_{-1}^{1} \log|x-y| f(y) v^{\frac{1}{2},-\frac{1}{2}}(y)dy$ acts on polynomials according to the rule:

$$\begin{aligned} Kp_0(v^{\frac{1}{2},-\frac{1}{2}})(x) &= (x - \log 2)/\sqrt{\pi}, \\ Kp_k(v^{\frac{1}{2},-\frac{1}{2}})(x) &= \frac{1}{2}\left[\frac{p_{k+1}(v^{-\frac{1}{2},\frac{1}{2}},x)}{k+1} - \frac{p_k(v^{-\frac{1}{2},\frac{1}{2}},x)}{k(k+1)} - \frac{p_{k-1}(v^{-\frac{1}{2},\frac{1}{2}},x)}{k}\right] \end{aligned}$$

A similar result holds for $Kq_k(v^{\frac{1}{2},-\frac{1}{2}})(x_{n,j})$ too, recalling the definition

$$q_k(w) := \begin{cases} p_k(w) & \text{if} \quad 0 \le k \le n-m \\ \dfrac{m+n-k}{2m} p_k(w) - \dfrac{m-n+k}{2m} p_{2n-k}(w) & \text{if} \quad n-m < k < n \end{cases}$$

and using $p_{2n-k}(v^{-\frac{1}{2},\frac{1}{2}}, x_{n,j}) = -p_k(v^{-\frac{1}{2},\frac{1}{2}}, x_{n,j})$.

Theorem 7 For all $n \in \mathbb{N}$ and any $k, j = 1, \ldots, n$, we have

$$Kq_0(v^{\frac{1}{2},-\frac{1}{2}})(x_{n,j}) = (x_{n,j} - \log 2)/\sqrt{\pi}$$
$$Kq_k(v^{\frac{1}{2},-\frac{1}{2}})(x_{n,j}) = \alpha_k p_{k+1}(v^{-\frac{1}{2},\frac{1}{2}}, x_{n,j}) - \beta_k p_k(v^{-\frac{1}{2},\frac{1}{2}}, x_{n,j}) - \gamma_k p_{k-1}(v^{-\frac{1}{2},\frac{1}{2}}, x_{n,j})$$

where for $k = 1, \ldots, n-m$, it is

$$\alpha_k := \frac{1}{2(k+1)}, \qquad \beta_k := \frac{1}{2k(k+1)}, \qquad \gamma_k := \frac{1}{2k}$$

while in the case $k = n-m+1, \ldots, n$, we have

$$\alpha_k := \frac{1}{4m}\left[\frac{n+m-k}{k+1} - \frac{m-n+k}{2n-k}\right] \qquad \gamma_k := \frac{1}{4m}\left[\frac{n+m-k}{k} - \frac{m-n+k}{2n-k+1}\right]$$
$$\beta_k := \frac{1}{4m}\left[\frac{n+m-k}{k(k+1)} + \frac{m-n+k}{(2n-k)(2n-k+1)}\right]$$

- **Matrix system from de la V.P. interpolation:** $M_n = I_n + \nu A_n$

$$A_n = \begin{pmatrix} -\beta_0 & -\gamma_1 & & & \mathsf{O} \\ \alpha_0 & -\beta_1 & -\gamma_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\gamma_{n-1} \\ \mathsf{O} & & & \alpha_{n-2} & -\beta_{n-1} \end{pmatrix} \quad \text{with} \quad \begin{cases} \beta_0 := \frac{1}{2} - \ln 2, \\ \alpha_0 := \frac{1}{2} \end{cases}$$

- **Matrix system from Lagrange interpolation:** $M_n = I_n + \nu B_n$

$$B_n = \begin{pmatrix} -\frac{1}{2} + \ln 2 & -\frac{1}{2} & & & \mathsf{O} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\frac{1}{2(n-1)} \\ \mathsf{O} & & & \frac{1}{2(n-1)} & -\frac{1}{2n(n-1)} \end{pmatrix}$$

SOME REFERENCES (Part 1: Generalized airfoil equation):

- K.E.Atkinson, **The numerical solution of integral equations of the second kind**, Cambridge University Press, Cambridge, 1997.
- D.Berthold, W.Hoppe, B.Silbermann, **A fast algorithm for solving the generalized airfoil equation**, in: *Orthogonal polynomials and numerical methods, J. Comput. Appl. Math.*, **43** (1992), 185–219.
- Capobianco M.R., **The stability and the convergence of a collocation method for a class of Cauchy singular integral equation**, Math. Nacr. **162** (1993), 45–58
- P.Junghanns,**Product integration for the generalized airfoil equation**, in: E.Schock (Ed.), *Beiträge zur Angewandten Analysis und Informatik*, Shaker Verlag, Aachen (1994), 171–188.
- G.Mastroianni, M.G.Russo, W.Themistoclakis, **Numerical methods for Cauchy singular integral equations in spaces of weighted continuous functions**, in *Operator Theory: Adv. and Appl.*, **160**, 311–336, Birkhuser Verlag, Basel, 2005
- G. Mastroianni, W. Themistoclakis, **A numerical method for the generalized airfoil equation based on the de la Valle Poussin interpolation**, *J. Comput. Appl. Math.*, **180** (2005), 71–105

POLYNOMIAL WAVELETS: some historical remarks

- **1993** - C.K.Chui, H.N.Mhaskar, **On trigonometric wavelets**. *Constr. Approx.*, **9** , 167–190.
- **1995** - G.Plonka, K.Selig, M.Tasche, **On the construction of wavelets on the interval**, *Adv. Comp. Math.*, **4** (1995), 357–388.
- **1995** - J.Prestin, K.Selig, **Interpolating and orthonormal trigonometric wavelets**, in: *Signal and Image Representation in Combined Spaces*, Academic Press, 1995.
- **1995** - M.Tasche, **Polynomial wavelets on [-1,1]**, *Approximation Theory, Wavelets and Applications* (Ed. S.P. Singh), Kluwer Academic Publ., Dordrecht, 497–512.
- **1996** - T.Kilgore, J.Prestin, **Polynomial wavelets on the interval**, *Constr. Approx.*, **12** (1996), 95–110.
- **1997** - B.Fischer, J.Prestin, **Wavelets based on orthogonal polynomials**, *Math. Comp.*, **66** (1997), 1593–1618.
- **2005** - M.R.Capobianco, W.Themistoclakis, **Interpolating polynomial wavelets on [- 1,1]**, *Adv. in Comput. Math.*, **23** (2005), 353-374.

Polynomial wavelets based on de la V. P. interpolation

In order to have a **multiresolution structure**, we take the integers $n > m$ as functions of the *resolution level* $j \in \mathbb{N}$, i.e. we assume

$$n := n_j \qquad \text{and} \qquad m := m_j.$$

▶ **The choice** of n_j and m_j is different in dependence on which Chebyshev weight w we consider. More precisely we set

$$n_j := 2 \cdot 3^j, \qquad m_j := 3^j \qquad \text{if} \quad w(x) = \frac{1}{\sqrt{1-x^2}},$$

$$n_j := 2^{j+2} - 1, \quad m_j := 2^j - 1 \qquad \text{if} \quad w(x) = \sqrt{1-x^2},$$

$$n_j := \frac{3^{j+1}-1}{2}, \quad m_j := \frac{3^j - 1}{2} \qquad \text{if} \quad w(x) = \sqrt{\frac{1 \pm x}{1 \mp x}}$$

▶ **Reason for this choice:** The zeros of $p_{n_j(w)}$ are also zeros of $p_{n_{j+1}(w)}$.

SIMPLIFIED NOTATIONS: For all resolution level $j \in \mathbb{N}$, we set $x_{j,k} := x_{n_j,k}(w)$, $\lambda_{j,k} := \lambda_{n_j,k}(w)$ and define:

- **Scaling functions:** $\Phi_{j,k}(x) := \lambda_{j,k} H_{n_j,m_j}(w, x, x_{j,k}), \qquad k = 1, \ldots, n_j$

- **Sample spaces:** $S_j := S_{n_j,m_j}(w) := \text{span}\{\Phi_{j,k} : \quad k = 1, \ldots, n_j\}$

- **De la V.P. projection:** $V_j f(x) := \tilde{V}_{n_j,m_j}(w, f, x) = \sum_{k=1}^{n_j} f(x_{j,k})\Phi_{j,k}(x)$

Properties: The choices of n_j and m_j guarantee that:

- The interpolation knots of level j are also knots of level $j+1$, i.e. we have the partition $\{x_{j+1,k}\}_{k=1,..,n_{j+1}} = \{x_{j,k}\}_{k=1,..,n_j} \cup \{y_{j,k}\}_{k=1,..,(n_{j+1}-n_j)}$,

- We have a nested sequence of polynomial spaces $S_j \subset S_{j+1}$

Figure 1: Scaling functions $\Phi_{n_j,k}^{m_j}$ of level $j = 3$

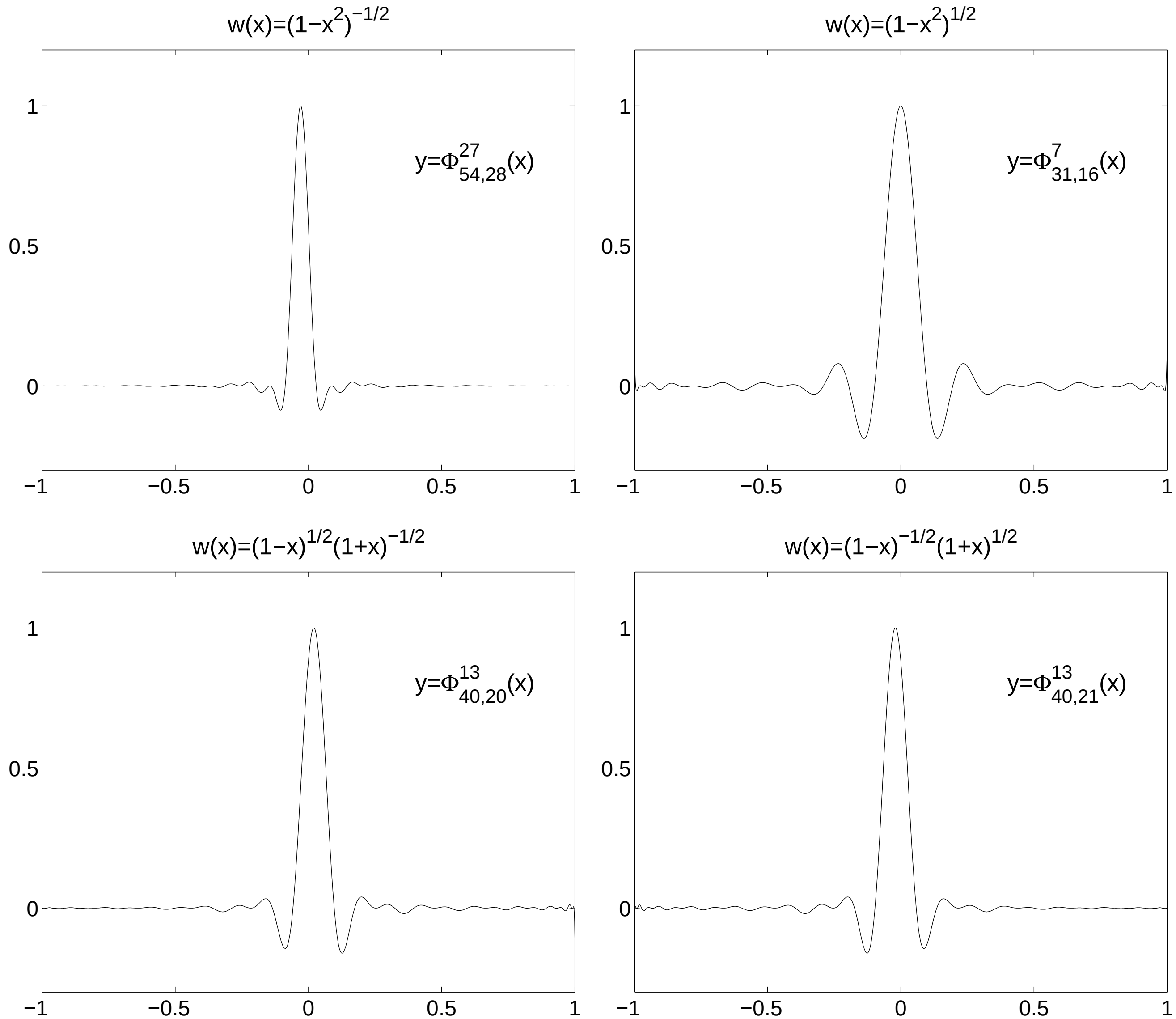
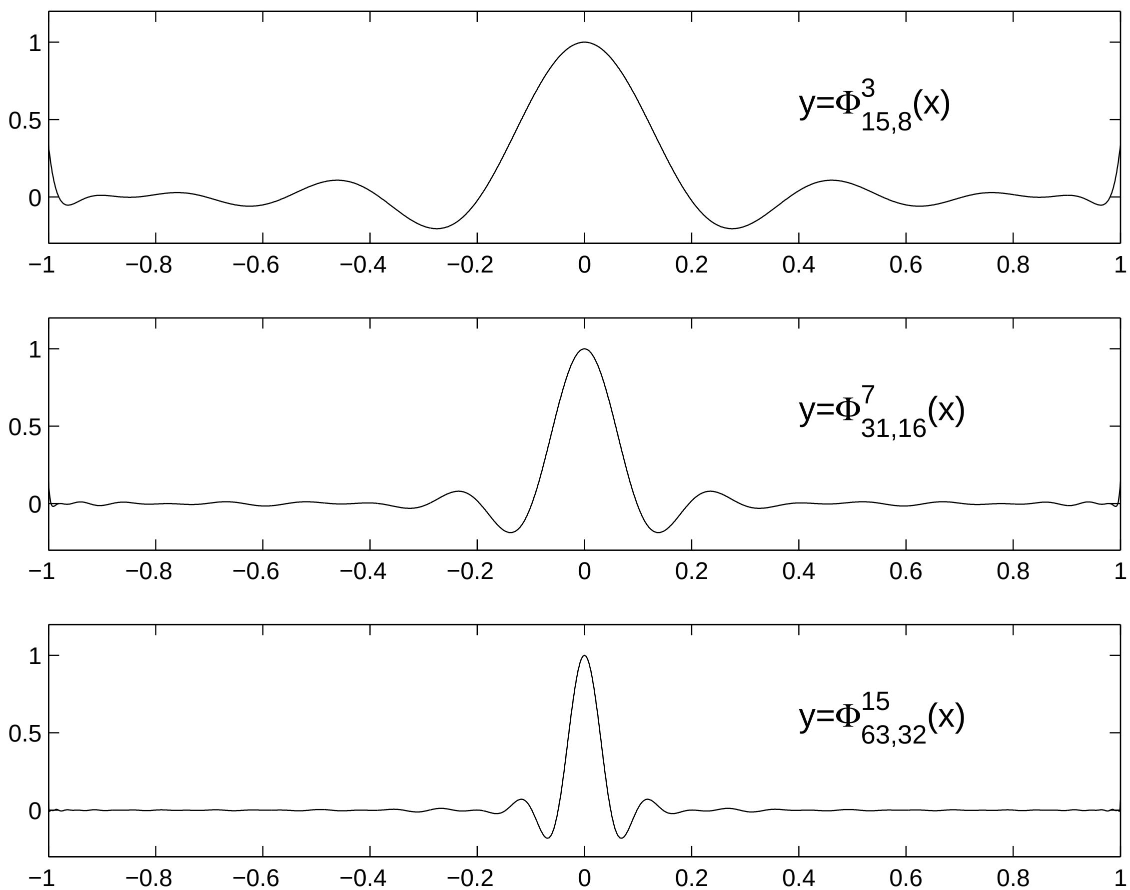


Figure 2: Scaling functions $\Phi_{n_j,k}^{m_j}$ associated with $w(x) = \sqrt{1-x^2}$ and $x_{j,k} = 0$ for increasing resolution levels $j = 1, 2, 3$



WAVELET SPACES W_j are defined as the orthogonal complement of S_j in S_{j+1}, i.e. $S_{j+1} = S_j \oplus W_j$ and $S_j \perp W_j$.

WAVELET FUNCTIONS $\Psi_{j,k}$ provide local bases in the spaces W_j. Generally they are orthogonal or interpolating. In our case, they are uniquely determined by the conditions:

- $< \Psi_{j,h}, \Phi_{j,k} >_w \; = \; 0, \qquad h = 1, .., n_{j+1} - n_j, \qquad k = 1, .., n_j,$
- $\Psi_{j,h}(y_{j,k}) \; = \; \delta_{h,k}, \quad h, k = 1, .., n_{j+1} - n_j.$

where $\{y_{j,k}\}_k$ are those zeros of $p_{n_{j+1}}(w)$ which are not zeros of $p_{n_j}(w)$.

The previous requirements allow us to compute uniquely the unknown coefficients in the expansion $\Psi_{j,h}(x) = \sum_{k=1}^{n_{j+1}} \Psi_{j,h}(x_{j+1,k}) \Phi_{j+1,k}(x)$

Computation of the wavelet functions

Since we partitioned $\{x_{j+1,k}\}_k = \{x_{j,k}\}_k \cup \{y_{j,k}\}_k$, set

$$\Phi_j(x,y) := \lambda_j(x) H_{n_j,m_j}(w,x,y), \qquad \lambda_j(x) = \left[\sum_{k=0}^{n_j-1} p_k^2(w,x) \right]^{-1}$$

we write $\boxed{\Psi_{j,h}(x) = \sum_{k=1}^{n_{j+1}} \Psi_{j,h}(x_{j+1,k}) \Phi_{j+1,k}(x)}$ as follows

$$\Psi_{j,h}(x) = \sum_{k=1}^{n_{j+1}-n_j} \underbrace{\Psi_{j,h}(y_{j,k})} \Phi_{j+1}(y_{j,k},x) + \sum_{k=1}^{n_j} \underbrace{\Psi_{j,h}(x_{j,k})} \Phi_{j+1}(x_{j,k},x)$$

where we required $\boxed{\Psi_{j,h}(y_{j,k}) = \delta_{h,k}}$ and

$$< \Psi_{j,h}, \Phi_{j,k} >_w = 0 \quad \Longrightarrow \quad \boxed{\Psi_{j,h}(x_{j,k}) = -\frac{\lambda_{j+1}(y_{j,h})}{\lambda_{j+1}(x_{j,k})} \Phi_{j,k}(y_{j,h})}$$

In fact, easy computations give

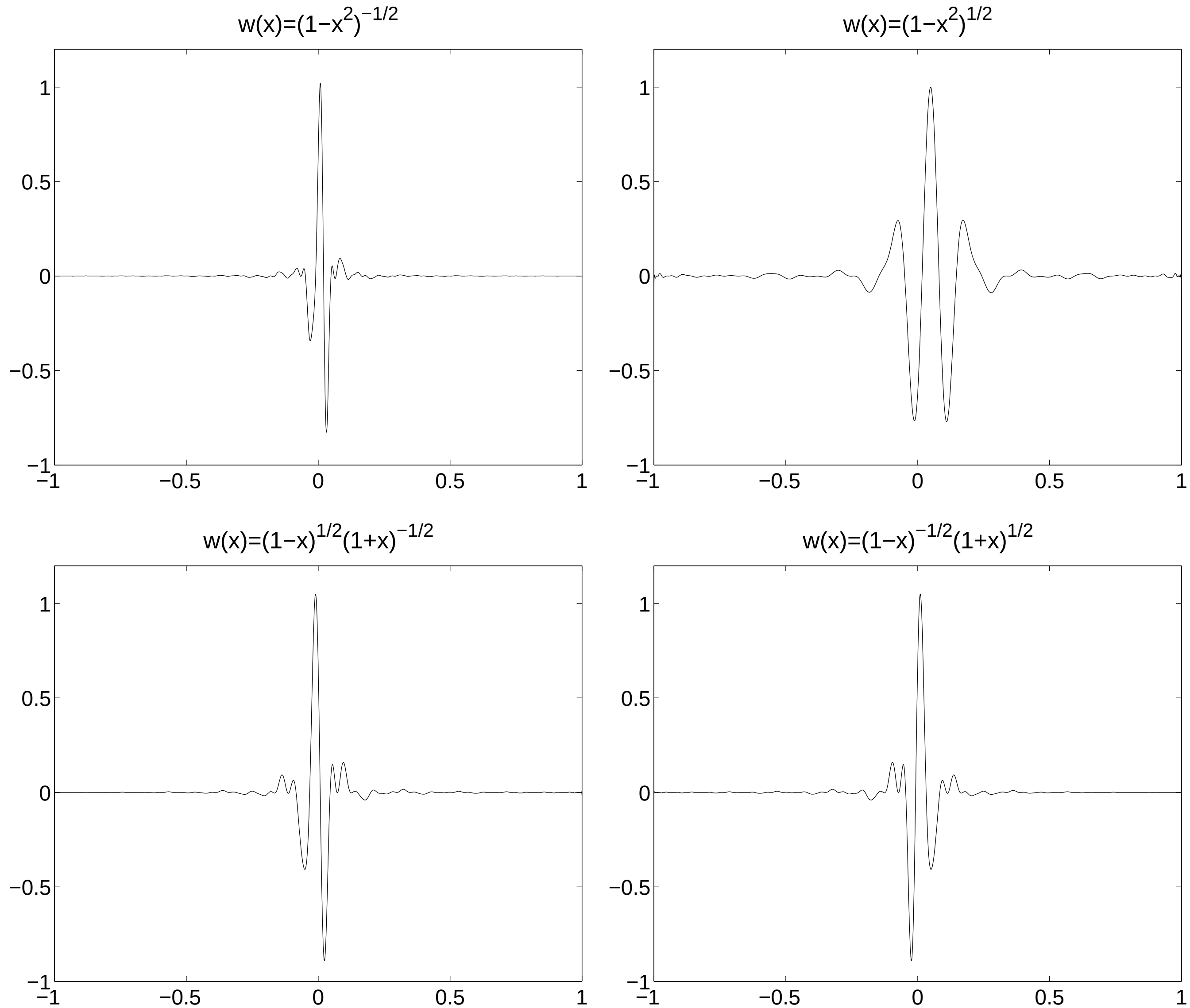
$$< \Phi_{j,k}, \Phi_{j+1,r} >_w = \lambda_{j+1}(x_{j+1,r})\Phi_{j,k}(x_{j+1,r})$$

Then by $\Psi_{j,h}(x) = \Phi_{j+1}(y_{j,h}, x) + \sum_{s=1}^{n_j} \Psi_{j,h}(x_{j,s})\Phi_{j+1}(x_{j,s}, x)$, we deduce

$$\begin{aligned} 0 &= < \Phi_{j,k}, \Psi_{j,h} >_w \\ &= \lambda_{j+1}(y_{j,h})\Phi_{j,k}(y_{j,h}) + \sum_{s=1}^{n_j} \Psi_{j,h}(x_{j,s})\lambda_{j+1}(x_{j,s})\Phi_{j,k}(x_{j,s}) \\ &= \lambda_{j+1}(y_{j,h})\Phi_{j,k}(y_{j,h}) + \Psi_{j,h}(x_{j,k})\lambda_{j+1}(x_{j,k}) \end{aligned}$$

i.e. $\boxed{\Psi_{j,h}(x_{j,k}) = -\frac{\lambda_{j+1}(y_{j,h})}{\lambda_{j+1}(x_{j,k})}\Phi_{j,k}(y_{j,h})}$. $\square$

Figure 3: Wavelet functions of level $j = 3$



WAVELET DECOMPOSITION AND RECONSTRUCTION

$$
\begin{aligned}
S_J &= S_{J-1} \oplus W_{J-1} \\
&= S_{J-2} \oplus W_{J-2} \oplus W_{J-1} = \ldots\ldots = S_{J-s} \oplus W_{J-s} \oplus \ldots \oplus W_{J-1}
\end{aligned}
$$

Consequently, any $f_J \in S_J$ can be uniquely decomposed:

$$
\begin{aligned}
f_J &= f_{J-1} + g_{J-1} \\
&= f_{J-2} + g_{J-2} + g_{J-1} = \ldots\ldots = f_{J-s} + g_{J-s} + \ldots + g_{J-1}
\end{aligned}
$$

where for $j = J-1, J-2, \ldots$:

$$
\begin{cases}
f_j(x) = \displaystyle\sum_{k=1}^{n_j} a_{j,k} \Phi_{j,k}(x) \in S_j & \text{lower degree approximations} \\
g_j(x) = \displaystyle\sum_{k=1}^{n_{j+1}-n_j} b_{j,k} \Psi_{j,k}(x) \in W_j & \text{details we lost}
\end{cases}
$$

TWO SCALE RELATIONS:

$$\Psi_{j,h}(x) = \Phi_{j+1}(y_{j,h}, x) - \sum_{k=1}^{n_j} \frac{\lambda_{j+1}(y_{j,h})}{\lambda_{j+1}(x_{j,k})} \Phi_{j,k}(y_{j,h}) \Phi_{j+1}(x_{j,k}, x)$$

$$j \in \mathbb{N}, \; h = 1, \ldots, n_{j+1} - n_j,$$

$$\Phi_{j,h}(x) = \Phi_{j+1}(x_{j,h}, x) + \sum_{k=1}^{n_{j+1}-n_j} \Phi_{j,h}(y_{j,k}) \Phi_{j+1}(y_{j,k}, x)$$

$$j \in \mathbb{N}, \; h = 1, \ldots, n_j$$

where, for each $j \in \mathbb{N}$, we set

$$\Phi_j(x, y) := \lambda_j(x) H_{n_j, m_j}(w, x, y), \qquad \lambda_j(x) = \left[\sum_{k=0}^{n_j - 1} p_k^2(w, x) \right]^{-1}$$

Matrix formulation:

$$\begin{pmatrix} \underline{\Phi}_j \\ \underline{\Psi}_j \end{pmatrix} = \left(\begin{array}{c|c} I & A_j \\ \hline B_j & I \end{array}\right) \begin{pmatrix} \underline{\Phi}'_{j+1} \\ \underline{\Phi}''_{j+1} \end{pmatrix}$$

where the matrices A_j and B_j are defined by

$$\begin{cases} (A_j)_{h,k} & := \Phi_{j,h}(y_{j,k}), \quad h = 1, .., n_j, \qquad\qquad k = 1, .., n_{j+1} - n_j, \\ (B_j)_{h,k} & := \Psi_{j,h}(x_{j,k}), \quad h = 1, .., n_{j+1} - n_j, \quad k = 1, .., n_j \end{cases}$$

I is the identity matrix and we set

$$\begin{aligned} \underline{\Phi}_j(x) &:= \left(\Phi_{j,1}(x), \ldots, \Phi_{j,n_j}(x)\right)^T, \\ \underline{\Psi}_j(x) &:= \left(\Psi_{j,1}(x), \ldots, \Psi_{j,n_{j+1}-n_j}(x)\right)^T, \\ \underline{\Phi}'_{j+1}(x) &:= \left(\Phi_{j+1}(x_{j,1}, x), \ldots, \Phi_{j+1}(x_{j,n_j}, x)\right)^T, \\ \underline{\Phi}''_{j+1}(x) &:= \left(\Phi_{j+1}(y_{j,1}, x), \ldots, \Phi_{j+1}(y_{j,n_{j+1}-n_j}, x)\right)^T, \end{aligned}$$

THEOREM: Under the previous notations, we have

$$\begin{pmatrix} \underline{\Phi}'_{j+1} \\ \underline{\Phi}''_{j+1} \end{pmatrix} = \left(\begin{array}{c|c} G_j^{-1} & -G_j^{-1}A_j \\ \hline -B_jG_j^{-1} & I + B_jG_j^{-1}A_j \end{array}\right) \begin{pmatrix} \underline{\Phi}_j \\ \underline{\Psi}_j \end{pmatrix}$$

where G_j^{-1} is the inverse matrix of G_j defined by

$$(G_j)_{h,k} := \frac{1}{\lambda_{j+1}(x_{j,k})} < \Phi_{j,h}, \Phi_{j,k} >_w, \qquad h,k = 1,\dots,n_j.$$

Proof. It is based on the identity $G_j = I - A_jB_j$, which follows from

$$\Phi_{j,h}(x) = \Phi_{j+1}(x_{j,h}, x) + \sum_{k=1}^{n_{j+1}-n_j} \Phi_{j,h}(y_{j,k})\Phi_{j+1}(y_{j,k}, x)$$

taking into account that $< \Phi_{j,k}, \Phi_{j+1,r} >_w = \lambda_{j+1}(x_{j+1,r})\Phi_{j,k}(x_{j+1,r})$. □

NOTATIONS: For all resolution level j, assume $\boxed{f_{j+1} = f_j + g_j}$ with

$$f_j(x) = \sum_{k=1}^{n_j} a_{j,k}\Phi_{j,k}(x) \in S_j, \qquad g_j(x) = \sum_{k=1}^{n_{j+1}-n_j} b_{j,k}\Psi_{j,k}(x) \in W_j$$

and recalling that $\{x_{j+1,k}\}_k = \{x_{j,k}\}_k \cup \{y_{j,k}\}_k$, set

$$\begin{aligned} f_{j+1}(x) &= \sum_{k=1}^{n_{j+1}} a_{j+1,k}\Phi_{j+1}(x_{j+1,k}, x) \\ &= \sum_{k=1}^{n_j} a'_{j+1,k}\Phi_{j+1}(x_{j,k}, x) + \sum_{k=1}^{n_{j+1}-n_j} a''_{j+1,k}\Phi_{j+1}(y_{j,k}, x) \end{aligned}$$

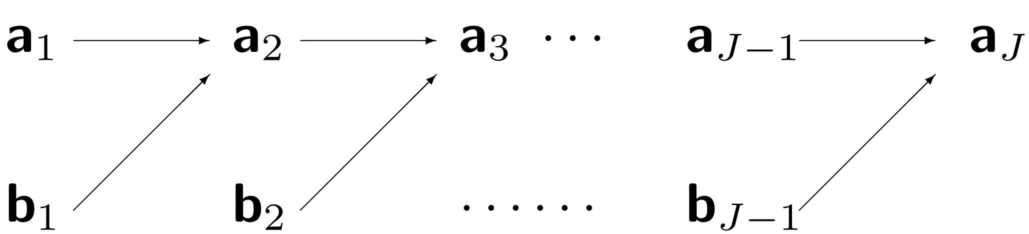
Basis coefficients:

$$\mathbf{a}_j := \left(a_{j,1}, \ldots, a_{j,n_j}\right), \qquad \mathbf{b}_j := \left(b_{j,1}, \ldots, b_{j,n_{j+1}-n_j}\right),$$

$$\mathbf{a}'_{j+1} := \left(a'_{j+1,1}, \ldots, a'_{j+1,n_j}\right), \qquad \mathbf{a}''_{j+1} := \left(a''_{j+1,1}, \ldots, a''_{j+1,n_{j+1}-n_j}\right)$$

RECONSTRUCTION FORMULA: $$\left(\mathbf{a}'_{j+1},\ \mathbf{a}''_{j+1}\right) = \left(\mathbf{a}_j,\ \mathbf{b}_j\right) \left(\begin{array}{c|c} I & A_j \\ \hline B_j & I \end{array}\right)$$

where
$$\begin{cases} (A_j)_{h,k} := \Phi_{j,h}(y_{j,k}) = \lambda_j(x_{j,h}) \displaystyle\sum_{k=0}^{n-1} p_k(w,,x_{j,h}) q_k(w, y_{j,k}) \\ (B_j)_{h,k} := \Psi_{j,h}(x_{j,k}) = -\dfrac{\lambda_{j+1}(y_{j,h})}{\lambda_{j+1}(x_{j,k})} \Phi_{j,k}(y_{j,h}) \end{cases}$$



Reconstruction scheme

DECOMPOSITION FORMULA:

$$(\mathbf{a}_j,\ \mathbf{b}_j) = (\mathbf{a}'_{j+1},\ \mathbf{a}''_{j+1}) \left(\begin{array}{c|c} G_j^{-1} & -G_j^{-1} A_j \\ \hline -B_j G_j^{-1} & I + B_j G_j^{-1} A_j \end{array} \right)$$

where

$$\begin{cases} (A_j)_{h,k} := \Phi_{j,h}(y_{j,k}), & (B_j)_{h,k} := \Psi_{j,h}(x_{j,k}), \\ (G_j)_{h,k} := \dfrac{< \Phi_{j,h}, \Phi_{j,k} >}{\lambda_{j+1}(x_{j,h})} & \end{cases}$$

$\leftarrow$ High computational cost!

$$\begin{array}{ccccccc} \mathbf{a}_J & \longrightarrow & \mathbf{a}_{J-1} & \longrightarrow & \mathbf{a}_{J-2} & \cdots \longrightarrow & \mathbf{a}_1 \\ & \searrow & & \searrow & & \cdots \searrow & \\ & & \mathbf{b}_{J-1} & & \mathbf{b}_{J-2} & \cdots & \mathbf{b}_1 \end{array}$$

Decomposition scheme

THEOREM: The elements of the matrix G_j^{-1} are given by

$$(G_j^{-1})_{r,s} := \lambda_{j+1}(x_{j,r}) \sum_{h=0}^{n_j-1} \nu_{j,h} p_h(w, x_{j,r}) p_h(w, x_{j,s}), \quad r, s = 1, .., n_j,$$

where $\nu_{j,h} := \begin{cases} 1 & \text{if} \quad 0 \le h \le n_j - m_j, \\ \dfrac{2m_j^2}{m_j^2 + (n_j - h)^2} & \text{if} \quad n_j - m_j < h < n_j. \end{cases}$

Proof. Recalling $\Phi_{j,h}(x) = \lambda_{j,h} \sum_{k=0}^{n_j-1} p_k(w, , x_{j,h}) q_k(w, x)$, we get

$$(G_j)_{r,s} := \frac{< \Phi_{j,r}, \Phi_{j,s} >}{\lambda_{j+1}(x_{j,s})} = \frac{\lambda_{j,r}\lambda_{j,s}}{\lambda_{j+1}(x_{j,s})} \sum_{h=0}^{n_j-1} \frac{1}{\nu_{j,h}} p_h(w, x_{j,r}) p_h(w, x_{j,s})$$

Thus $G_j = \Delta_j C_j^T M_j C_j D_j$ holds, where Δ_j, M_j, D_j are diagonal and $(C_j)_{r,s} := \sqrt{\lambda_{j,s}}\, p_{r-1}(w, x_{j,s})$, $r, s = 1, .., n_j$, is orthogonal. $\square$

Decomposition formulas:

For $k = 1, \ldots, n_j$

$$a_{j,k} = \sum_{s=1}^{n_j} a'_{j+1,s} \lambda_{j+1}(x_{j,s}) \left[\sum_{r=0}^{n_j-1} \nu_{j,r} p_r(w, x_{j,k}) p_r(w, x_{j,s}) \right] + \sum_{s=1}^{n_{j+1}-n_j} a''_{j+1,s} \lambda_{j+1}(y_{j,s}) \left[\sum_{r=0}^{n_j-1} \nu_{j,r} p_r(w, x_{j,k}) q_r(w, y_{j,s}) \right]$$

For $k = 1, \ldots, n_{j+1} - n_j$

$$b_{j,k} = a''_{j+1,k} - \sum_{s=1}^{n_j} a'_{j+1,s} \lambda_{j+1}(x_{j,s}) \left[\sum_{r=0}^{n_j-1} \nu_{j,r} q_r(w, y_{j,k}) p_r(w, x_{j,s}) \right] - \sum_{s=1}^{n_{j+1}-n_j} a''_{j+1,s} \lambda_{j+1}(y_{j,s}) \left[\sum_{r=0}^{n_j-1} \nu_{j,r} q_r(w, y_{j,k}) q_r(w, y_{j,s}) \right]$$

Reconstruction formulas:

For $k = 1, \dots, n_j$

$$a'_{j+1,k} = a_{j,k} - \frac{\lambda_j(x_{j,k})}{\lambda_{j+1}(x_{j,k})} \sum_{s=1}^{n_{j+1}-n_j} b_{j,s}\lambda_{j+1}(y_{j,s}) \left[\sum_{r=0}^{n_j-1} p_r(w, x_{j,k}) q_r(w, y_{j,s}) \right]$$

For $k = 1, \dots, n_{j+1} - n_j$

$$a''_{j+1,k} = b_{j,k} + \sum_{s=1}^{n_j} a_{j,s}\lambda_j(x_{j,s}) \left[\sum_{r=0}^{n_j-1} q_r(w, y_{j,k}) p_r(w, x_{j,s}) \right]$$

where we recall that

$$q_k(w) := \begin{cases} p_k(w) & \text{if} \quad 0 \le k \le n-m \\ \dfrac{m+n-k}{2m} p_k(w) - \dfrac{m-n+k}{2m} p_{2n-k}(w) & \text{if} \quad n-m < k < n \end{cases}$$

DECOMPOSITION ALGORITHM:

1. Compute $\alpha_r = \sum_{s=1}^{n_j} \lambda_{j+1}(x_{j,s}) a'_{j+1,s}\, p_r(w, x_{j,s}), \qquad r = 0, \ldots, n_j - 1$
2. Compute $\beta_r = \sum_{s=1}^{n_{j+1}-n_j} \lambda_{j+1}(y_{j,s}) a''_{j+1,s}\, q_r(w, y_{j,s}), \quad r = 0, \ldots, n_j - 1$
3. Compute $a_{j,k} = \sum_{r=0}^{n_j - 1} \nu_{j,r}(\alpha_r + \beta_r) p_r(w, x_{j,k}), \qquad k = 1, \ldots, n_j$
4. Compute $b_{j,k} = \sum_{r=0}^{n_j - 1} \nu_{j,r}(\alpha_r + \beta_r) q_r(w, y_{j,k}), \qquad k = 1, \ldots, n_{j+1} - n_j$

Reconstruction Algorithm:

1. Compute $\alpha_r = \sum_{s=1}^{n_j} \lambda_j(x_{j,s}) a_{j,s}\; p_r(w, x_{j,s}), \qquad r = 0, \ldots, n_j - 1$

2. Compute $\beta_r = \sum_{s=1}^{n_{j+1}-n_j} \lambda_{j+1}(y_{j,s}) b_{j,s}\; q_r(w, y_{j,s}), \qquad r = 0, \ldots, n_j - 1$

3. Compute $a'_{j+1,k} = a_{j,k} - \frac{\lambda_j(x_{j,k})}{\lambda_{j+1}(x_{j,k})} \sum_{r=0}^{n_j-1} \beta_r p_r(w, x_{j,k}), \quad k = 1, .., n_j$

4. Compute $a''_{j+1,k} = b_{j,k} + \sum_{r=0}^{n_j-1} \alpha_r q_r(w, y_{j,k}), \qquad k = 1, \ldots, n_{j+1} - n_j$

SOME REFERENCES (Part 2: Polynomial wavelets):

- M.R.Capobianco, W.Themistoclakis, **Interpolating polynomial wavelets on [- 1,1]**, *Adv. in Comput. Math.*, **23** (2005), 353-374.
- C.K.Chui, H.N.Mhaskar, **On trigonometric wavelets** *Constr. Approx.*, **9** (1993), 167–190.
- B.Fischer, J.Prestin, **Wavelets based on orthogonal polynomials**, *Math. Comp.*, **66** (1997), 1593–1618.
- B.Fischer, W.Themistoclakis, **Orthogonal polynomial wavelets**, *Numer. Algorithms*, **30** (2002), 37–58.
- T.Kilgore, J.Prestin, **Polynomial wavelets on the interval**, *Constr. Approx.*, **12** (1996), 95–110.
- D.Potts, G.Steidl, M.Tasche **Fast algorithms for discrete polynomial transforms**, *Math. Comp,*, **67** (1998), 1577–1590.
- J.Prestin, K.Selig, **Interpolating and orthonormal trigonometric wavelets**, in: *Signal and Image Representation in Combined Spaces*, Academic Press, 1995.
- M.Tasche, **Polynomial wavelets on [-1,1]**, *Approximation Theory, Wavelets and Applications* (Ed. S.P. Singh), Kluwer Academic Publ., Dordrecht, 497–512, 1995.