Summer School on Applied Analysis 2011

TU Chemnitz, September 26-30, 2011

Polynomial approximation via de la Vallée Poussin means

Lecture 2: Discrete operators

Woula Themistoclakis

CNR - National Research Council of Italy

Institute for Computational Applications "Mauro Picone", Naples, Italy.

De la Vallée Poussin means: $$V_{n,m}(w,f,x) := \frac{1}{2m}\sum_{k=n-m}^{n+m-1} S_k(w,f,x)$$

where
$$\begin{cases} S_n(w,f,x) := \sum_{k=0}^{n} c_k(w,f)p_k(w,x) & \text{Fourier sum} \\ c_k(w,f) := \int_{-1}^{1} p_k(w,y)f(y)w(y)dy & \text{Fourier coefficients} \end{cases}$$

$w(x) := v^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha,\beta > -1$, being a Jacobi weight and $\{p_j(w,x)\}_j$ the corresponding system of orthonormal Jacobi polynomials

- **Quasi-projection:** $$V_{n,m}(w,P) = P, \quad \forall P \in \mathbb{P}_{n-m}$$

- **Near best polynomial:** $$\|(f - V_{n,m}(w,f))u\|_p \le CE_{n-m}(f)_{u,p}$$

holds for any $1 \le p \le \infty$, $m = \theta n$ ($\theta \in]0,1[$ fixed) and suitable u, w.

Other forms:

$$V_{n,m}(w,f,x) = \int_{-1}^{1} H_{n,m}(w,x,y)f(y)w(y)dy \quad (1)$$

$$V_{n,m}(w,f,x) = \sum_{k=0}^{n+m-1} \mu_{n,k}^{m}\, c_k(w,f)p_k(w,x) \quad (2)$$

$$H_{n,m}(w,x,y) := \frac{1}{2m}\sum_{r=n-m}^{n+m-1} K_r(w,x,y)$$ de la Vallée Poussin kernel

$$K_r(w,x,y) := \sum_{j=0}^{r} p_j(w,x)p_j(w,y)$$ Darboux kernel

$$c_k(w,f) := \int_{-1}^{1} p_k(w,y)f(y)w(y)dy$$ Fourier coefficients

$$\mu_{n,k}^{m} := \begin{cases} 1 & \text{if} \quad 0 \le k \le n-m, \\ \dfrac{n+m-k}{2m} & \text{if} \quad n-m < k < n+m. \end{cases}$$

Lagrange interpolation at the zeros of orthogonal polynomials

▶ Take the **Fourier sum:**

$$S_{n-1}(w,f,x) := \int_{-1}^{1} K_{n-1}(w,x,y) f(y) w(y) dy$$

▶ Apply the **Gaussian rule:**

$$\int_{-1}^{1} g(x)w(x)dx = \sum_{j=1}^{n} \lambda_{n,j}\; g(x_{n,j}), \qquad g \in \mathbb{P}_{2n-1}$$

In this way we obtain the **Lagrange polynomial** of degree $n-1$ interpolating f at the n zeros $\{x_{n,j}\}_j$ of $p_n(w)$, i.e.

$$L_n(w,f,x) := \sum_{j=1}^{n} \lambda_{n,j}\; K_{n-1}(w,x,x_{n,j}) f(x_{n,j})$$

with $\lambda_{n,j} K_{n-1}(w, x_{n,i}, x_{n,j}) = \delta_{i,j}, \qquad j = 1, \ldots, n.$

DISCRETE DE LA VALLEE POUSSIN MEANS

By applying the previous Gaussian rule to the integrals in

$$(1) \qquad V_{n,m}(w,f,x) = \int_{-1}^{1} H_{n,m}(w,x,y)f(y)w(y)dy$$

$$(2) \qquad V_{n,m}(w,f,x) = \sum_{k=0}^{n+m-1} \mu_{n,k}^{m}\, c_k(w,f)p_k(w,x)$$

we obtain the following discrete operator:

$$(1') \qquad \tilde{V}_{n,m}(w,f,x) = \sum_{j=1}^{n} \lambda_{n,j}\, H_{n,m}(w,x,x_{n,j})f(x_{n,j})$$

$$(2') \qquad \tilde{V}_{n,m}(w,f,x) = \sum_{k=0}^{n+m-1} \mu_{n,k}^{m}\, \tilde{c}_{n,k}(w,f)\, p_k(w,x)$$

Discrete Fourier coefficients: $\tilde{c}_{n,k}(w,f) := \sum_{j=1}^{n} \lambda_{n,j} f(x_{n,j}) p_k(w,x_{n,j})$

COMPARISON WITH LAGRANGE INTERPOLATION

$$L_n(w,f,x) = \sum_{j=1}^{n} \lambda_{n,j}\ K_{n-1}(w,x,x_{n,j}) f(x_{n,j}),$$

$$\tilde{V}_{n,m}(w,f,x) = \sum_{j=1}^{n} \lambda_{n,j}\ H_{n,m}(w,x,x_{n,j}) f(x_{n,j}), \qquad m < n$$

COMPARISON WITH LAGRANGE INTERPOLATION

$$L_n(w,f,x) = \sum_{j=1}^{n} \lambda_{n,j}\ K_{n-1}(w,x,x_{n,j}) f(x_{n,j}),$$

$$\tilde{V}_{n,m}(w,f,x) = \sum_{j=1}^{n} \lambda_{n,j}\ H_{n,m}(w,x,x_{n,j}) f(x_{n,j}), \qquad m < n$$

- **Computation:** Both computable from the data $f(x_{n,j}),\ j = 1,..,n$

COMPARISON WITH LAGRANGE INTERPOLATION

$$L_n(w,f,x) = \sum_{j=1}^{n} \lambda_{n,j}\ K_{n-1}(w,x,x_{n,j}) f(x_{n,j}),$$

$$\tilde{V}_{n,m}(w,f,x) = \sum_{j=1}^{n} \lambda_{n,j}\ H_{n,m}(w,x,x_{n,j}) f(x_{n,j}), \qquad m < n$$

- **Computation:** Both computable from the data $f(x_{n,j}),\ j = 1,..,n$
- **Invariance:** $\begin{cases} L_n(w,P) & = S_{n-1}(w,P) & = P, & \forall P \in \mathbb{P}_{n-1} \\ \tilde{V}_{n,m}(w,P) & = V_{n,m}(w,P) & = P, & \forall P \in \mathbb{P}_{n-m} \end{cases}$

COMPARISON WITH LAGRANGE INTERPOLATION

$$L_n(w,f,x) = \sum_{j=1}^{n} \lambda_{n,j}\ K_{n-1}(w,x,x_{n,j}) f(x_{n,j}),$$

$$\tilde{V}_{n,m}(w,f,x) = \sum_{j=1}^{n} \lambda_{n,j}\ H_{n,m}(w,x,x_{n,j}) f(x_{n,j}), \qquad m < n$$

- **Computation:** Both computable from the data $f(x_{n,j}),\ j = 1,..,n$

- **Invariance:** $\begin{cases} L_n(w,P) & = S_{n-1}(w,P) & = P, & \forall P \in \mathbb{P}_{n-1} \\ \tilde{V}_{n,m}(w,P) & = V_{n,m}(w,P) & = P, & \forall P \in \mathbb{P}_{n-m} \end{cases}$

- **Approximation:** $\begin{cases} \|L_n(w)\|_{C_u^0 \to C_u^0} \geq C \log n \ \text{ for any } u \\ \textbf{Is } \sup_{n,m} \|\tilde{V}_{n,m}(w)\|_{C_u^0 \to C_u^0} < \infty \ \textbf{ true for suitable } u \textbf{ ??} \end{cases}$

COMPARISON WITH LAGRANGE INTERPOLATION

$$L_n(w,f,x) = \sum_{j=1}^n \lambda_{n,j}\ K_{n-1}(w,x,x_{n,j})f(x_{n,j}),$$

$$\tilde{V}_{n,m}(w,f,x) = \sum_{j=1}^n \lambda_{n,j}\ H_{n,m}(w,x,x_{n,j})f(x_{n,j}), \qquad m<n$$

- **Computation:** Both computable from the data $f(x_{n,j}),\ j=1,..,n$

- **Invariance:** $$\begin{cases} L_n(w,P) = S_{n-1}(w,P) = P, & \forall P\in\mathbb{P}_{n-1} \\ \tilde{V}_{n,m}(w,P) = V_{n,m}(w,P) = P, & \forall P\in\mathbb{P}_{n-m} \end{cases}$$

- **Approximation:** $$\begin{cases} \|L_n(w)\|_{C^0_u\to C^0_u} \geq C\log n \quad \text{for any } u \\ \textbf{Is } \sup_{n,m}\|\tilde{V}_{n,m}(w)\|_{C^0_u\to C^0_u} < \infty \ \textbf{true for suitable } u\ \textbf{??} \end{cases}$$

- **Interpolation:** $$\begin{cases} L_n(w,f,x_{n,j}) = f(x_{n,j}), \quad j=1,\dots,n \\ \textbf{Is it true that } \tilde{V}_{n,m}(w,f,x_{n,j}) = f(x_{n,j})\ ?? \end{cases}$$

Connection with the continuous de la Vallée Poussin operator

- **Case $1 \le p < \infty$:** We have

$$\|\tilde{V}_{n,m}(w,f)u\|_p \le C\|V_{n,m}(w)\|_{L^{p'}_{\frac{w}{u}} \to L^{p'}_{\frac{w}{u}}} \left(\sum_{k=1}^{n} \lambda_n(u^p, x_{n,k}) |f(x_{n,k})|^p \right)^{\frac{1}{p}}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\lambda_n(v, x_{n,k}) := \left[\sum_{k=0}^{n-1} p_k^2(v, x_{n,k}) \right]^{-1} \sim v(x_{n,k}) \frac{\sqrt{1 - x_{n,k}^2}}{n}$

- **Case $p = \infty$:** We have

$$\|\tilde{V}_{n,m}(w,f)u\|_\infty \le C\|V_{n,m}(w)\|_{C^0_u \to C^0_u} \left(\max_{1 \le k \le n} |f(x_{n,k})| u(x_{n,k}) \right)$$

$C > 0$ being independent of n, m, f in both the cases.

Proof for $p = \infty$. By $\lambda_{n,k} = \lambda_n(w, x_{n,k}) \sim w(x_{n,k})\frac{\sqrt{1-x_{n,k}^2}}{n}$, and Marcinkiewicz inequality, we get

$$\|\tilde{V}_{n,m}(w,f)u\|_\infty = \max_{|x|\le 1}\left[u(x)\left|\sum_{k=1}^{n}\lambda_n(w,x_{n,k})H_{n,m}(w,x,x_{n,k})f(x_{n,k})\right|\right]$$

$$\le \quad C\max_{|x|\le 1}\left[u(x)\sum_{k=1}^{n}\lambda_n\left(\frac{w}{u},x_{n,k}\right)|H_{n,m}(w,x,x_{n,k})(fu)(x_{n,k})|\right]$$

$$\le \quad C\left(\max_{1\le k\le n}|(fu)(x_{n,k})|\right)\max_{|x|\le 1}\left[u(x)\sum_{k=1}^{n}\lambda_n\left(\frac{w}{u},x_{n,k}\right)|H_{n,m}(w,x,x_{n,k})|\right]$$

$$\le \quad C\left(\max_{1\le k\le n}|(fu)(x_{n,k})|\right)\max_{|x|\le 1}\left[u(x)\int_{-1}^{1}|H_{n,m}(w,x,y)|\frac{w(y)}{u(y)}dy\right]$$

$$= \quad C\left(\max_{1\le k\le n}|(fu)(x_{n,k})|\right)\ \|V_{n,m}(w)\|_{C_u^0\to C_u^0}. \ \square$$

▶ APPROXIMATION IN C^0_u

Theorem 1: Let $w = v^{\alpha,\beta} \in L^1[-1,1]$ and $u = v^{\gamma,\delta} \in C[-1,1]$ satisfy

$$\begin{cases} \dfrac{\alpha}{2} - \dfrac{1}{4} < \gamma \le \dfrac{\alpha}{2} + \dfrac{5}{4}, \\[2ex] \dfrac{\beta}{2} - \dfrac{1}{4} < \delta \le \dfrac{\beta}{2} + \dfrac{5}{4} \end{cases} \quad \text{and} \quad \left| \gamma - \delta - \frac{\alpha - \beta}{2} \right| \le 1$$

Then for all integers n and any $m = \theta n$ with $0 < \theta < 1$ arbitrarily fixed, the map $\tilde{V}_{n,m}(w) : C^0_u \to C^0_u$ is uniformly bounded w.r.t. n, m and

$$E_{n+m}(f)_{u,\infty} \le \|[f - \tilde{V}_{n,m}(w,f)]u\|_\infty \le C E_{n-m}(f)_{u,\infty}$$

holds for every $f \in C^0_u$, $C > 0$ being independent of f, n, m.

Theorem 2: Let $w = v^{\alpha,\beta}$ $(\alpha, \beta > -1)$ and $u = v^{\gamma,\delta}$ $(\gamma, \delta \geq 0)$ satisfy

$$\begin{cases} \frac{\alpha}{2} + \frac{1}{4} - \nu & < \gamma \leq \quad \frac{\alpha}{2} + \frac{5}{4} - \nu, \\ \frac{\beta}{2} + \frac{1}{4} - \nu & < \delta \leq \quad \frac{\beta}{2} + \frac{5}{4} - \nu, \end{cases} \qquad \text{for some} \qquad 0 \leq \nu \leq \frac{1}{2}$$

Then for all integers n and $m = \theta n$ with $0 < \theta < 1$ arbitrarily fixed, the map $\tilde{V}_{n,m}(w) : C_u^0 \to C_u^0$ is uniformly bounded w.r.t. n, m and

$$E_{n+m}(f)_{u,\infty} \leq \|[f - \tilde{V}_{n,m}(w, f)]u\|_\infty \leq C E_{n-m}(f)_{u,\infty}$$

holds for every $f \in C_u^0$, $C > 0$ being independent of f, n, m.

▶ COMPARISON WITH LAGRANGE INTERPOLATION

Let $w = v^{\alpha,\beta} \in L^1[-1,1]$ and $u = v^{\gamma,\delta} \in C[-1,1]$ be such that:

$$\frac{\alpha}{2}+\frac{1}{4} \leq \gamma \leq \frac{\alpha}{2}+\frac{5}{4},$$
$$\frac{\beta}{2}+\frac{1}{4} \leq \delta \leq \frac{\beta}{2}+\frac{5}{4}.$$

Then for all sufficiently large pair of integers n and $m = \theta n$ ($0 < \theta < 1$ fixed) and for each $f \in C_u^0$, we have

Lagrange error: $\|[f - L_n(w,f)]u\|_\infty \leq C \log n \; E_n(f)_{u,\infty}$

De la V.P. error: $\|[f - \tilde{V}_n^m(w,f)]u\|_\infty \leq C E_{n-m}(f)_{u,\infty}$

where $C > 0$ is independent of n, f in both the cases.

▶ APPROXIMATION IN L^p_u

Theorem 3: Let $1 \le p < \infty$ and assume that $w = v^{\alpha,\beta} \in L^1[-1,1]$ and $u = v^{\gamma,\delta} \in L^p[-1,1]$, with $\frac{w}{u} \in L^{p'}[-1,1]$, satisfy the bounds

$$\begin{cases} \frac{\alpha}{2} + \frac{1}{4} - \nu & < \gamma + \frac{1}{p} \le \quad \frac{\alpha}{2} + \frac{5}{4} - \nu \\ \frac{\beta}{2} + \frac{1}{4} - \nu & < \delta + \frac{1}{p} \le \quad \frac{\beta}{2} + \frac{5}{4} - \nu \end{cases} \qquad \text{for some} \qquad 0 \le \nu \le \frac{1}{2}$$

Then for all $n, m \in \mathbb{N}$ with $m = \theta n$ ($0 < \theta < 1$ fixed) and each $f \in L^p_u$ (everywhere defined on $]-1,1[$), we have

$$\|\tilde{V}_{n,m}(w,f)u\|_p \le C \left(\sum_{k=1}^{n} \lambda_n(u^p, x_{n,k}) |f(x_{n,k})|^p \right)^{\frac{1}{p}}$$

where we recall that $\lambda_n(u^p, x_{n,k}) \sim u^p(x_{n,k}) \frac{\sqrt{1-x_{n,k}^2}}{n}$.

Lagrange case $L_n(w,f)$: The same estimate holds with $p \ne 1$ and $\nu = 0$.

Error estimates in Sobolev–type spaces

$$\begin{cases} W_r^p(u) & := \quad \{f \in L_u^p : \ f^{(r-1)} \in AC_{loc}, \text{ and } f^{(r)}\varphi^r \in L_u^p\} \\ \|f\|_{W_r^p(u)} & := \quad \|fu\|_p + \|f^{(r)}\varphi^r u\|_p, \qquad \varphi(x) := \sqrt{1-x^2} \end{cases}$$

Note: For any $f \in W_r^p(u)$, we have $E_n(f)_{u,p} \leq \frac{C}{n^r} \|f^{(r)}\varphi^r u\|_p$

Theorem 4: Under the assumptions of Theorem 3, for all $f \in W_r^p(u)$, we have

$$\begin{aligned} \|[f - \tilde{V}_{n,m}(w,f)]u\|_p &\leq \frac{C}{n^r} \|f^{(r)}\varphi^r u\|_p, \\ \|f - \tilde{V}_{n,m}(w,f)\|_{W_s^p(u)} &\leq \frac{C}{n^{r-s}} \|f\|_{W_r^p(u)}, \qquad 0 < s \leq r \end{aligned}$$

where $C > 0$ is independent of f, n, m and $1 \leq p \leq \infty$ (setting $L_u^\infty := C_u^0$).

Lagrange case: $L_n(w,f)$ verifies the same estimates, but for $p \notin \{1, \infty\}$ and $\nu = 0$.

COMPARISON WITH LAGRANGE INTERPOLATION

$$L_n(w,f,x) := \sum_{k=1}^{n} \lambda_{n,k}\, K_{n-1}(w,x,x_{n,k}) f(x_{n,k}),$$

$$\tilde{V}_{n,m}(w,f,x) := \sum_{k=1}^{n} \lambda_{n,k}\, H_{n,m}(w,x,x_{n,k}) f(x_{n,k}), \quad n > m$$

- **Invariance:** $\tilde{V}_{n,m}(w) : f \to \tilde{V}_{n,m}(w,f) \in \mathbb{P}_{n+m-1}$ is a *quasi–projection*

- **Approximation:** $\tilde{V}_{n,m}(w,f)$ solves the “critical” cases $p = 1, \infty$.

- **Interpolation:** **Is it true that** $\tilde{V}_{n,m}(w,f,x_{n,k}) = f(x_{n,k})$, $k = 1,..,n$??

Theorem 5 Let w be such that, for all $x \in]-1,1[$, we have

$$(3) \quad p_{n+s}(w,x)+p_{n-s}(w,x) = p_n(w,x)Q(x), \qquad deg(Q) \le s < n,$$

Then $\tilde{V}_{n,m}(w,f,x_{n,i}) = f(x_{n,i}),\ i=1,..,n,$ holds for all $n \ge m > 0$.

Examples: Bernstein–Szego weights defined by

$$w(x) := (1-x)^\alpha(1+x)^\beta, \quad |\alpha| = |\beta| = \frac{1}{2}, \qquad \text{Chebyshev weights}$$

$$w(x) := \frac{1}{p(x)}\frac{1}{\sqrt{1-x^2}}, \qquad w(x) := \frac{1}{p(x)}\sqrt{\frac{1-x}{1+x}}, \qquad \deg(p) \le 1$$

$$w(x) := \frac{1}{p(x)}\sqrt{1-x^2}, \qquad \deg(p) \le 2$$

provide polynomials satisfying (3).

Proof. Note that we can write

$$H_{n,m}(w,x_{n,i},x_{n,j}) = \frac{1}{2m}\sum_{r=0}^{m-1}\left[K_{n+r}(w,x_{n,i},x_{n,j}) + K_{n-(r+1)}(w,x_{n,i},x_{n,j})\right]$$

where:

$$\begin{array}{rcl} K_{n+r}(w,x_{n,i},x_{n,j}) & = & K_n(w,x_{n,i},x_{n,j}) + \sum_{s=1}^{r} p_{n+s}(w,x_{n,i})p_{n+s}(w,x_{n,j}), \\ + & = & + \qquad\qquad\qquad + \\ K_{n-(r+1)}(w,x_{n,i},x_{n,j}) & = & K_n(w,x_{n,i},x_{n,j}) - \sum_{s=1}^{r} p_{n-s}(w,x_{n,i})p_{n-s}(w,x_{n,j}). \end{array}$$

Hence by (3) we get $p_{n+s}(w,x_{n,i}) = -p_{n-s}(w,x_{n,i}),\ \ i = 1,..,n$, and the kernel in

$$\tilde{V}_{n,m}(w,f,x_{n,i}) = \sum_{j=1}^{n}\lambda_{n,j}H_{n,m}(w,x_{n,i},x_{n,j})f(x_{n,j})$$

reduces to $H_{n,m}(w,x_{n,i},x_{n,j}) = K_n(w,x_{n,i},x_{n,j}) = \delta_{i,j}[\lambda_{n,j}]^{-1}$. $\square$

COMPARISON WITH LAGRANGE INTERPOLATION

$$
\begin{aligned}
L_n(w,f,x) &:= \sum_{k=1}^{n} \lambda_{n,k}\, K_{n-1}(w,x,x_{n,k}) f(x_{n,k}), \\
\tilde{V}_{n,m}(w,f,x) &:= \sum_{k=1}^{n} \lambda_{n,k}\, H_{n,m}(w,x,x_{n,k}) f(x_{n,k}), \qquad n > m
\end{aligned}
$$

▶ **Invariance:** $\begin{cases} L_n(w) & : f \to L_n(w,f) \in \mathbb{P}_{n-1} \\ \tilde{V}_{n,m}(w) & : f \to \tilde{V}_{n,m}(w,f) \in \mathbb{P}_{n+m-1} \end{cases}$ projection / quasi–projection

▶ **Approximation:** $\begin{cases} \|[f - L_n(w,f)]u\|_\infty & \leq C \log n \; E_n(f)_{u,\infty} \\ \|[f - \tilde{V}_{n,m}(w,f)]u\|_\infty & \leq C E_{n-m}(f)_{u,\infty} \end{cases}$

▶ **Interpolation:** $\begin{cases} L_n(w,f,x_{n,k}) = f(x_{n,k}), \\ \tilde{V}_{n,m}(w,f,x_{n,k}) = f(x_{n,k}), \end{cases}$ for all $w = v^{\alpha,\beta}$, $|\alpha| = |\beta| = \frac{1}{2}$, $n \geq m$

De la Vallée Poussin type polynomial spaces

DEF: $S_{n,m}(w) :=$ span$\{\ \lambda_{n,k}\ H_{n,m}(w, x, x_{n,k}) :\ \ k = 1, \ldots, n\}$

- **Interpolation property:** $\lambda_{n,k}\ H_{n,m}(w, x_{n,h}, x_{n,k}) = \delta_{h,k}$

$$\Downarrow$$

dim $S_{n,m}(w) = n$

- **Invariance property:** $\tilde{V}_{n,m}(w, P) = P, \quad \deg(P) \le n - m$

$$\Downarrow$$

$$\mathbb{P}_{n-m} \subset S_{n,m}(w) \subset \mathbb{P}_{n+m-1}$$

Theorem 6: In the interpolating case, $w = v^{\alpha,\beta}$ with $|\alpha| = |\beta| = \frac{1}{2}$, the operator $\tilde{V}_{n,m}(w) : f \to \tilde{V}_{n,m}(w, f)$ is a projection on $S_{n,m}(w)$, i.e. we have

$$f \in S_{n,m}(w) \Leftrightarrow f = \tilde{V}_{n,m}(w, f)$$

INTERPOLATING BASIS OF $S_{n,m}(w)$:

$$S_{n,m}(w) := \text{span}\left\{\Phi^m_{n,k}(w,x) := \lambda_{n,k}\ H_{n,m}(w,x,x_{n,k}), \qquad k=1,\dots,n\right\}$$

De la V. P. interpolating polynomial:

$$\tilde{V}_{n,m}(w,f,x) = \sum_{k=1}^{n} f(x_{n,k})\Phi^m_{n,k}(w,x)$$

Under the assumptions of Theorem 3, for all $a_k \in \mathbb{R}$, $k = 1,..,n$, we have

$$\left\| u\left(\sum_{k=1}^{n} a_k \Phi^m_{n,k}(w)\right)\right\|_p \sim \begin{cases} \left(\sum_{k=1}^{n} \lambda_k(u^p, x_{n,k})|a_k|^p\right)^{\frac{1}{p}} & \text{if} \quad 1 \le p < \infty \\ \max_{1\le k\le n} |a_k| u(x_{n,k}) & \text{if} \quad p = \infty \end{cases}$$

i.e. $\{\Phi^m_{n,k}(w)\}_k$ is a *Marcinkiewicz basis* in L^p_u, for all $1 \le p \le \infty$.

ORTHOGONAL BASIS OF $S_{n,m}(w)$

$$q_k(w) := \begin{cases} p_k(w) & \text{if} \quad 0 \le k \le n-m \\ \dfrac{m+n-k}{2m} p_k(w) - \dfrac{m-n+k}{2m} p_{2n-k}(w) & \text{if} \quad n-m < k < n \end{cases}$$

Theorem 7: The set $\{q_k(w)\}_k$ is an orthogonal basis of $S_{n,m}(w)$, i.e. we have $S_{n,m}(w) := \text{span}\{q_k(w) : \ k = 0, 1, \ldots, n-1\}$ with

$$\int_{-1}^{1} q_h(w,x) q_k(w,x) w(x) dx = \delta_{h,k} \cdot \begin{cases} 1 & \text{if} \quad 0 \le k \le n-m \\ \dfrac{m^2 + (n-k)^2}{2m^2} & \text{if} \quad n-m < k < n \end{cases}$$

De la Vallée Poussin interpolating polynomial:

$$\tilde{V}_{n,m}(w, f, x) = \sum_{k=0}^{n-1} \left[\sum_{i=1}^{n} \lambda_{n,i} p_k(w, x_{n,i}) f(x_{n,i}) \right] q_k(w, x)$$

Proof. We are going to state the basis transformation

$$\Phi_{n,k}^m(w,x) = \lambda_{n,k} \sum_{j=0}^{n-1} p_j(w, x_{n,k}) q_j(w,x), \qquad k = 1, \ldots, n.$$

Recall that

$$\Phi_{n,k}^m(w,x) := \lambda_{n,k}\ H_{n,m}(w,x,x_{n,k}) = \lambda_{n,k} \sum_{j=0}^{n+m-1} \mu_{n,j}^m\ p_j(w,x_{n,k}) p_j(w,x)$$

$$= \lambda_{n,k} \left[\sum_{j=0}^{n-m} p_j(w,x_{n,k}) p_j(w,x) + \sum_{j=n-m+1}^{n-1} \frac{n+m-j}{2m} p_j(w,x_{n,k}) p_j(w,x) \right.$$
$$\left. + \sum_{j=n+1}^{n+m-1} \frac{n+m-j}{2m} p_j(w,x_{n,k}) p_j(w,x) \right]$$

i.e., by changing the summation variables, we have

$$\Phi_{n,k}^m = \lambda_{n,k} \left[\sum_{j=0}^{n-m} p_j(w, x_{n,k}) p_j(w) + \sum_{s=1}^{m-1} \frac{m+s}{2m} p_{n-s}(w, x_{n,k}) p_{n-s}(w) \right.$$
$$\left. + \sum_{s=1}^{m-1} \frac{m-s}{2m} p_{n+s}(w, x_{n,k}) p_{n+s}(w) \right]$$

and using $p_{n+s}(w, x_{n,k}) = -p_{n-s}(w, x_{n,k})$, we get

$$\begin{aligned}\Phi_{n,k}^m &= \lambda_{n,k} \sum_{j=0}^{n-m} p_j(w, x_{n,k}) p_j(w) + \\ &\quad + \lambda_{n,k} \sum_{s=1}^{m-1} p_{n-s}(w, x_{n,k}) \left[\frac{m+s}{2m} p_{n-s}(w) - \frac{m-s}{2m} p_{n+s}(w) \right] \\ &= \lambda_{n,k} \sum_{j=0}^{n-1} p_j(w, x_{n,k}) q_j(w). \ \square\end{aligned}$$

COMPARISON WITH LAGRANGE INTERPOLATION

Chebyshev case:

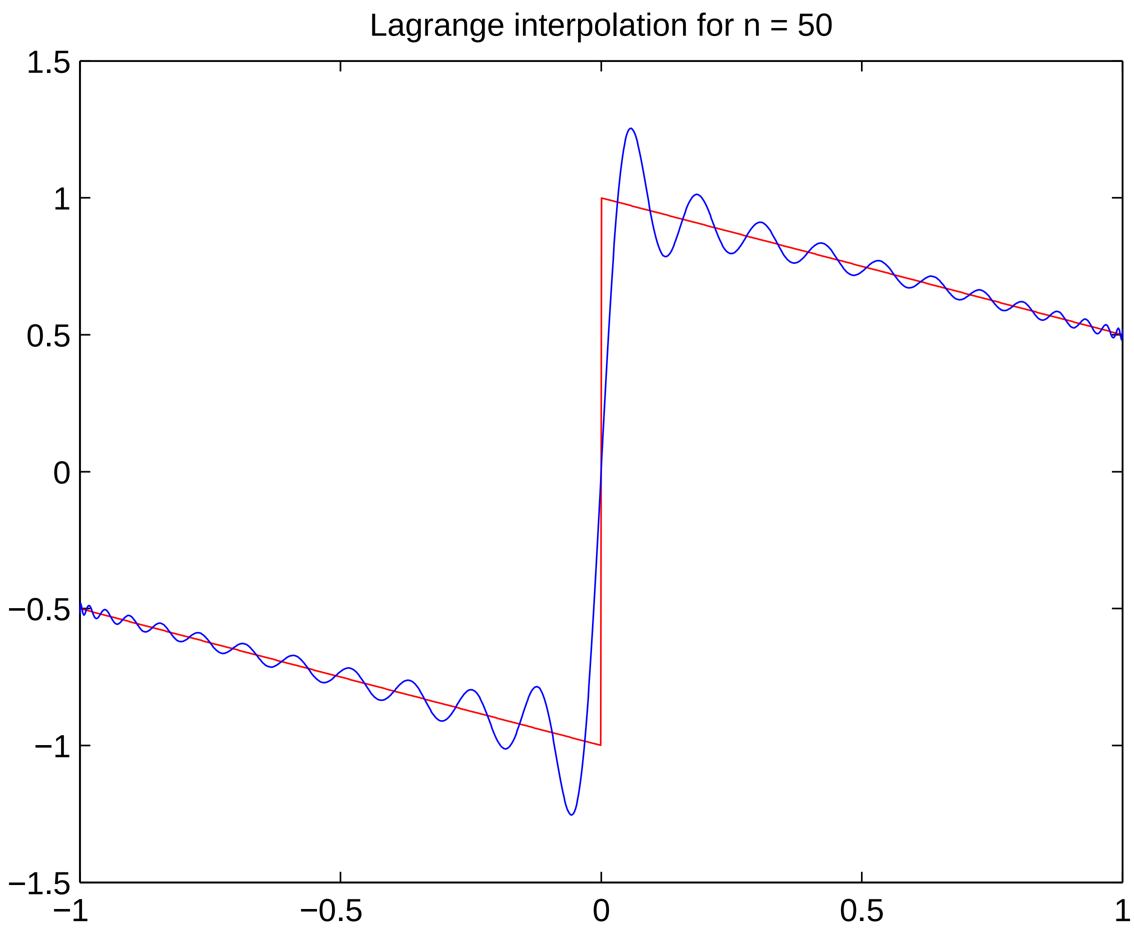
$$L_n(w,f,x) := \sum_{k=0}^{n-1} \tilde{c}_{n,k}(w,f)\; p_k(w,x),$$

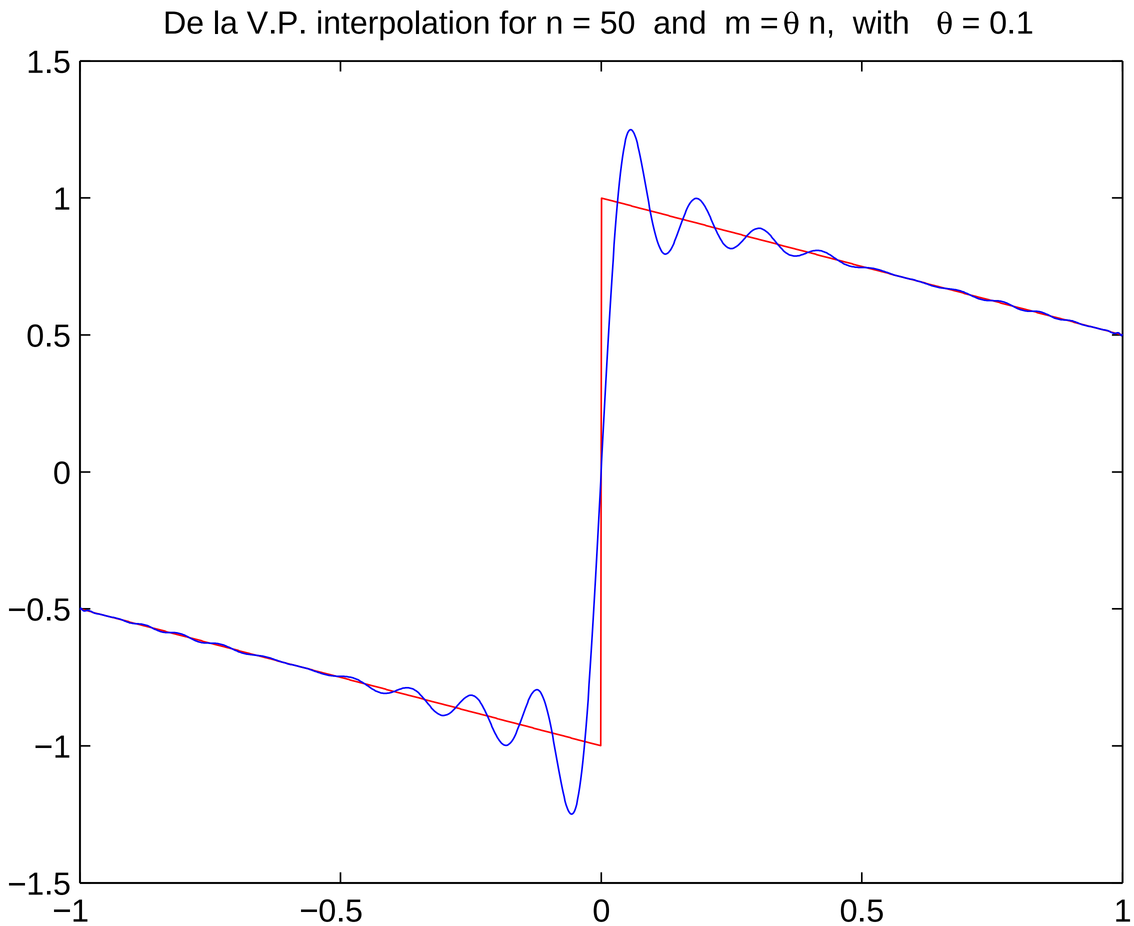
$$\tilde{V}_{n,m}(w,f,x) := \sum_{k=0}^{n-1} \tilde{c}_{n,k}(w,f)\; q_k(w,x), \qquad n > m$$

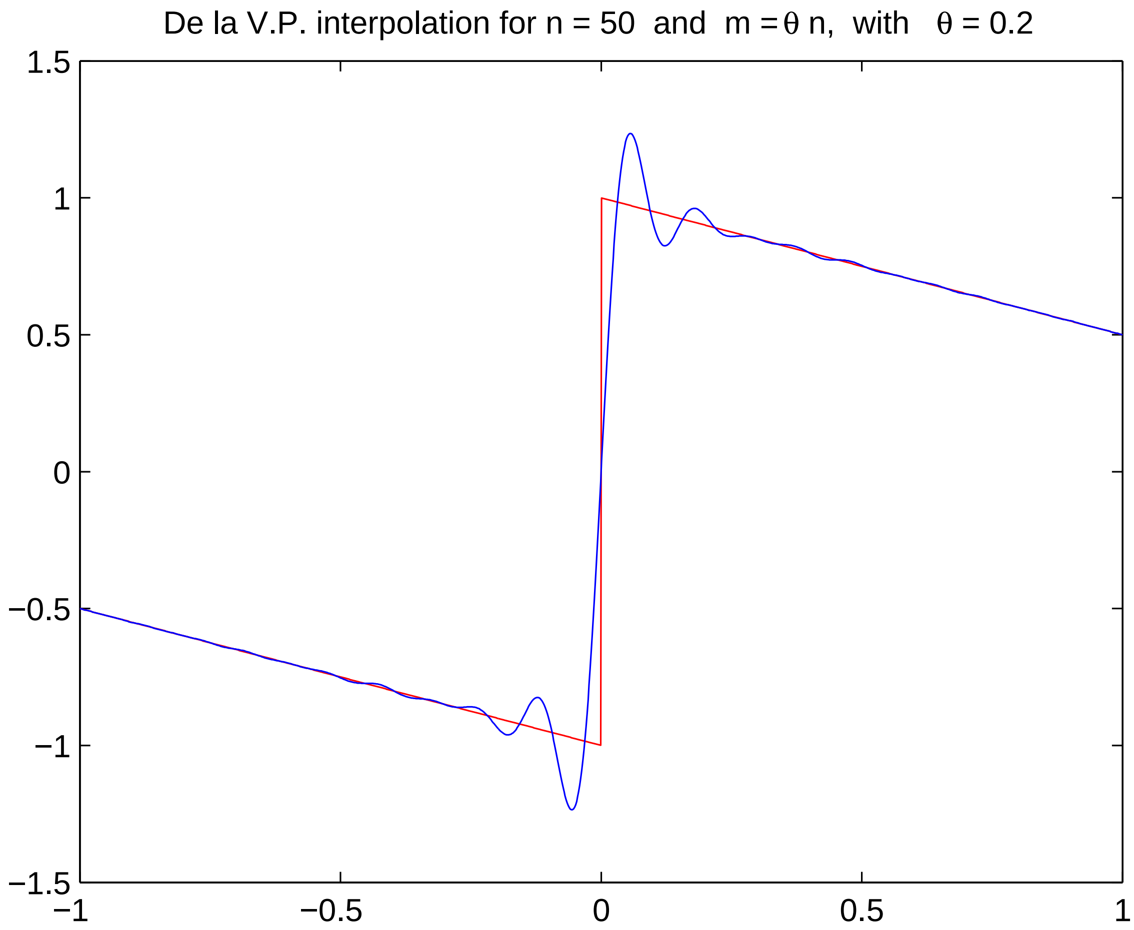
- **Interpolation:** $\begin{cases} L_n(w,f,x_{n,k}) = f(x_{n,k}), & k=1,\dots,n \\ \tilde{V}_{n,m}(w,f,x_{n,k}) = f(x_{n,k}), & k=1,\dots,n \end{cases}$

- **Invariance:** $\begin{cases} L_n(w) : f \to L_n(w,f) \in \mathbb{P}_{n-1} & \text{projection} \\ \tilde{V}_{n,m}(w) : f \to \tilde{V}_{n,m}(w,f) \in S_{n,m}(w) & \text{projection} \end{cases}$

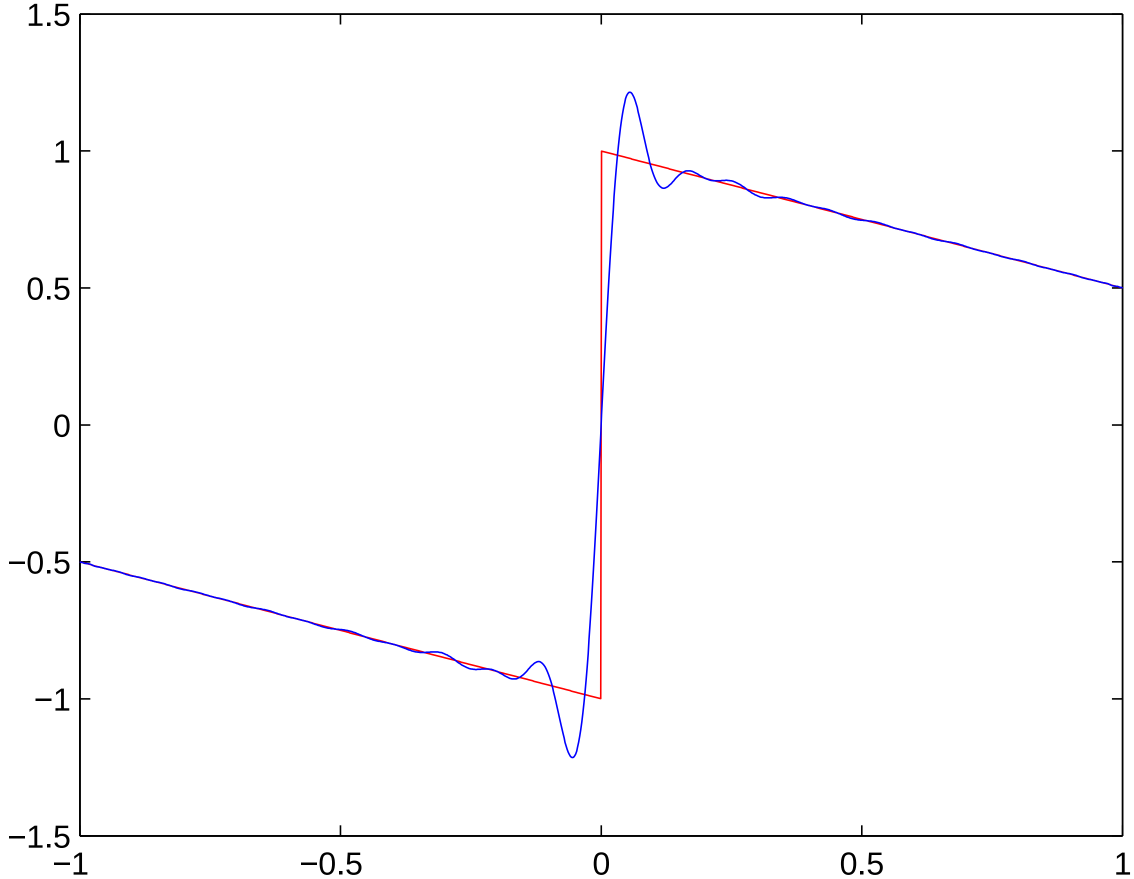
- **Approximation:** $\begin{cases} \|[f - L_n(w,f)]u\|_\infty \le C \log n \; E_n(f)_{u,\infty} \\ \|[f - \tilde{V}_{n,m}(w,f)]u\|_\infty \le C E_{n-m}(f)_{u,\infty} \end{cases}$

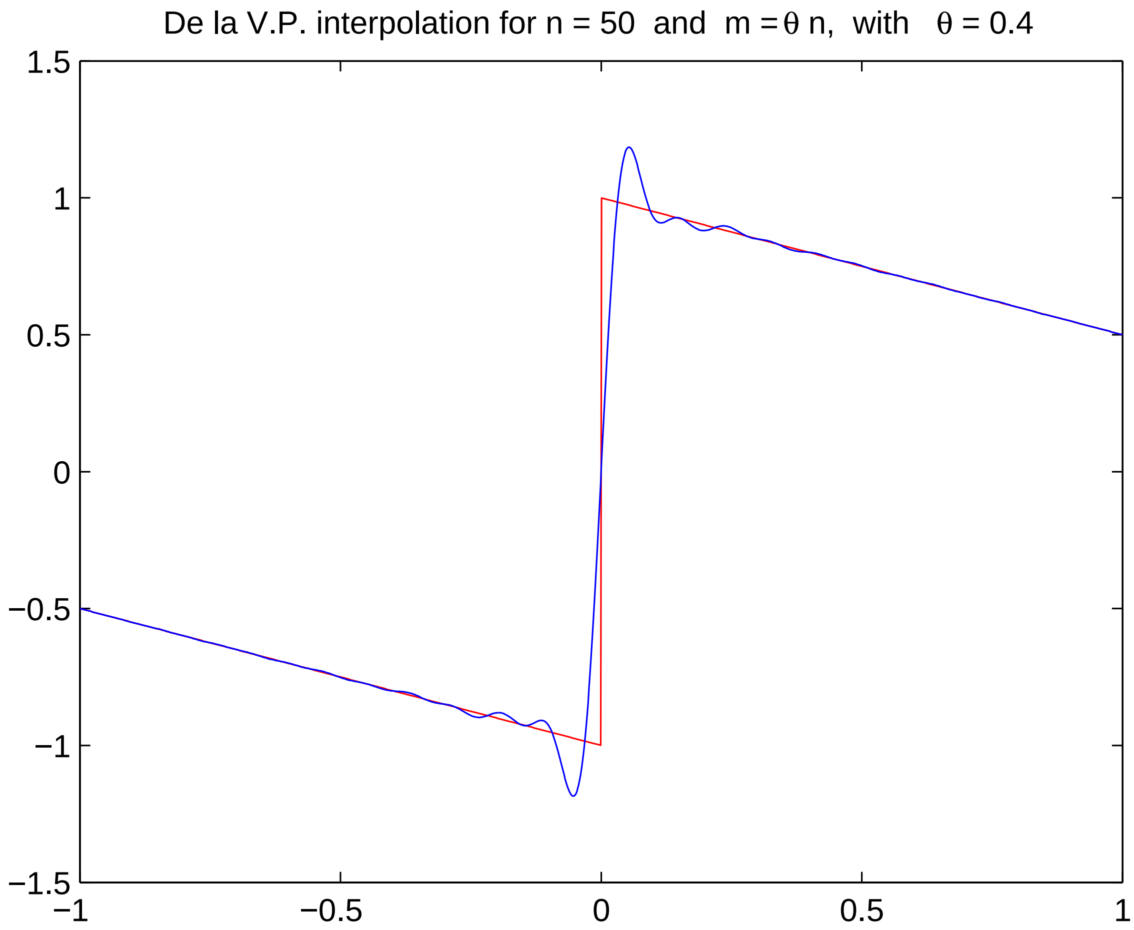

Lagrange interpolation for n = 50
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

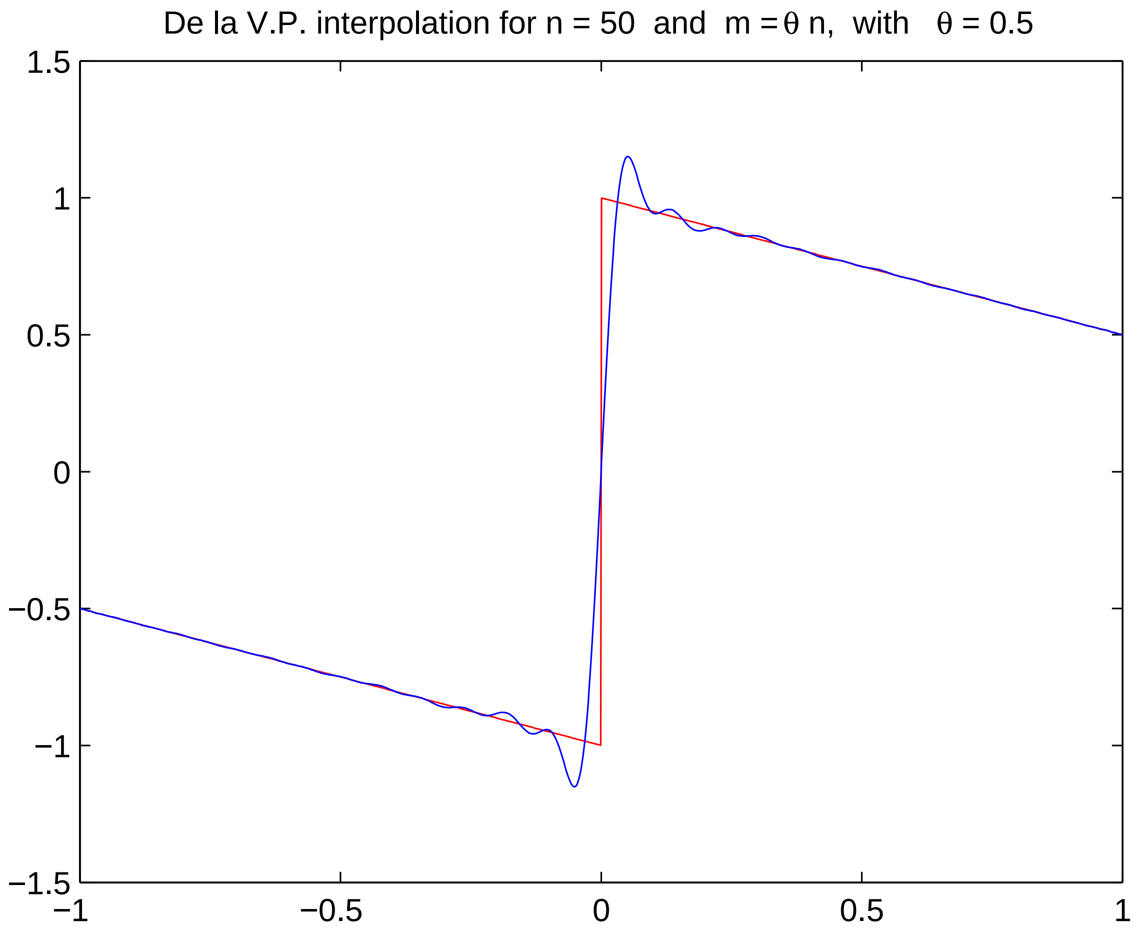

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.1
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

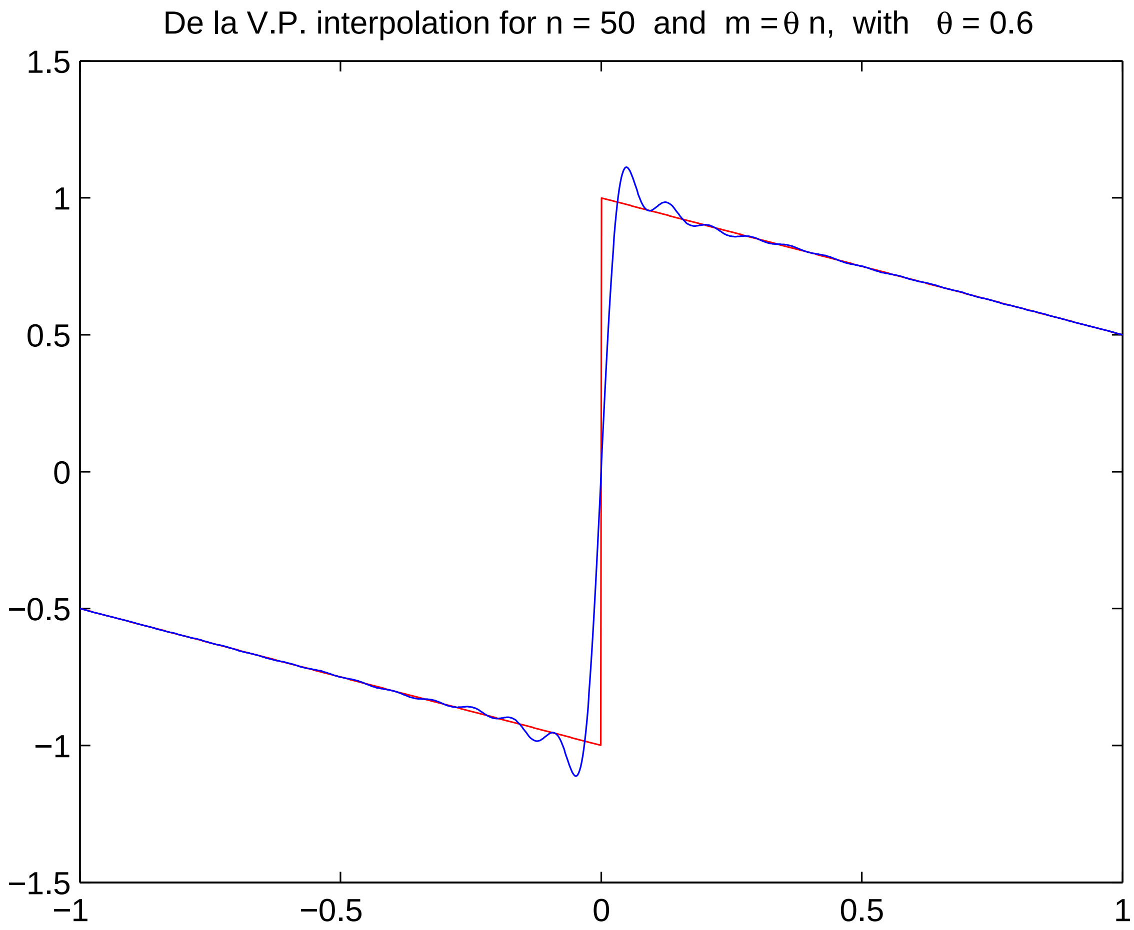

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.2
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

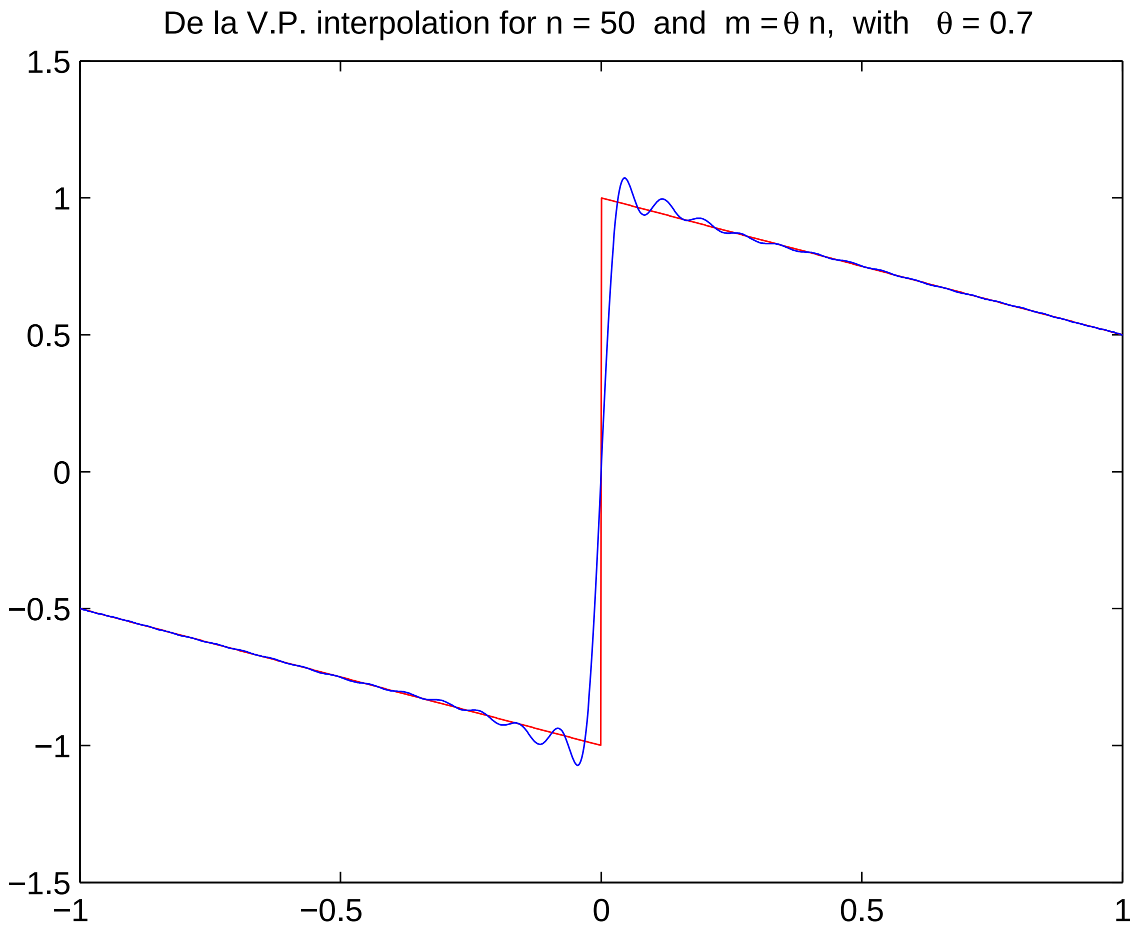
De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.3

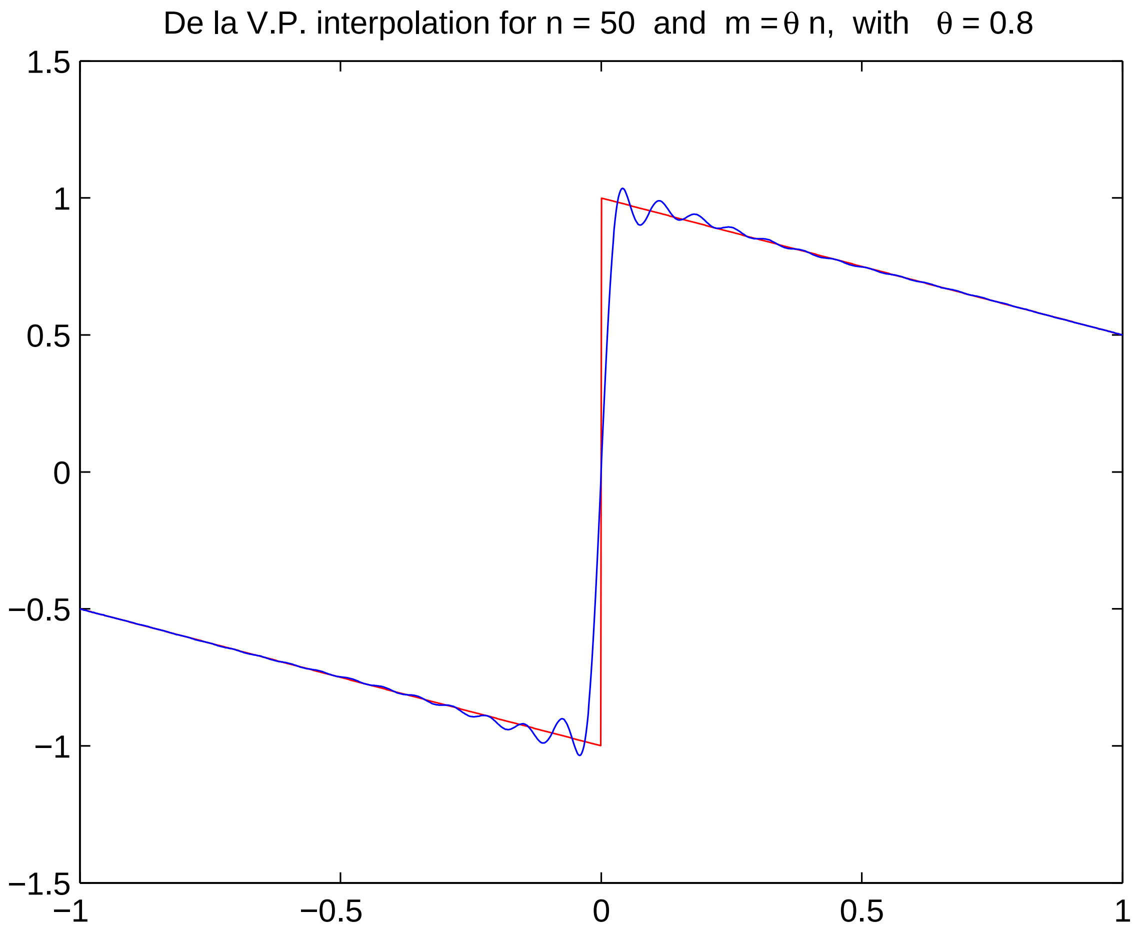


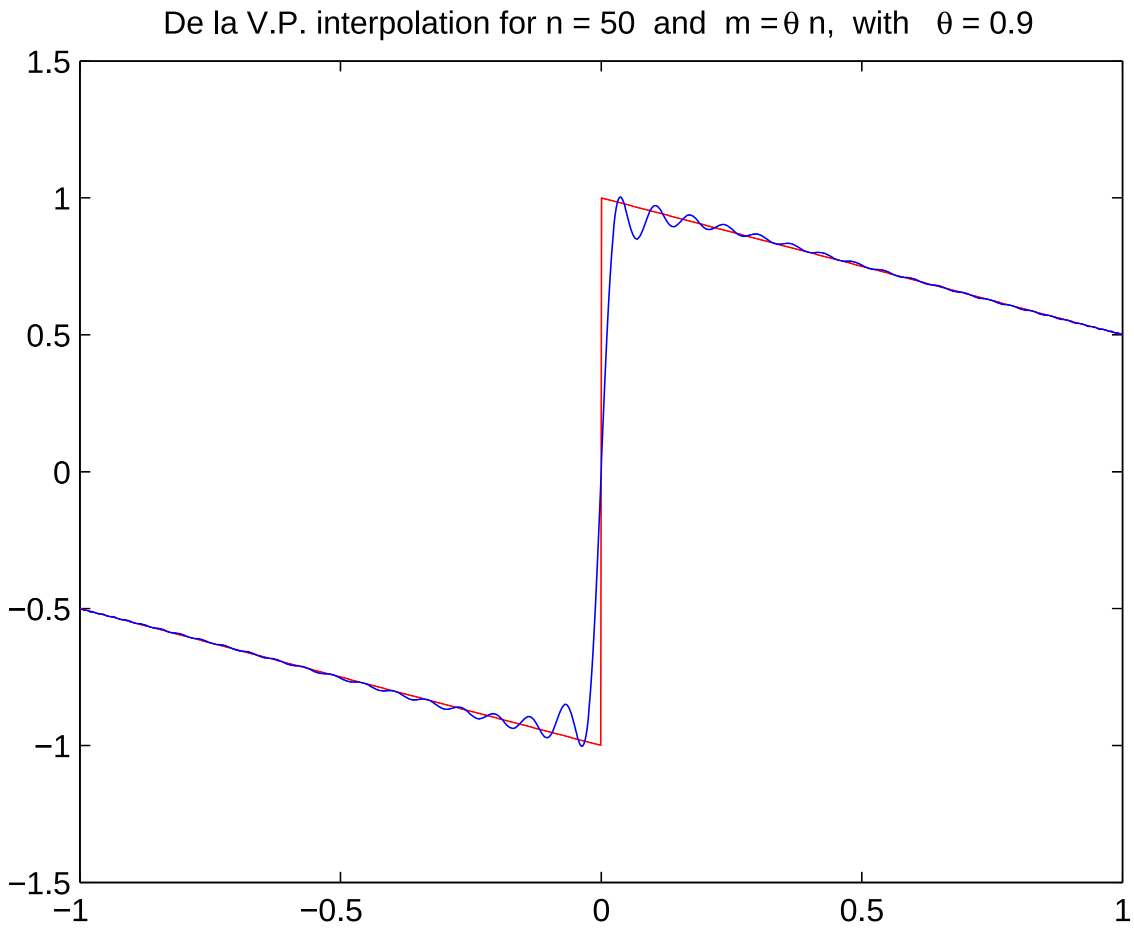

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.4
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

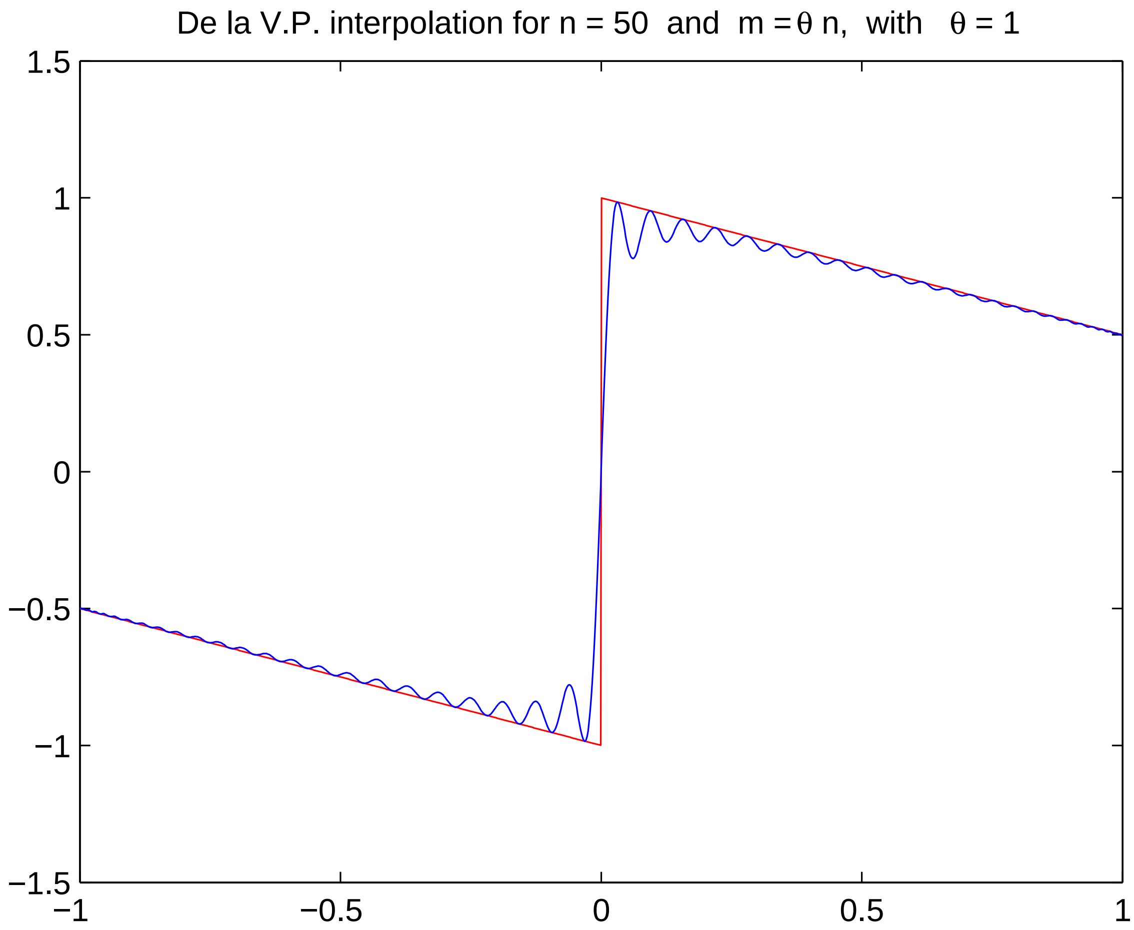

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.5
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

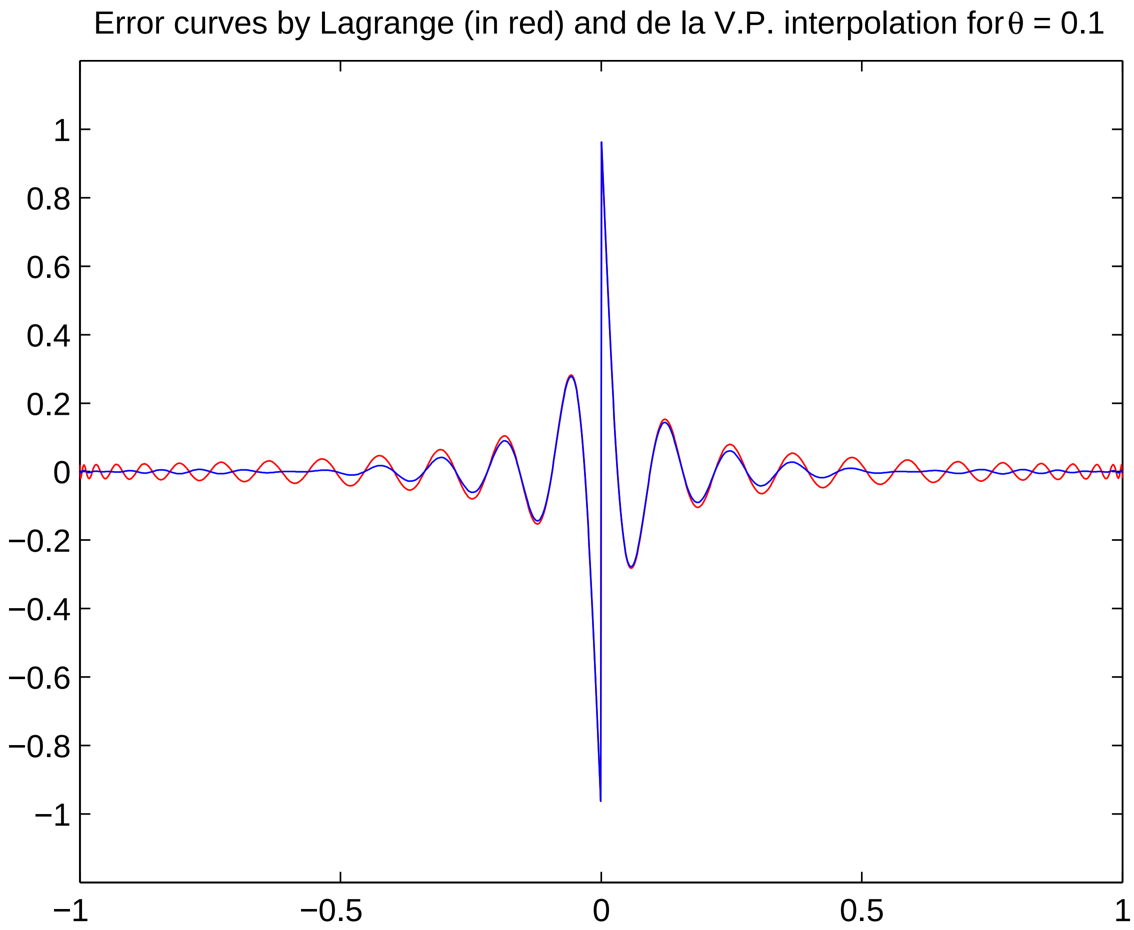

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.6
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

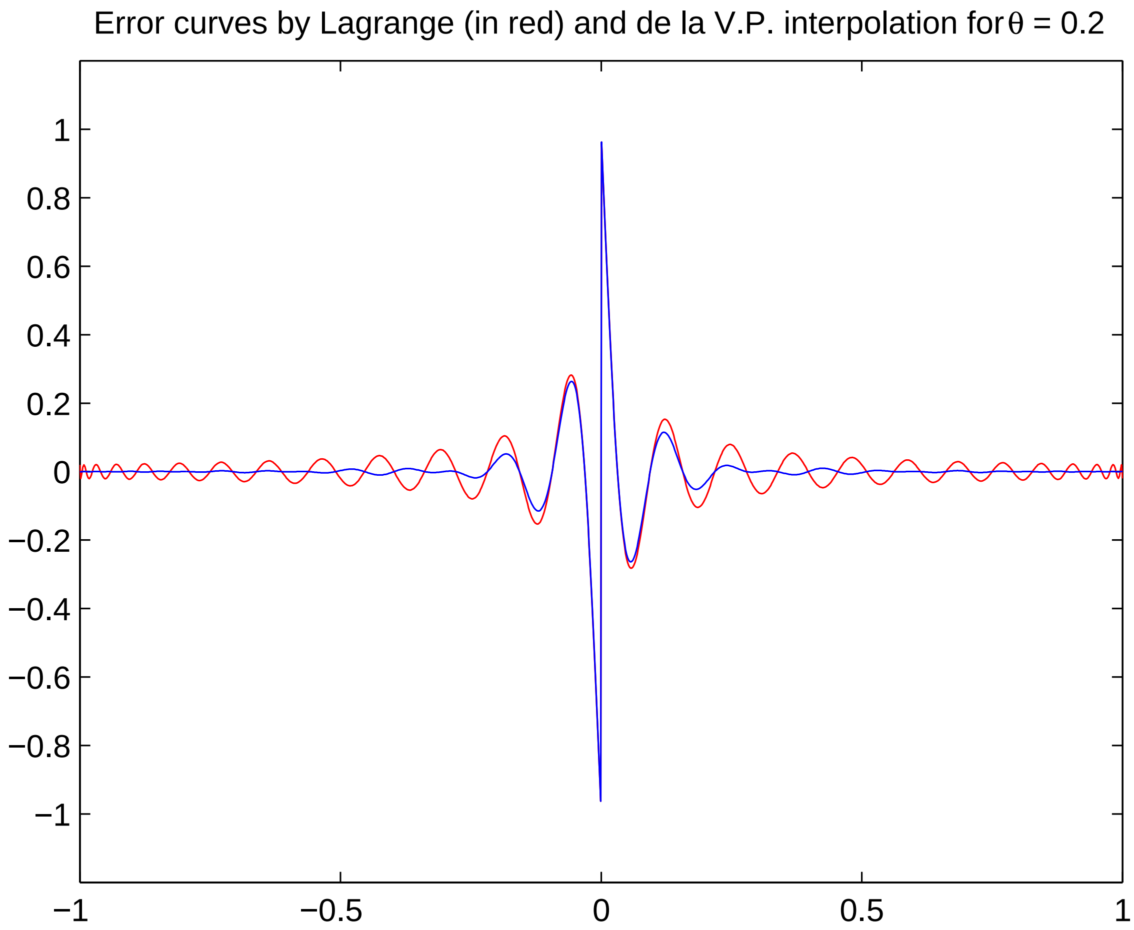

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.7
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

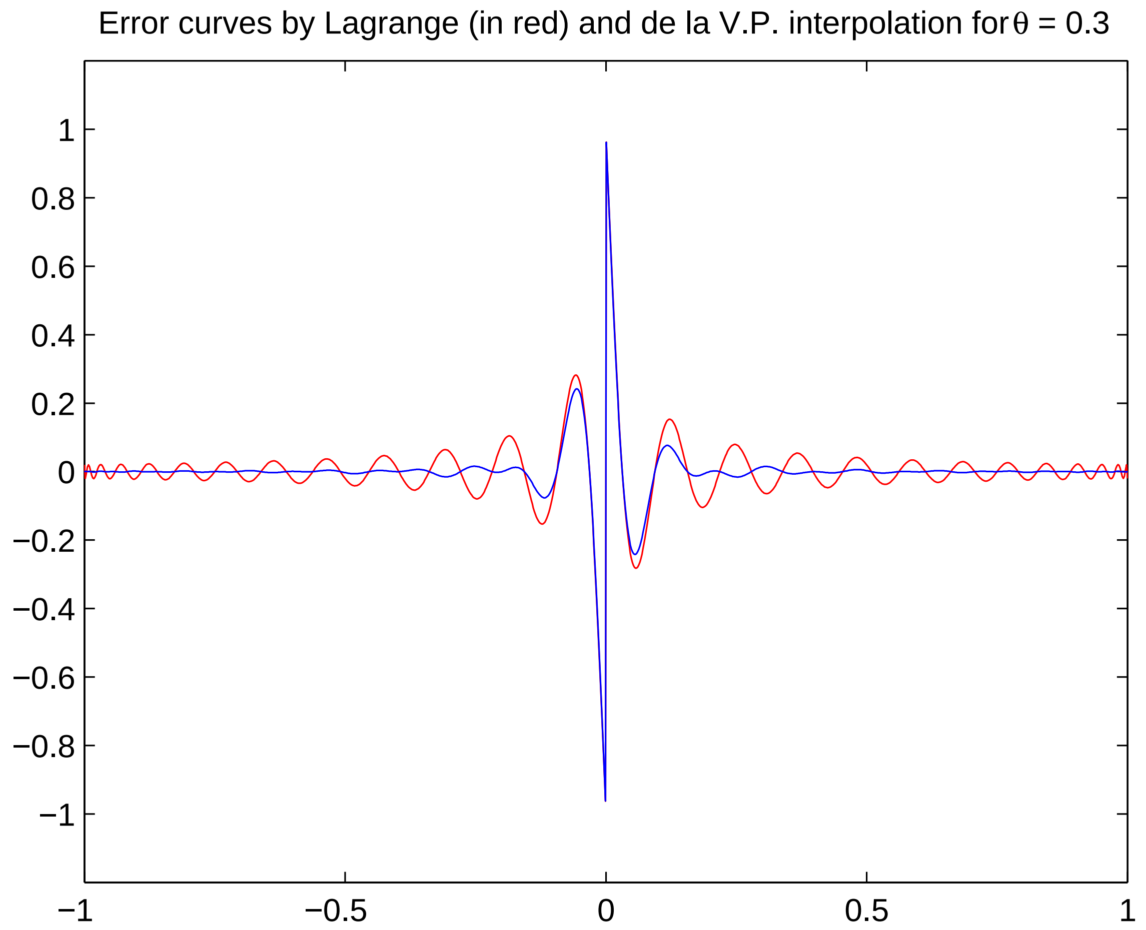

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.8
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

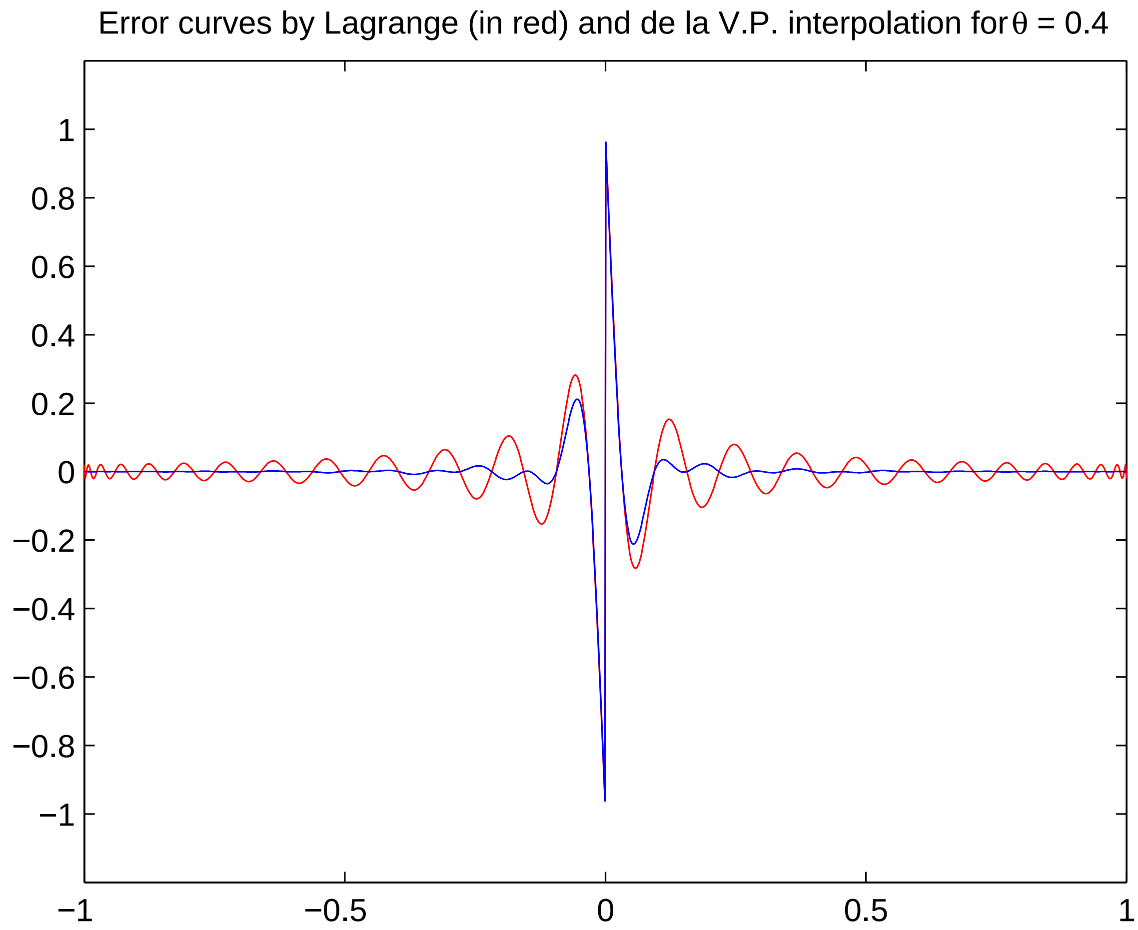

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.9
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1


De la V.P. interpolation for n = 50 and m = θ n, with θ = 1
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

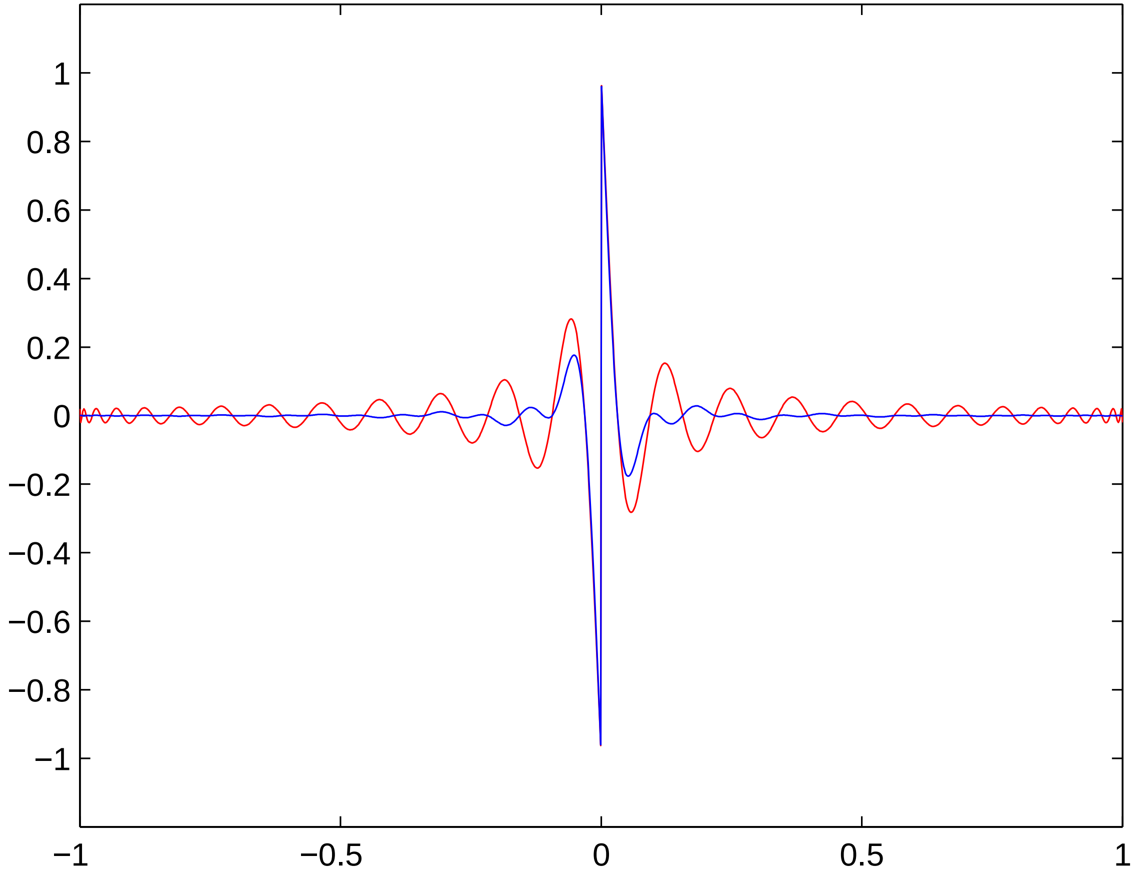

Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.5
0
0.5
1


Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.5
0
0.5
1

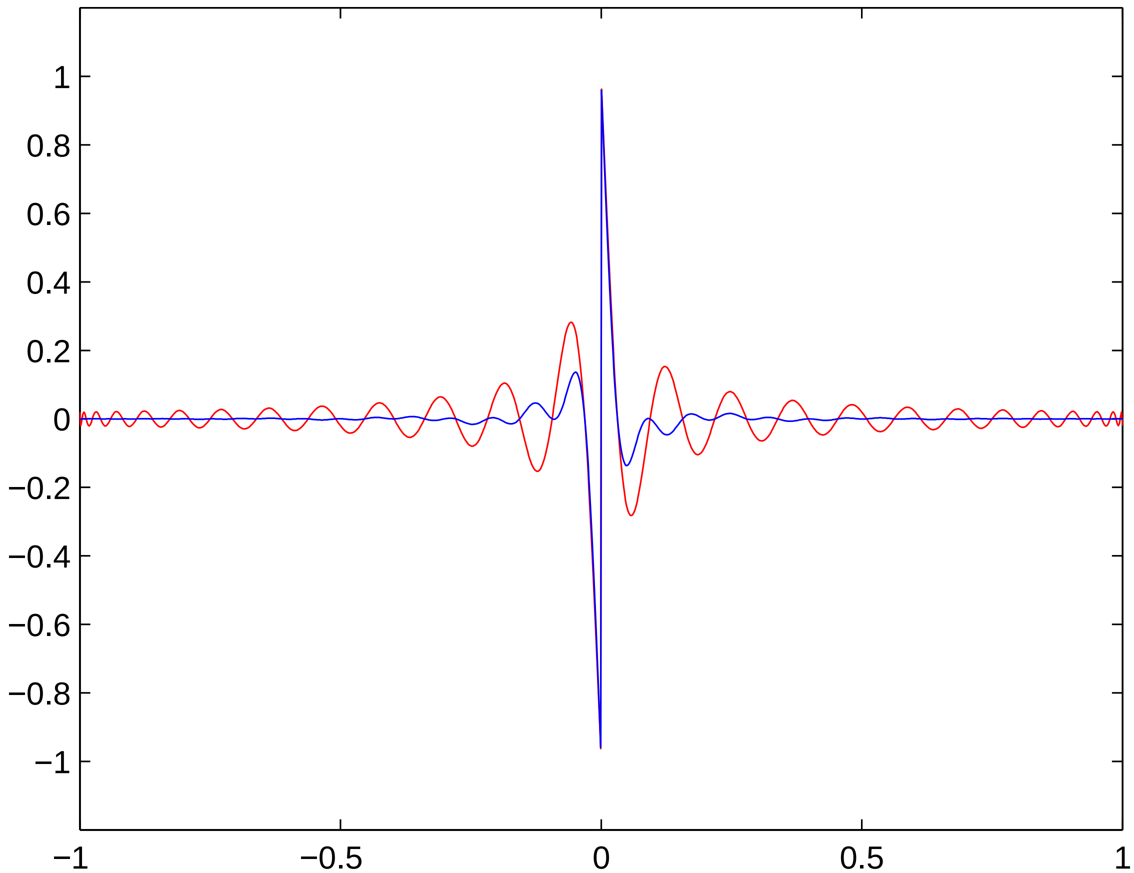

Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.3
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.5
0
0.5
1


Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.4
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.5
0
0.5
1

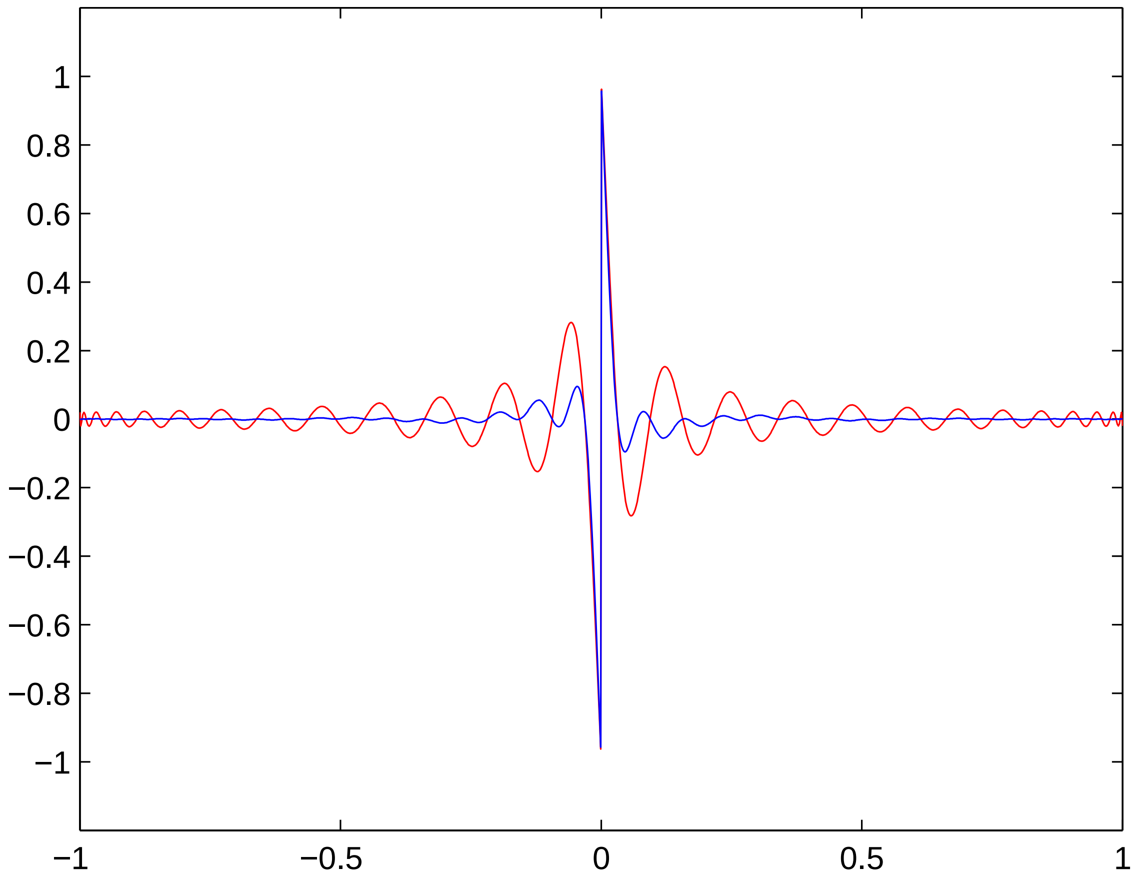
Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.5



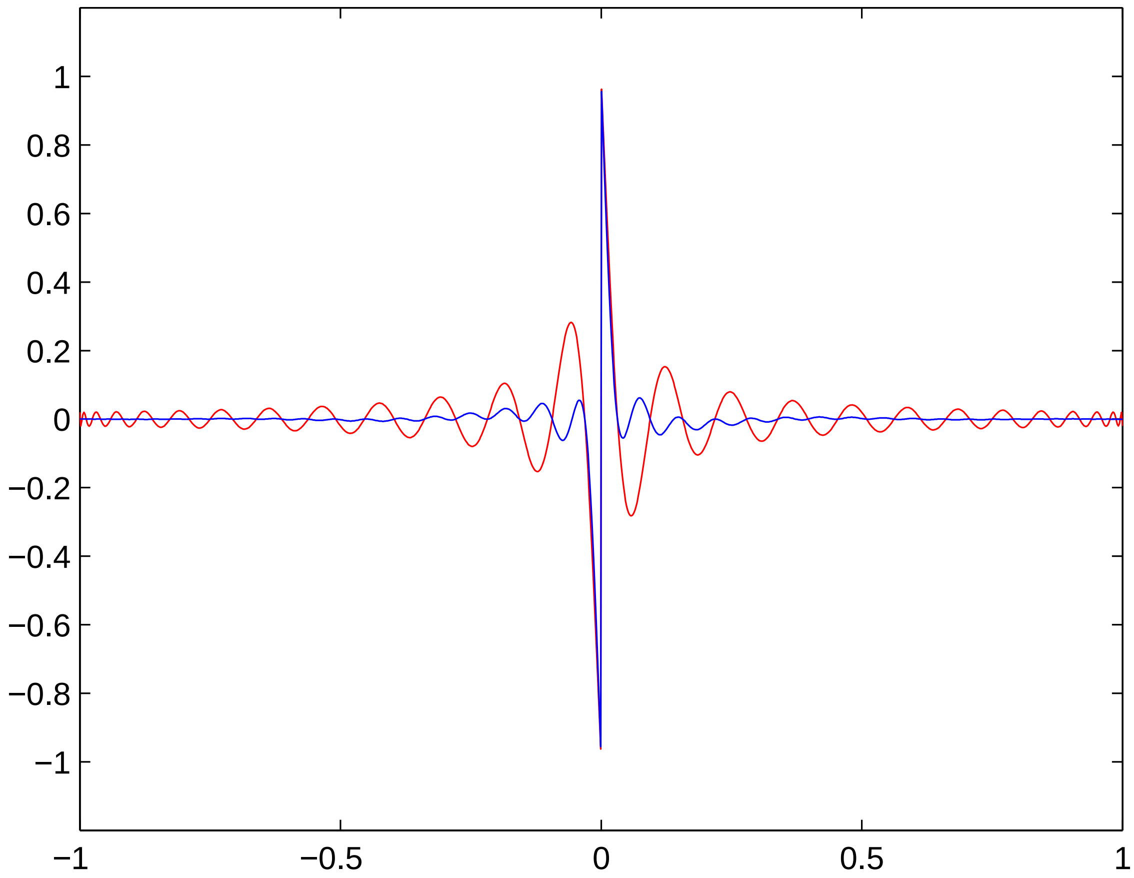
Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.6



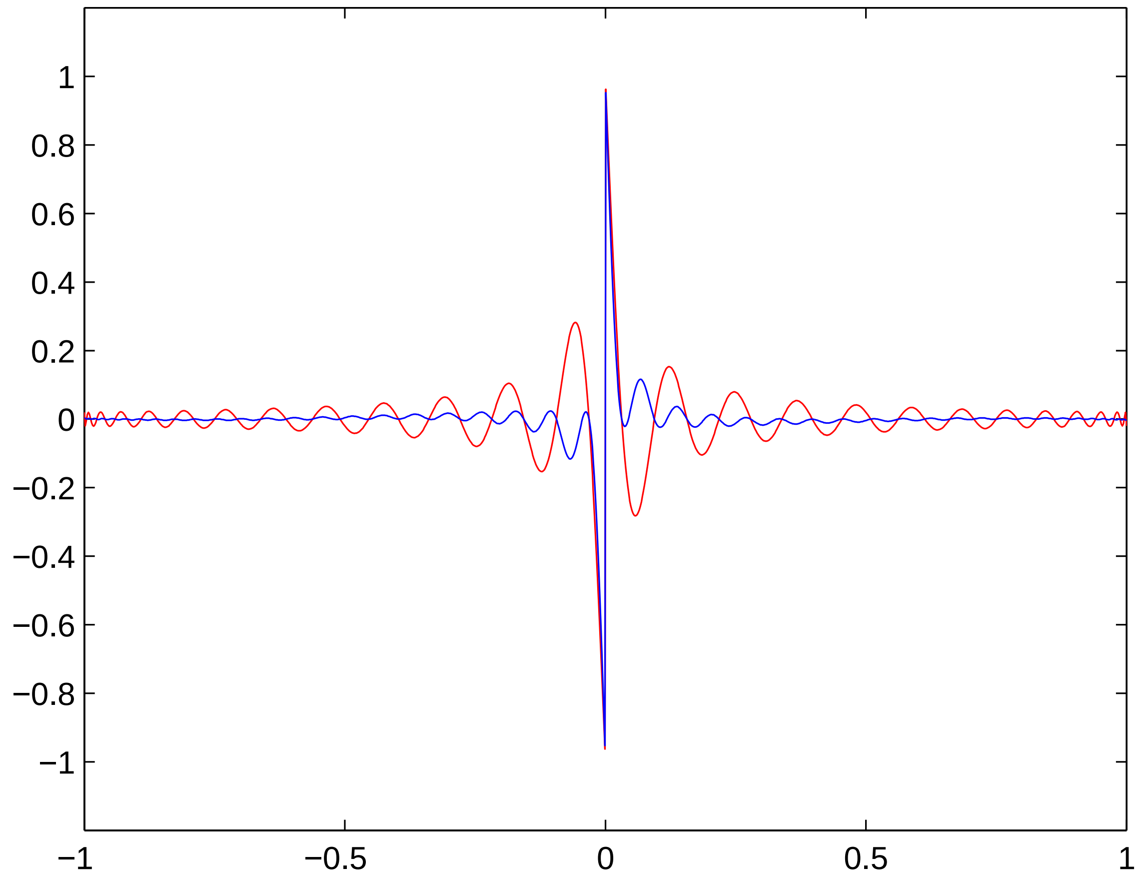
Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.7



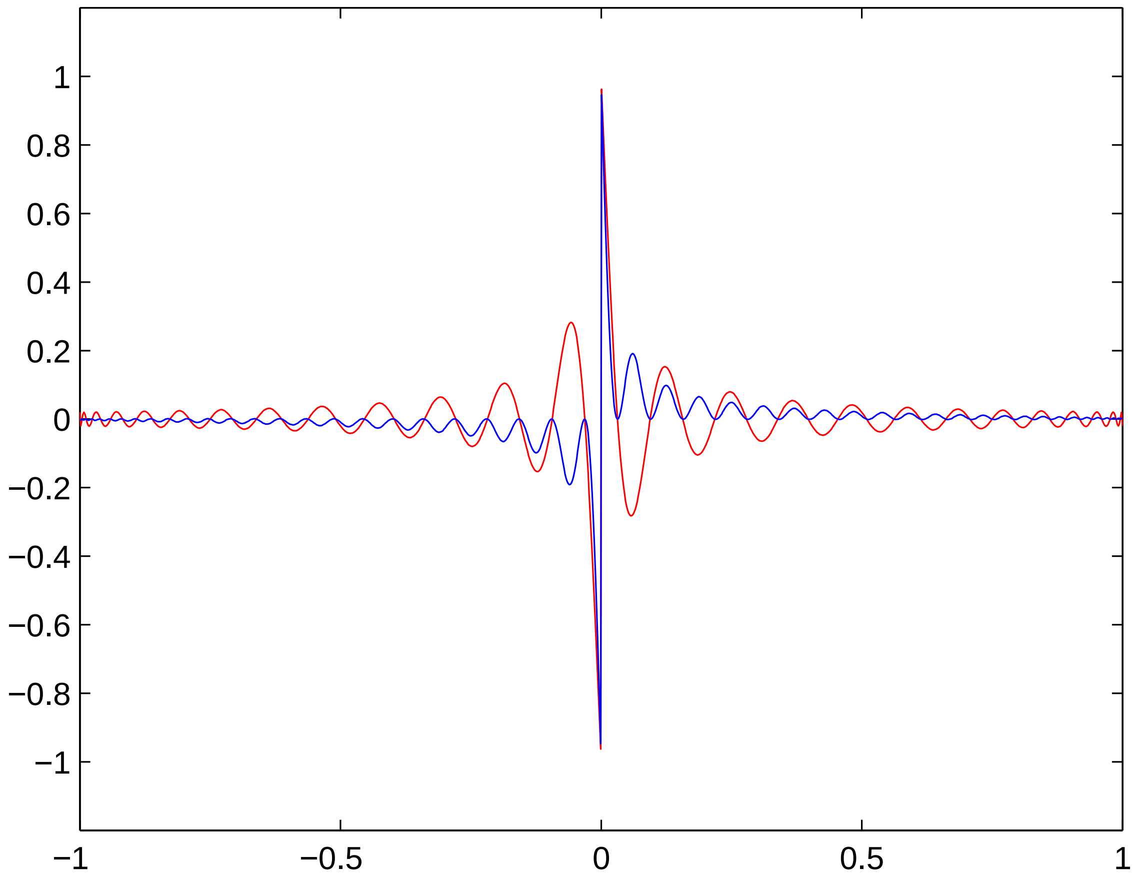
Error curves by Lagrange (in red) and de la V.P. interpolation for $\theta = 0.8$

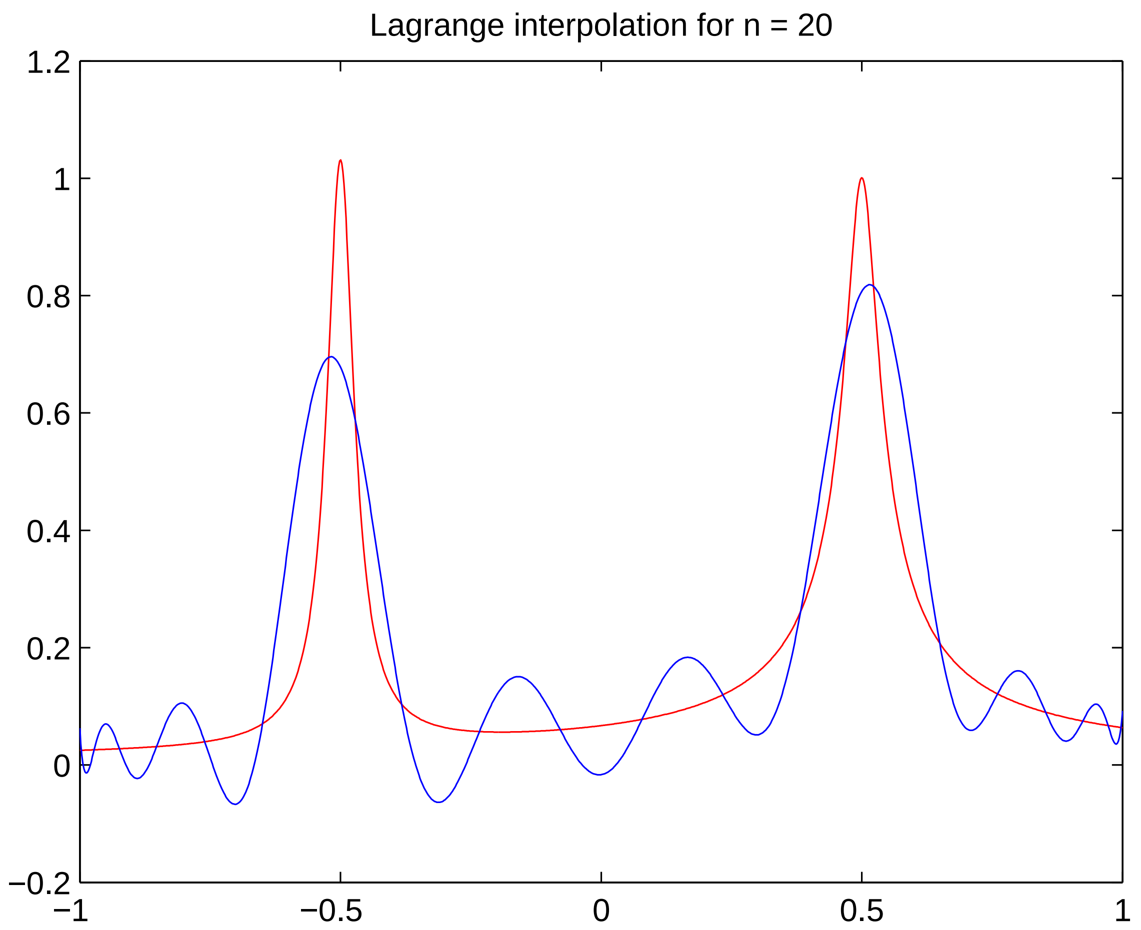


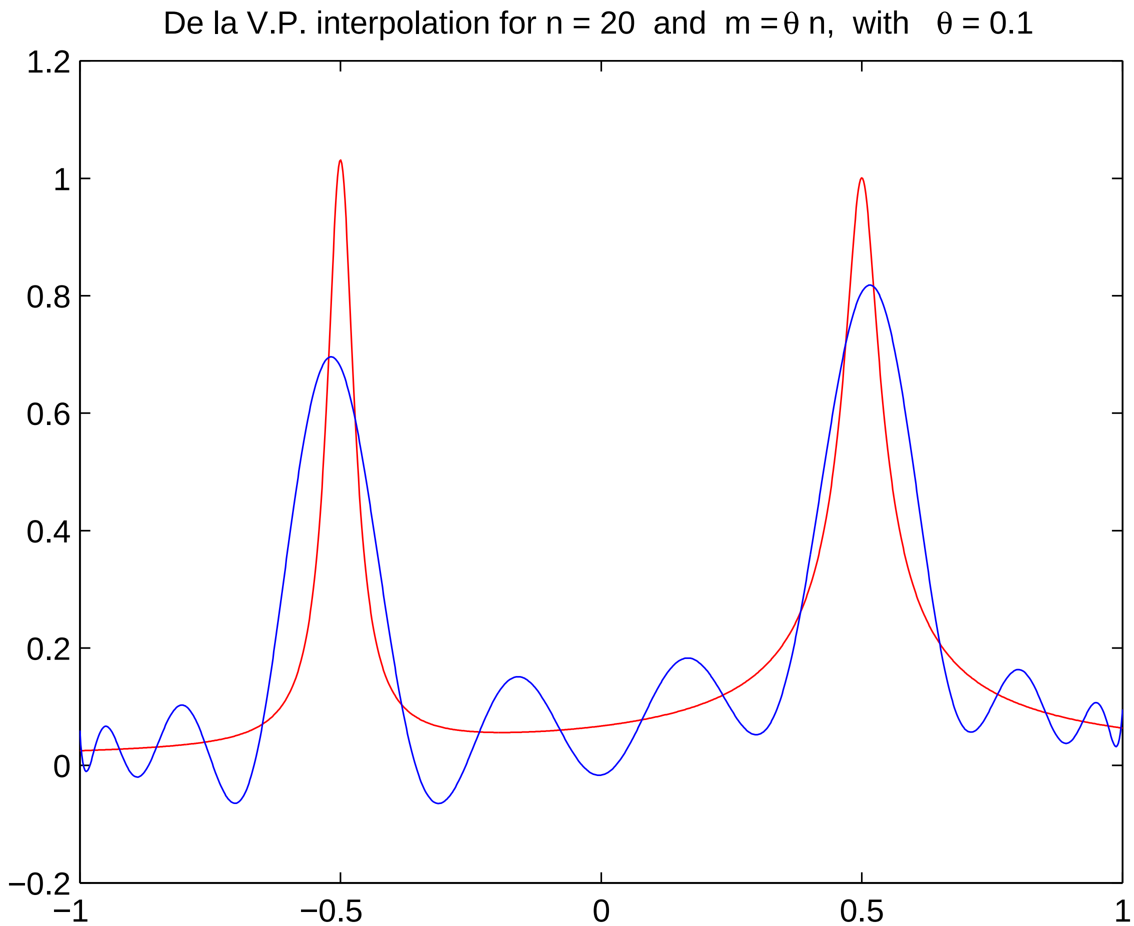
Error curves by Lagrange (in red) and de la V.P. interpolation for $\theta = 0.9$

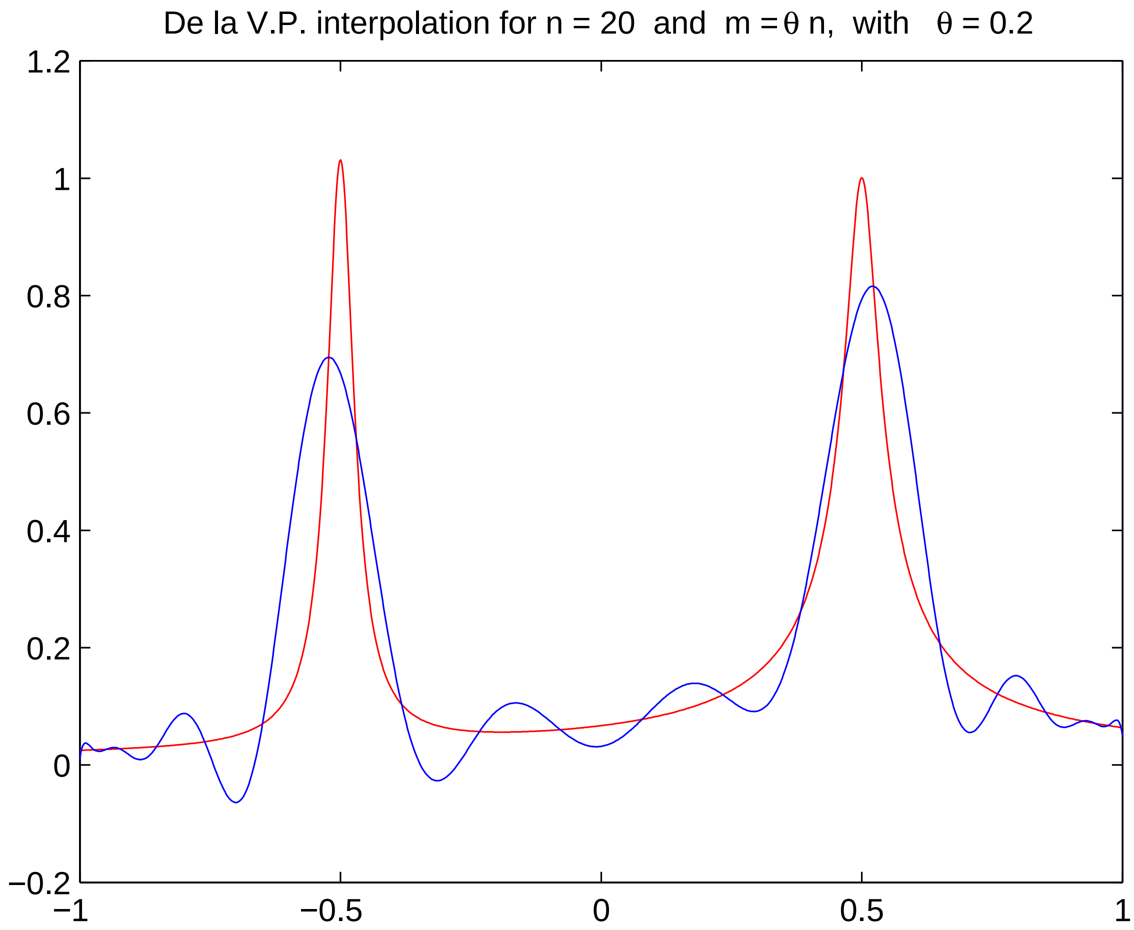


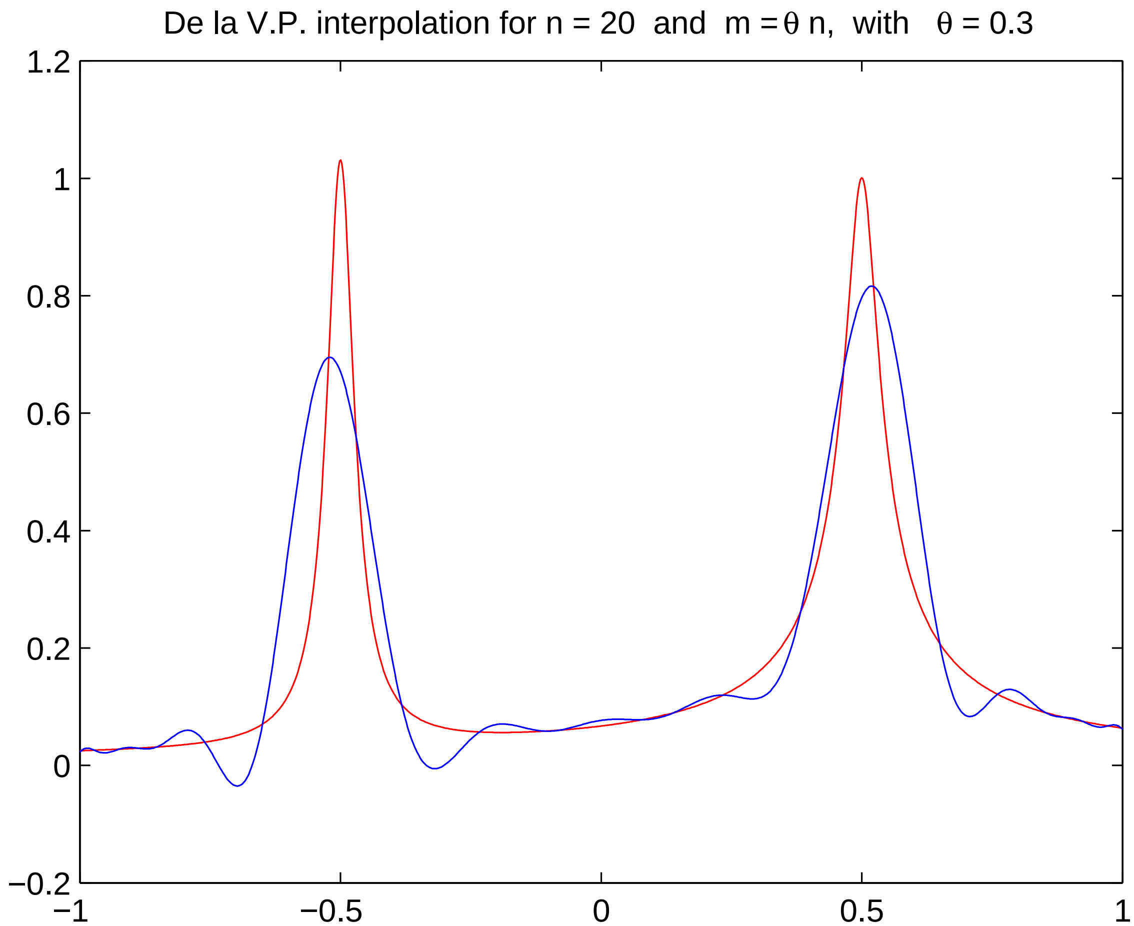
Error curves by Lagrange (in red) and de la V.P. interpolation for $\theta = 1$



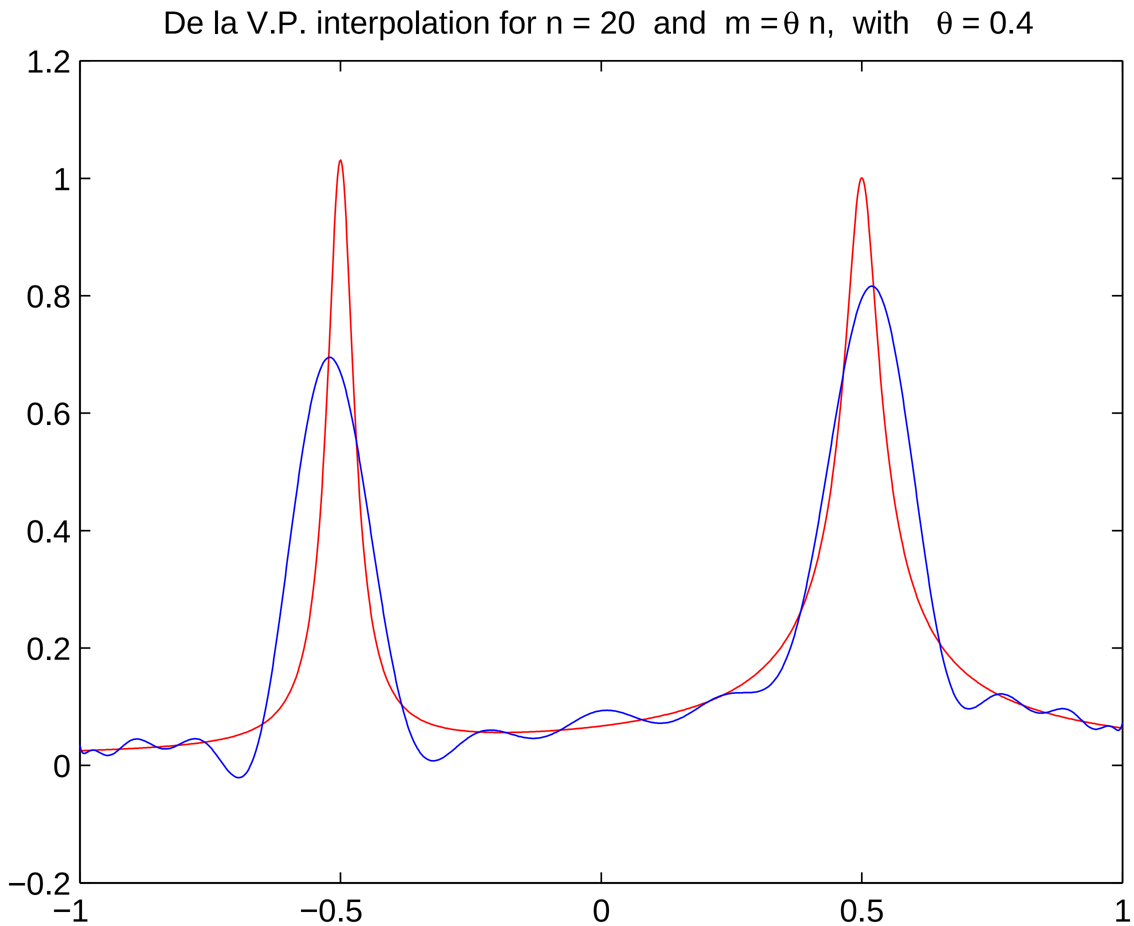

Lagrange interpolation for n = 20
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1


De la V.P. interpolation for n = 20 and m = θ n, with θ = 0.1
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1

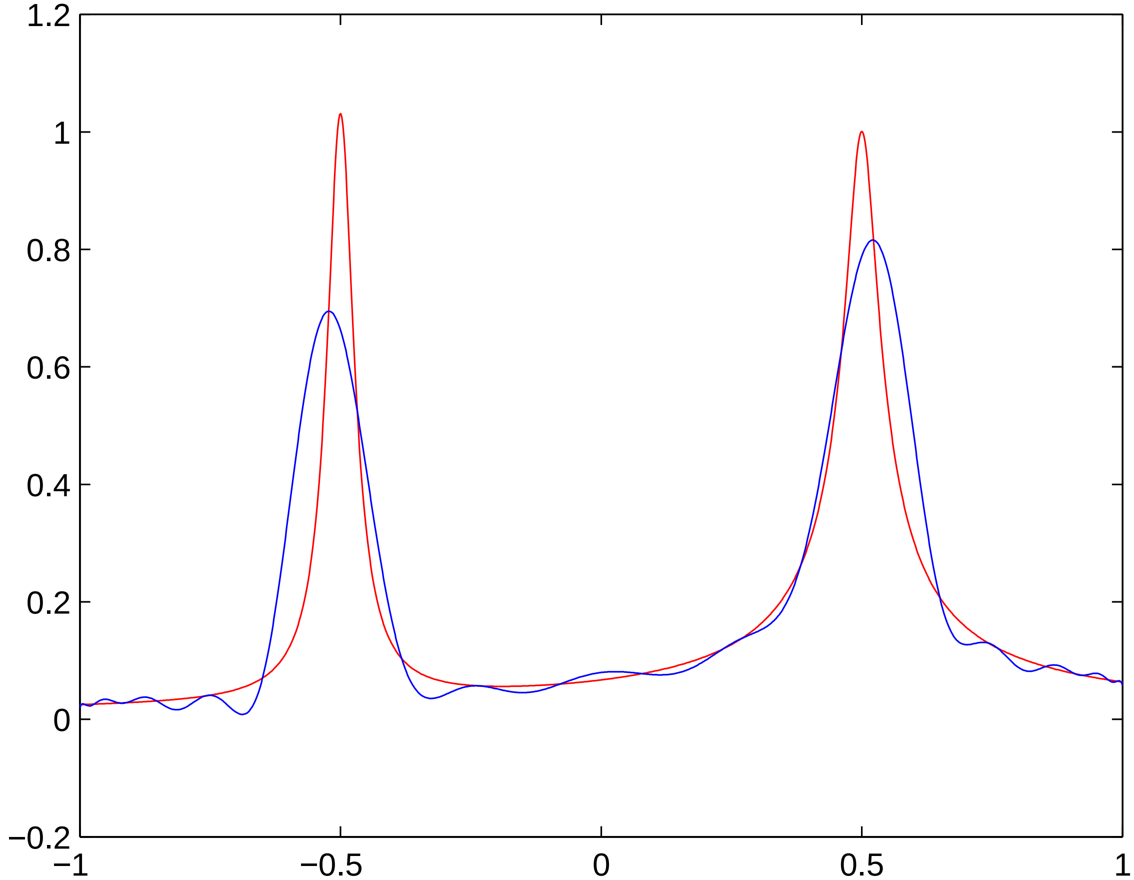

De la V.P. interpolation for n = 20 and m = θ n, with θ = 0.2
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1

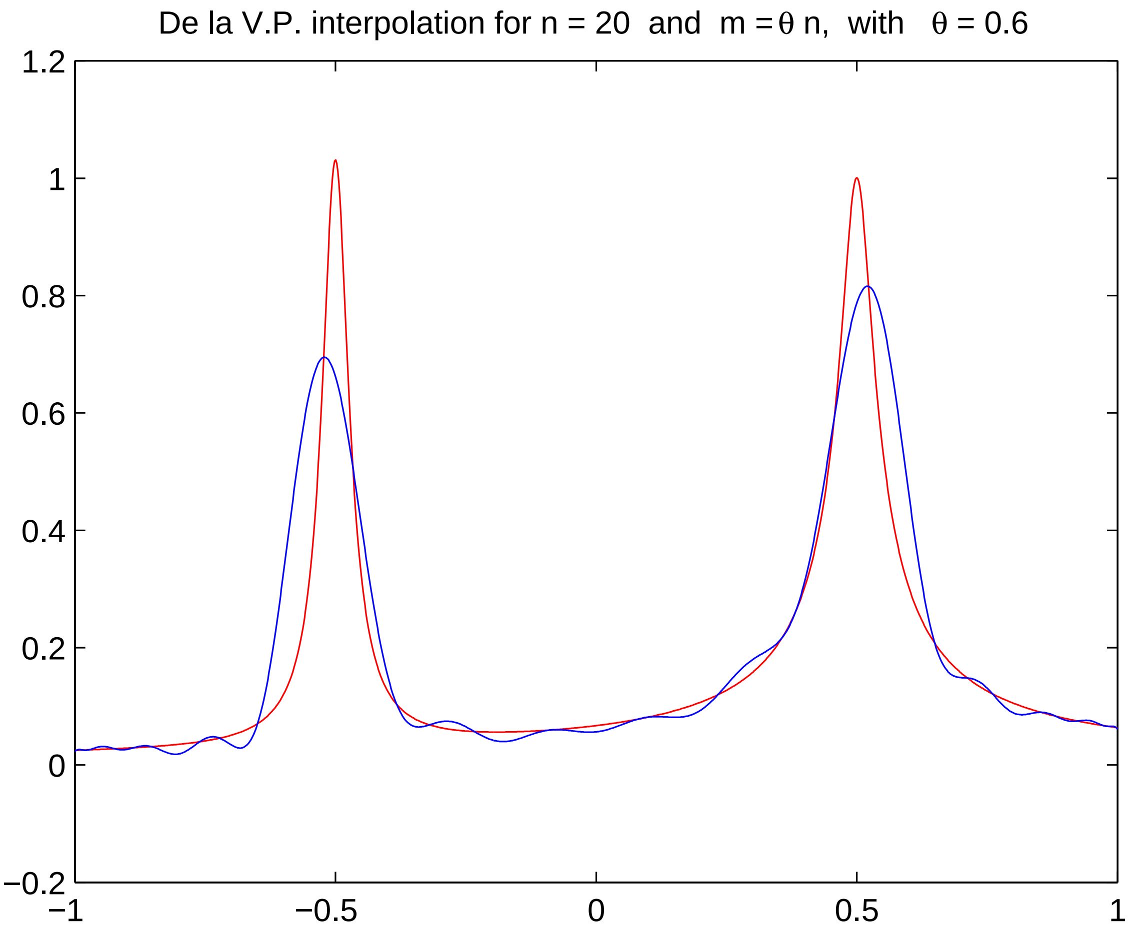


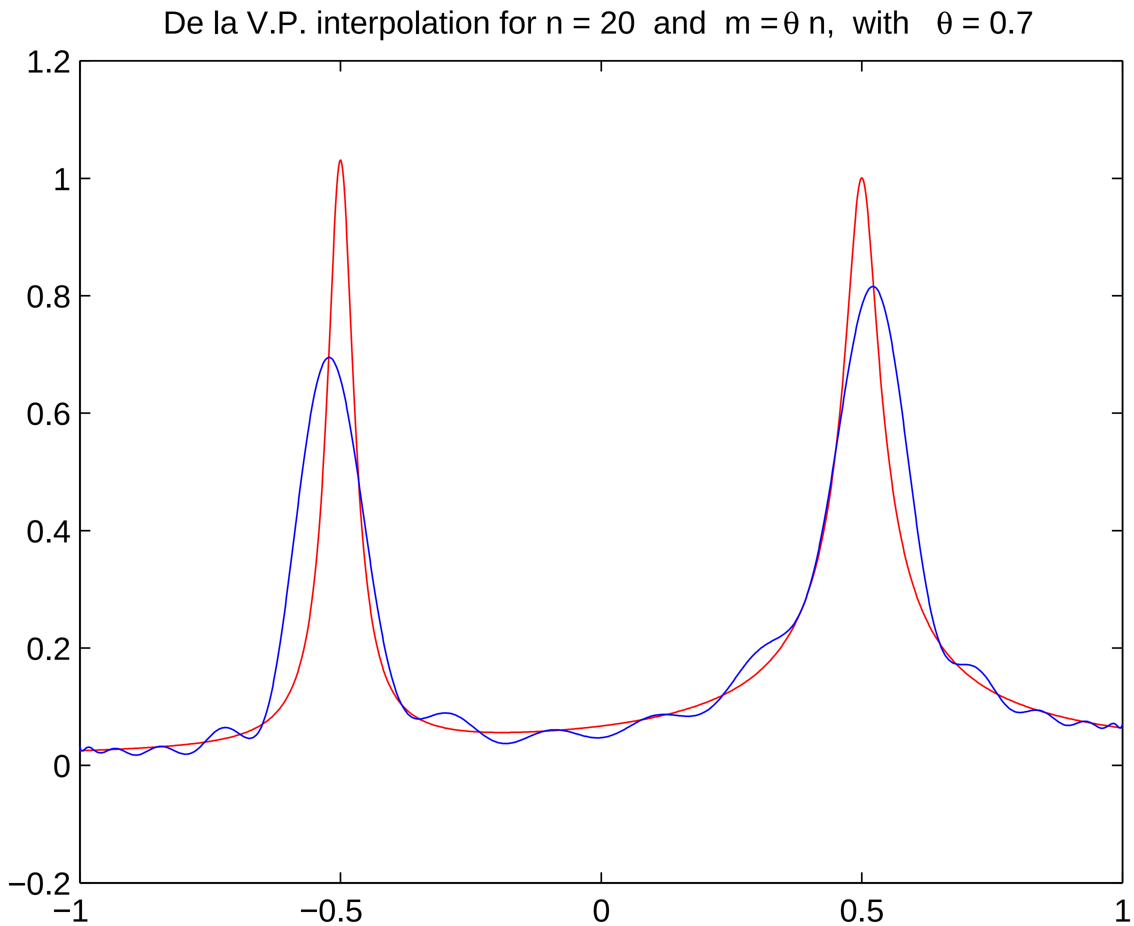
De la V.P. interpolation for n = 20 and m = θ n, with θ = 0.3
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1

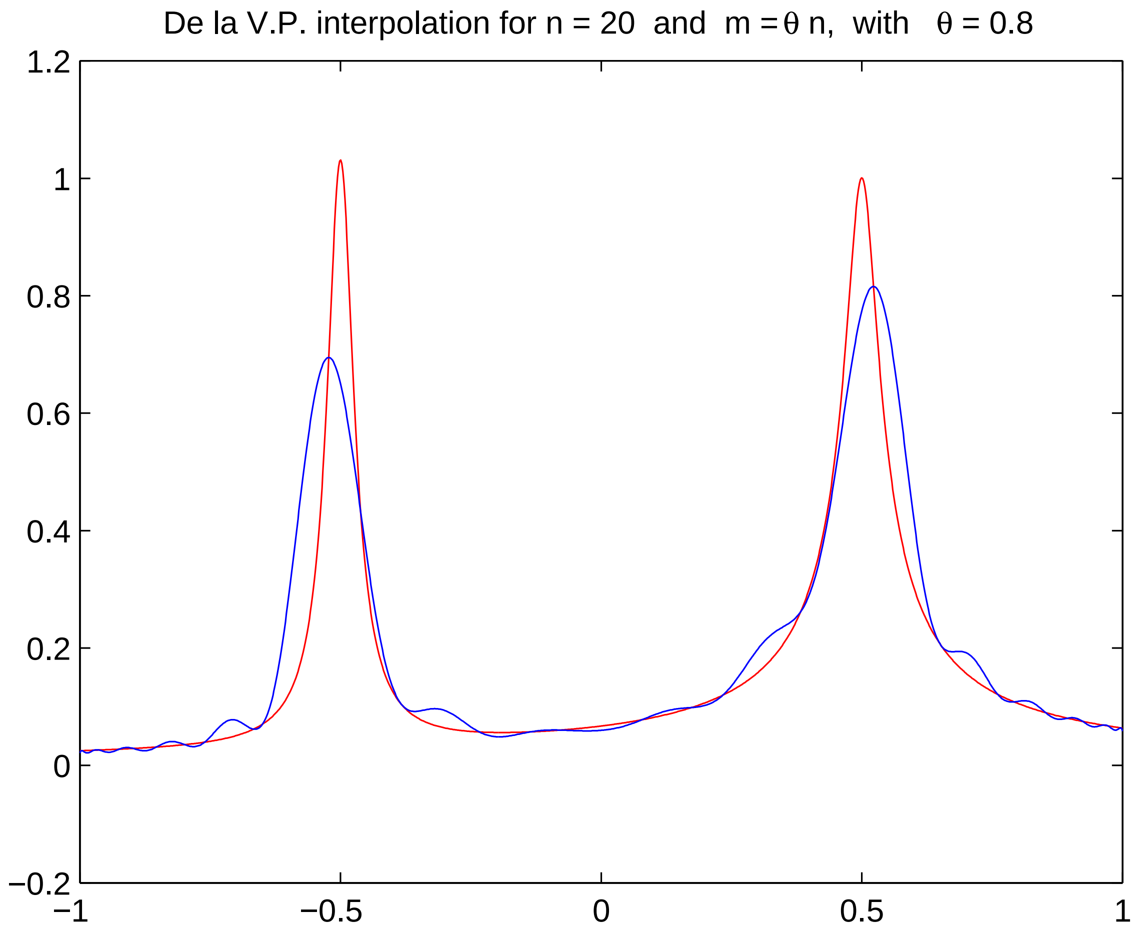

De la V.P. interpolation for n = 20 and m = θ n, with θ = 0.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1

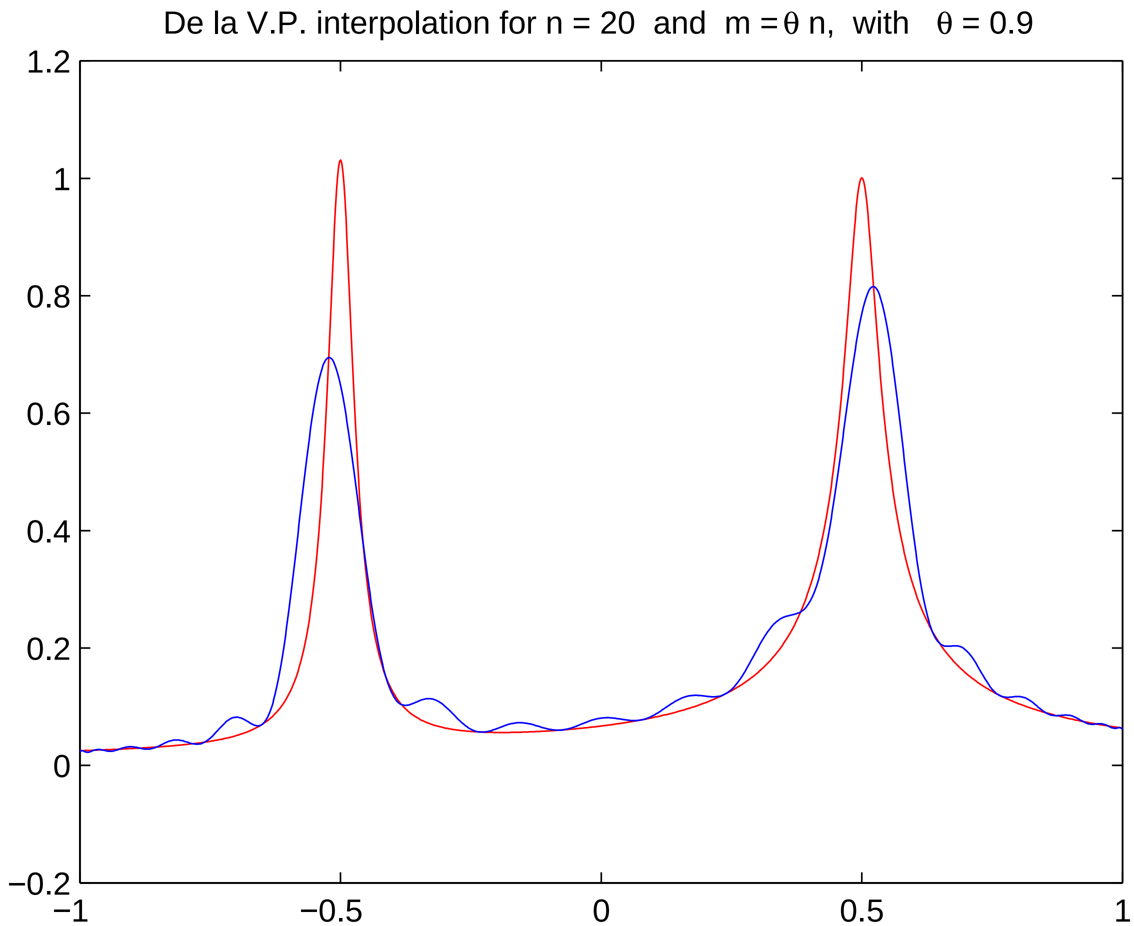
De la V.P. interpolation for n = 20 and m = θ n, with θ = 0.5

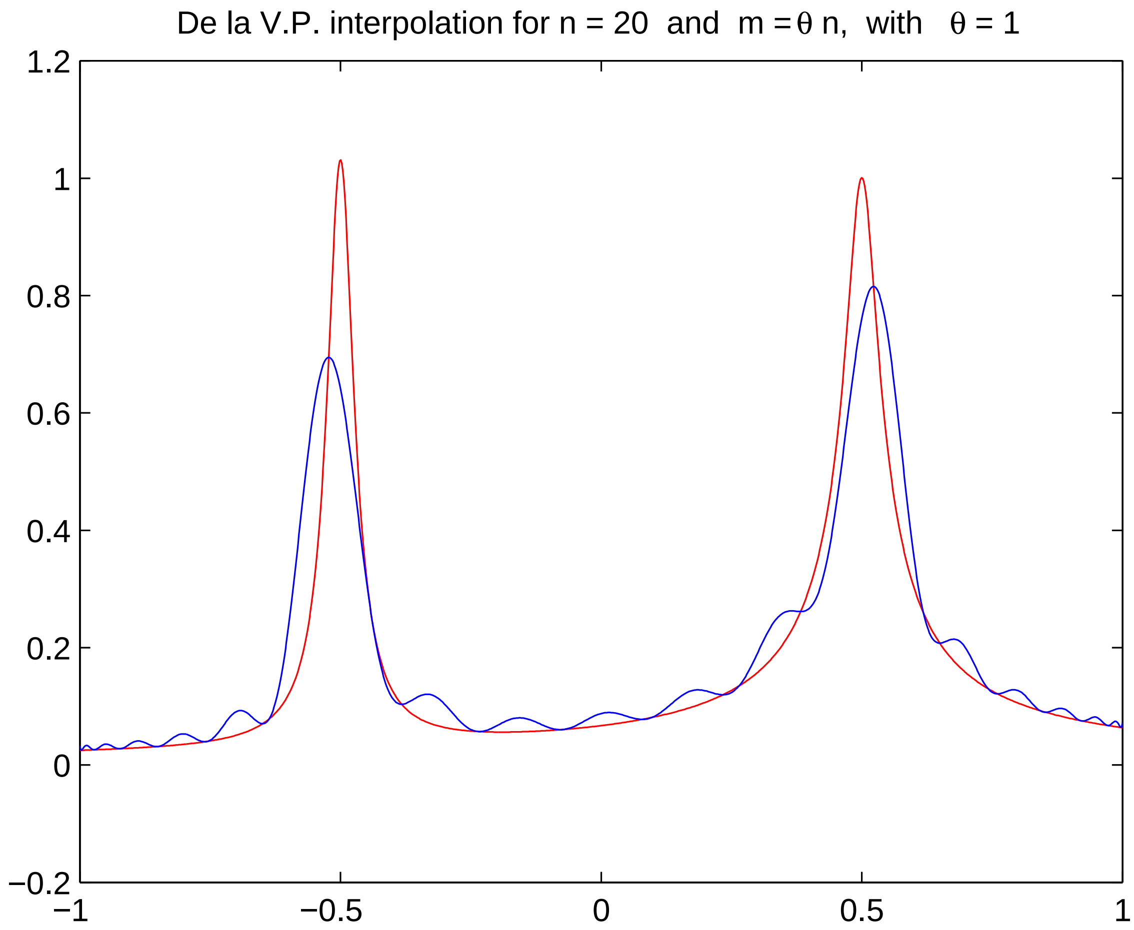



De la V.P. interpolation for n = 20 and m = θ n, with θ = 0.6
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1

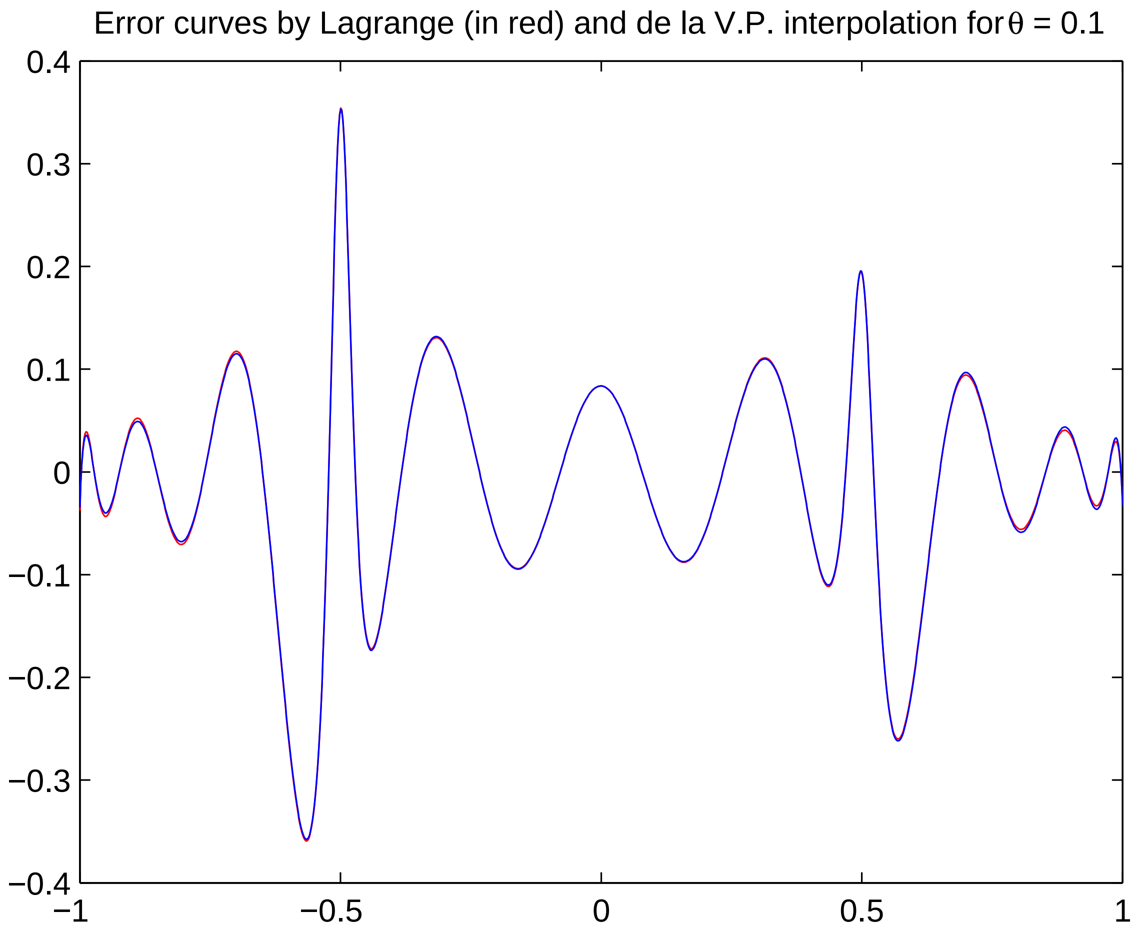

De la V.P. interpolation for n = 20 and m = θ n, with θ = 0.7
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1


De la V.P. interpolation for n = 20 and m = θ n, with θ = 0.8
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1


De la V.P. interpolation for n = 20 and m = θ n, with θ = 0.9
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1

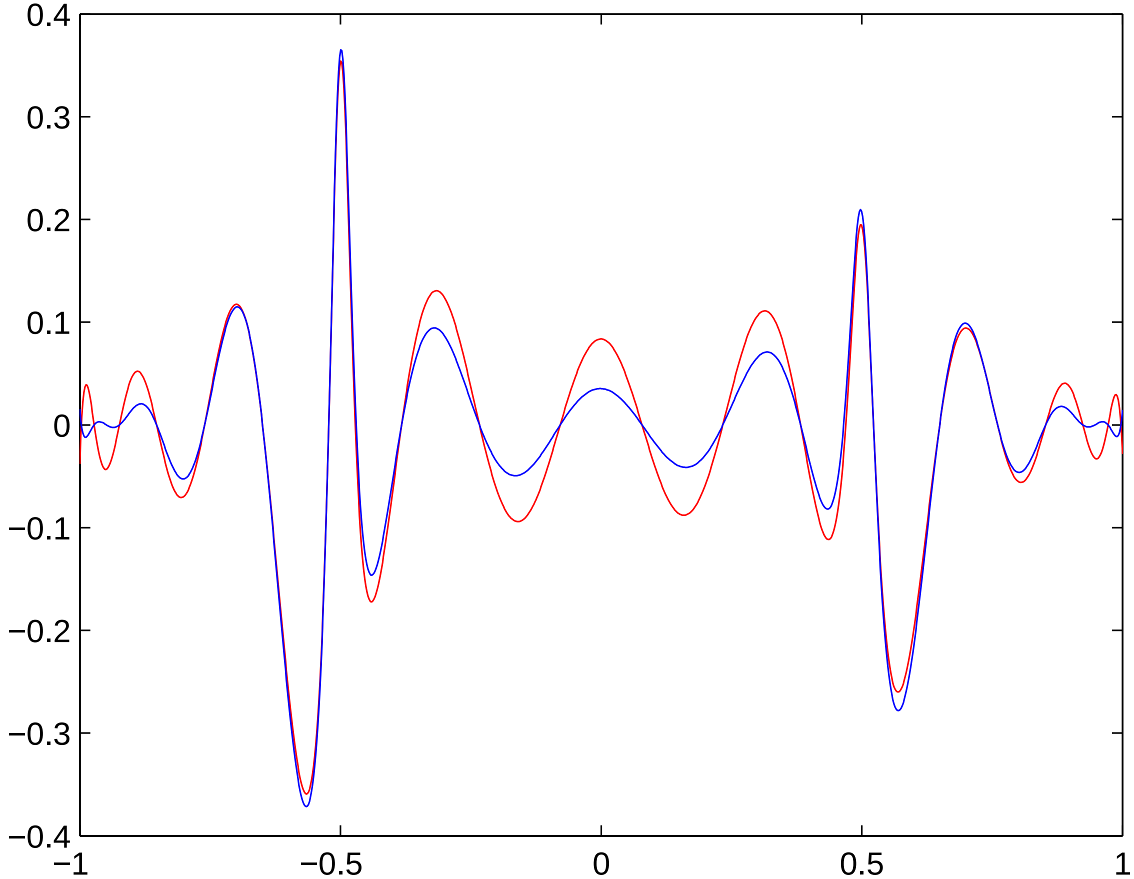


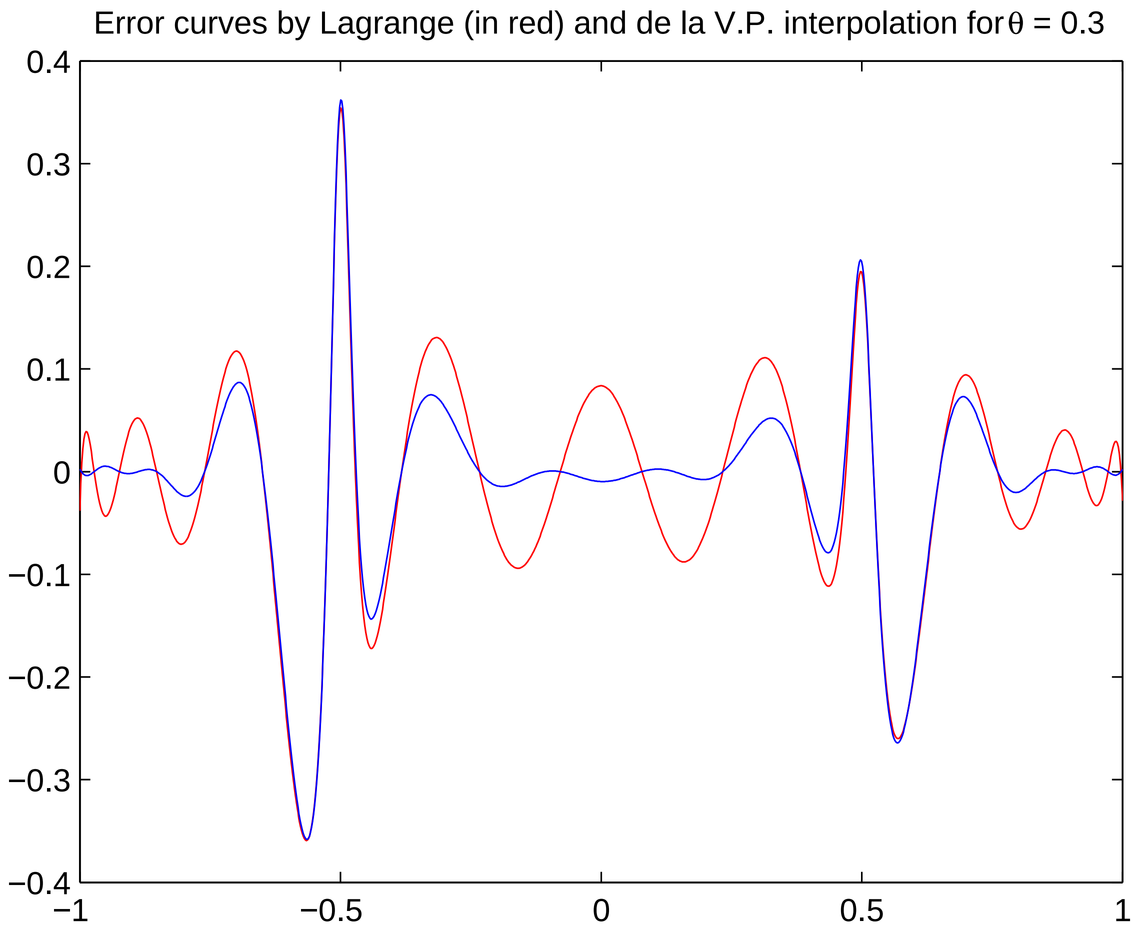
De la V.P. interpolation for n = 20 and m = θ n, with θ = 1
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1



Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.1
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−1
−0.5
0
0.5
1

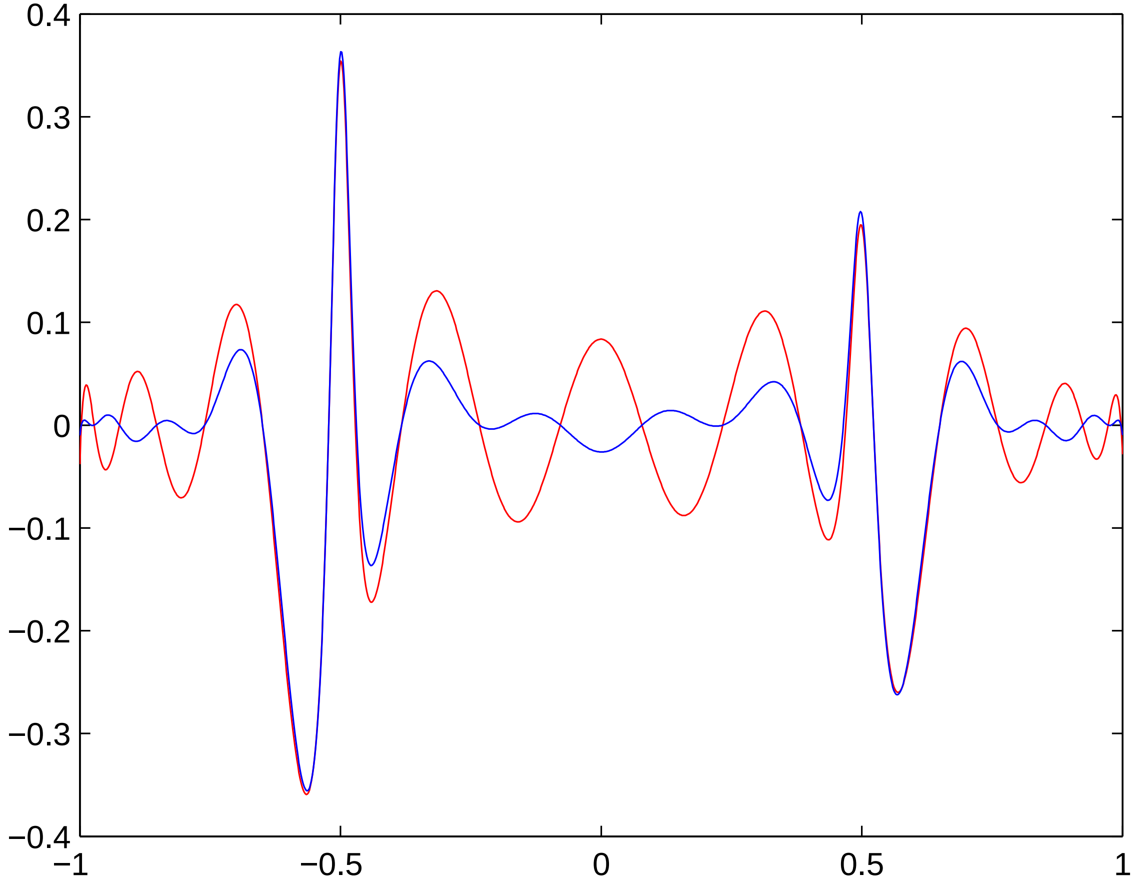
Error curves by Lagrange (in red) and de la V.P. interpolation for $\theta = 0.2$

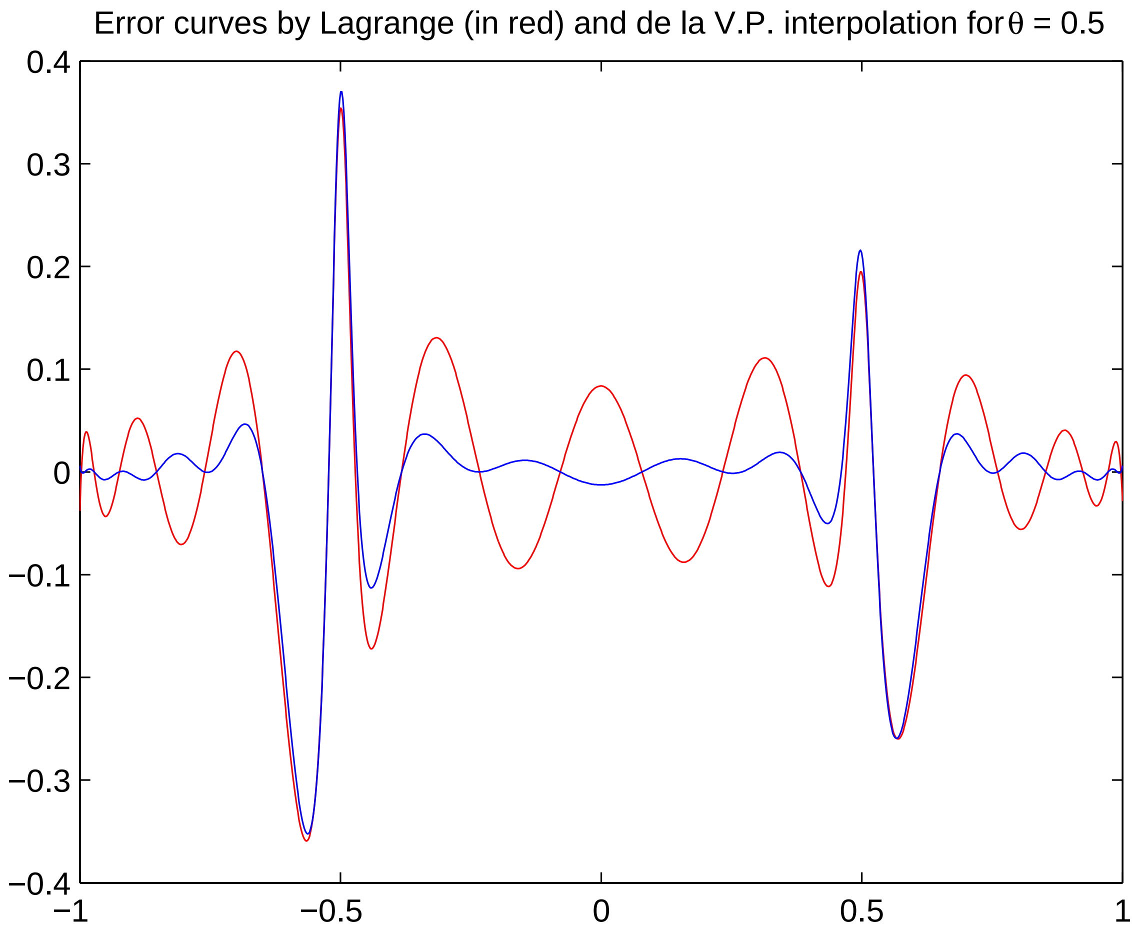


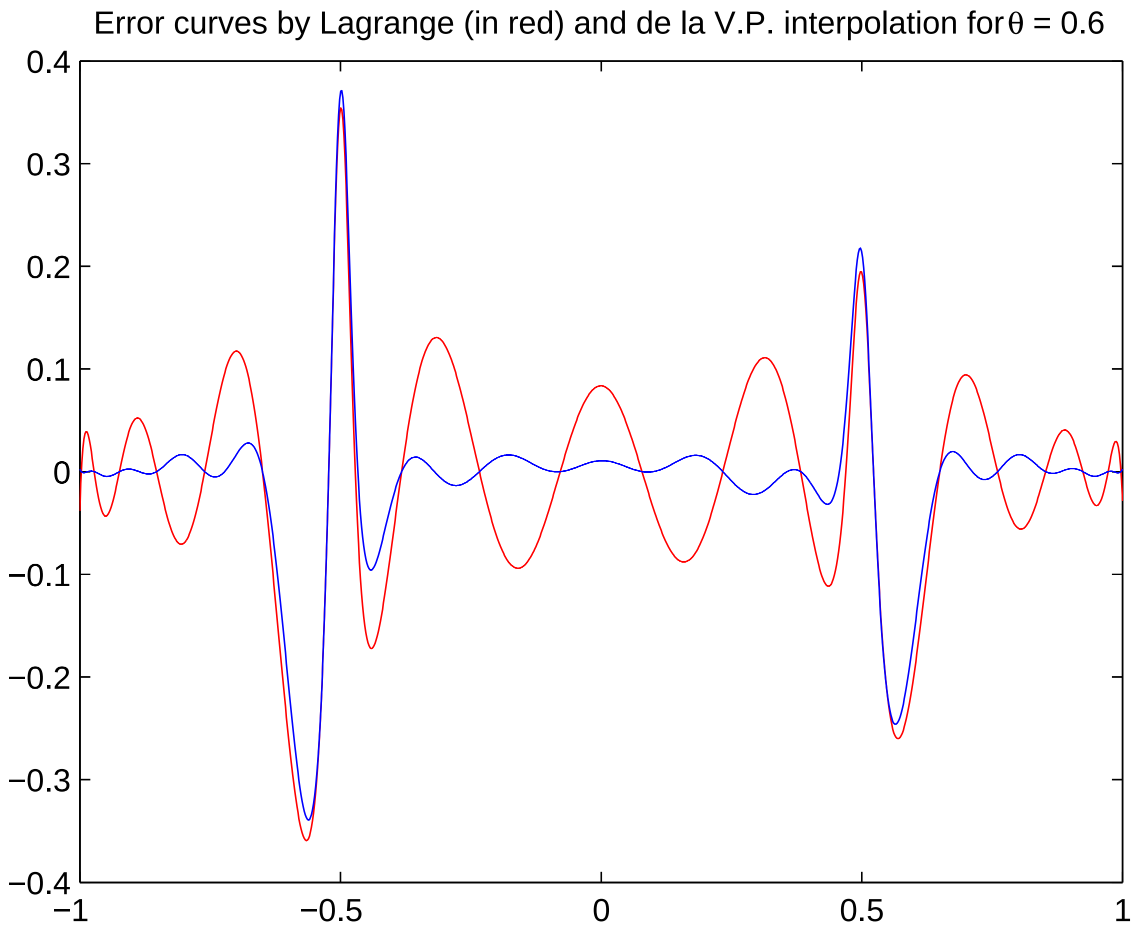


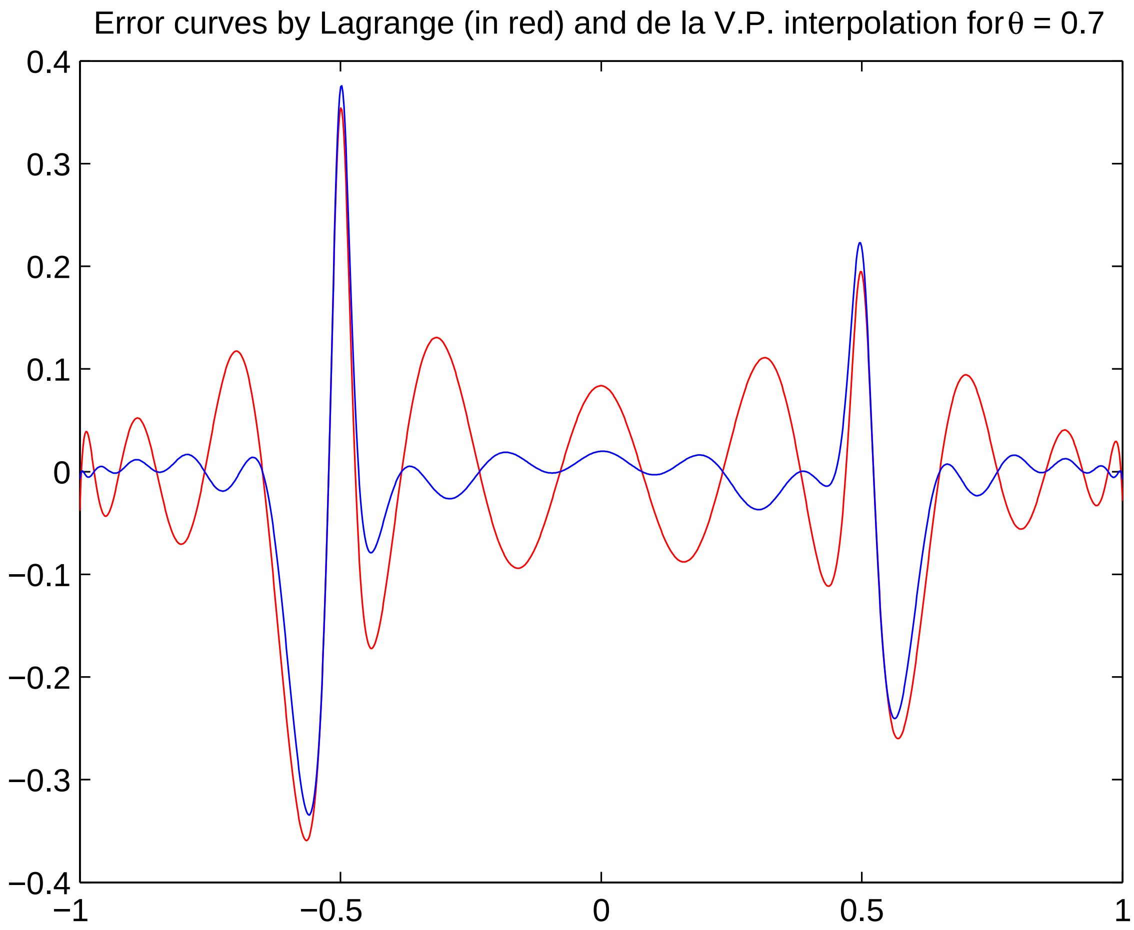
Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.3
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−1
−0.5
0
0.5
1

Error curves by Lagrange (in red) and de la V.P. interpolation for $\theta = 0.4$

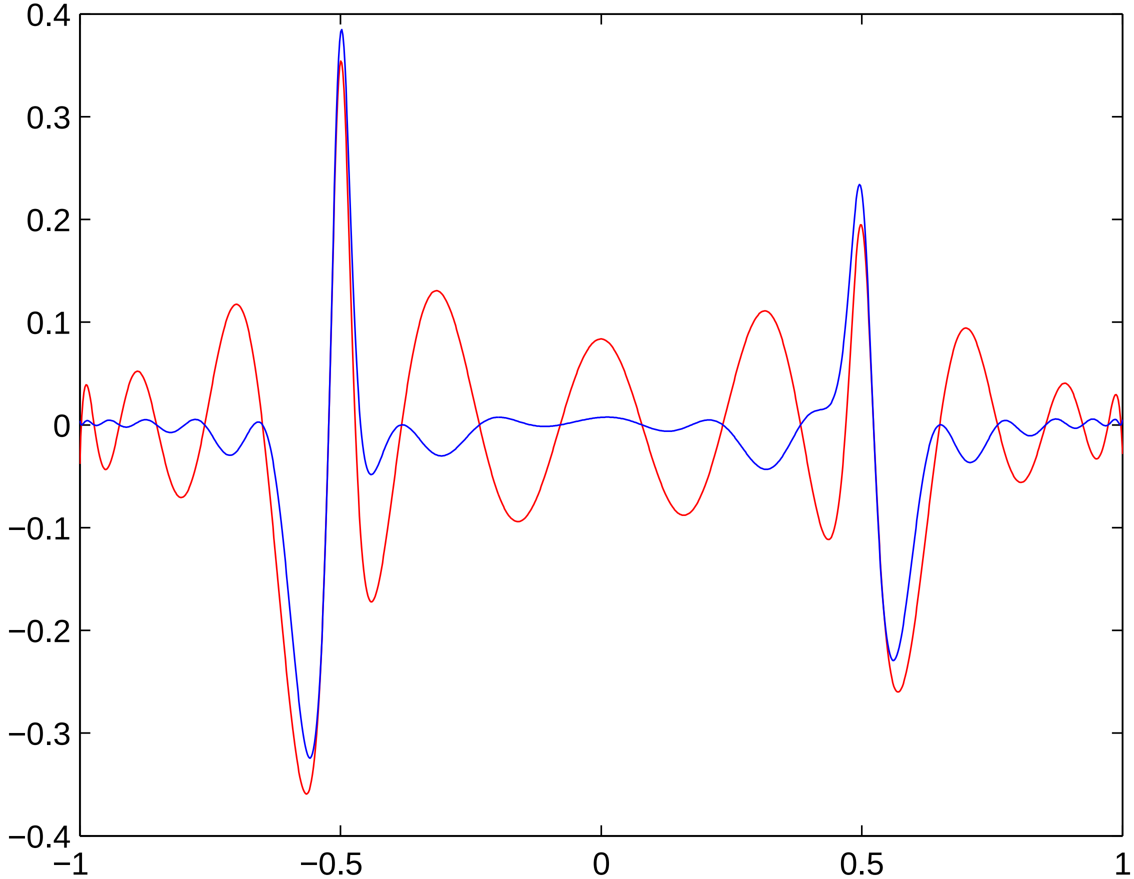


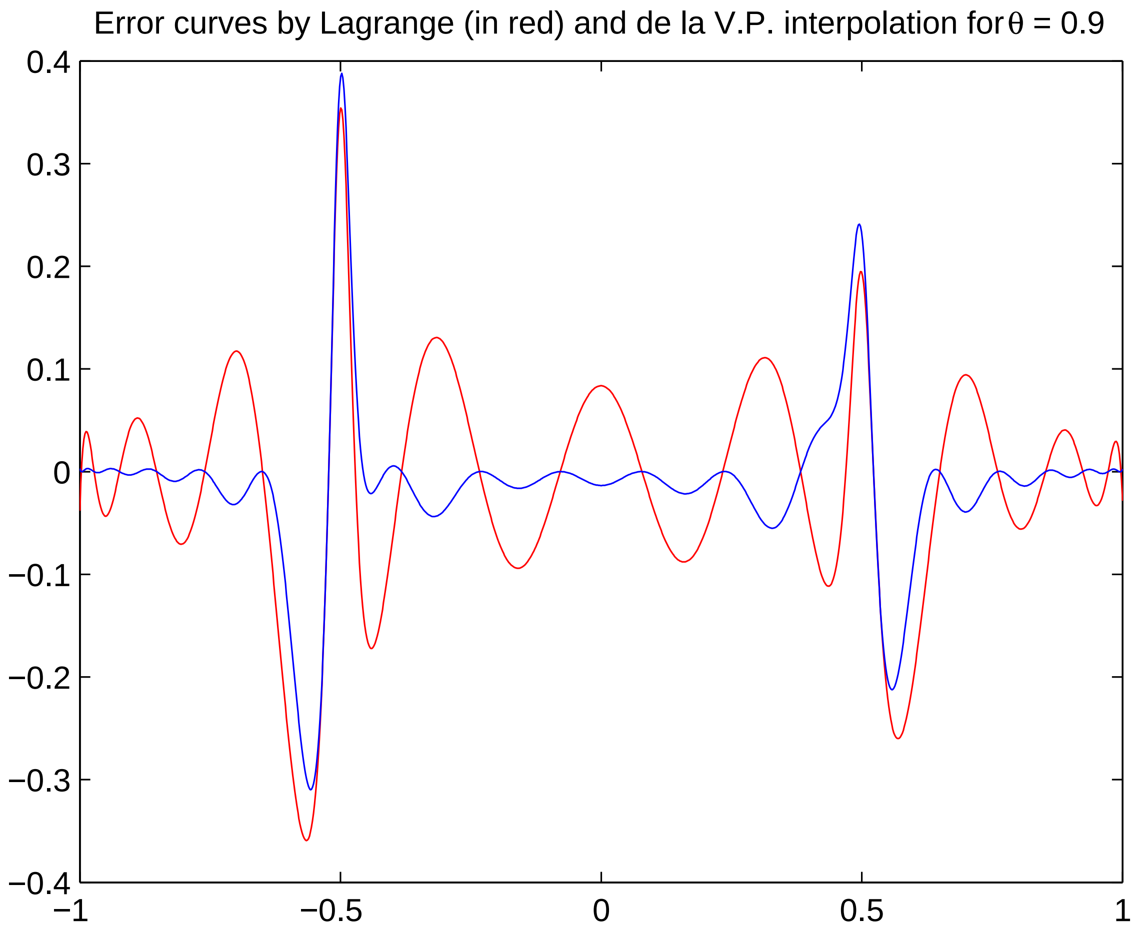

Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−1
−0.5
0
0.5
1


Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.6
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−1
−0.5
0
0.5
1

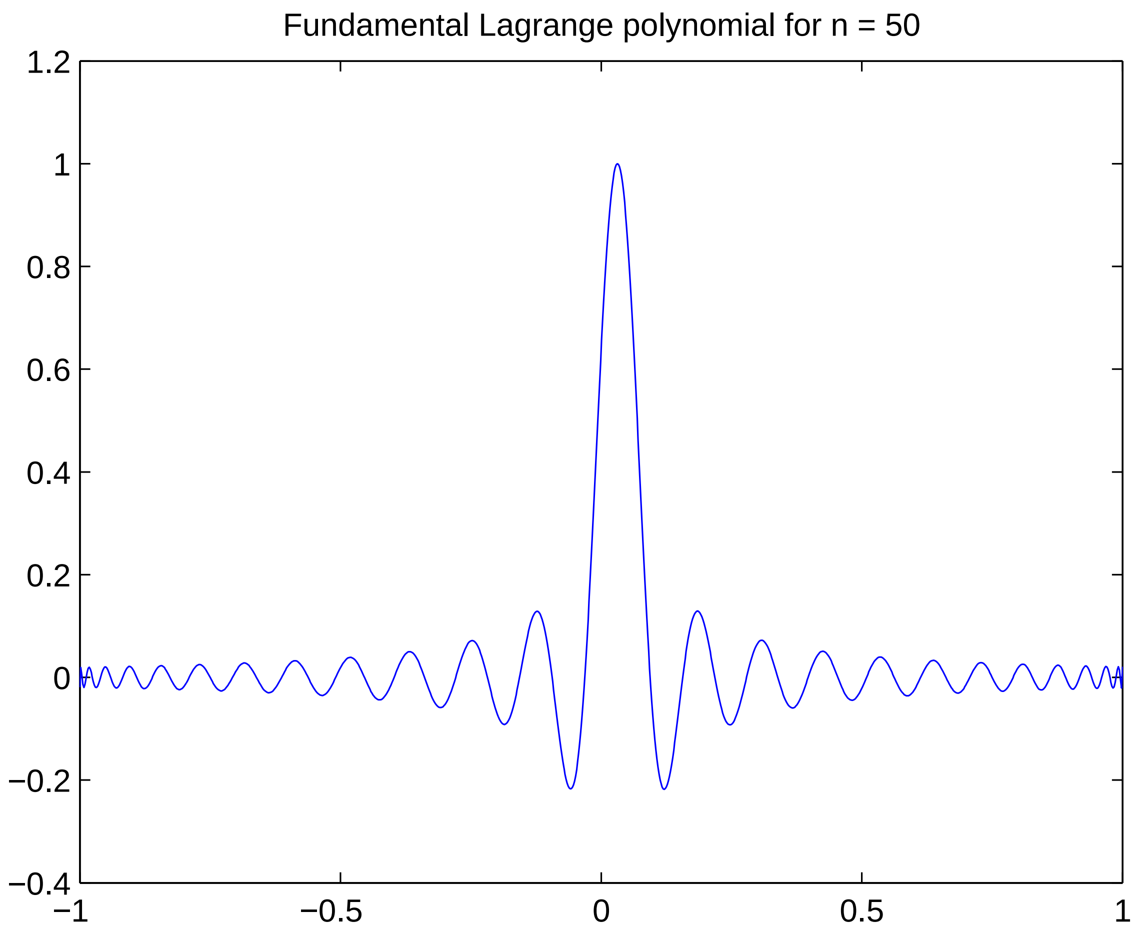

Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.7
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−1
−0.5
0
0.5
1

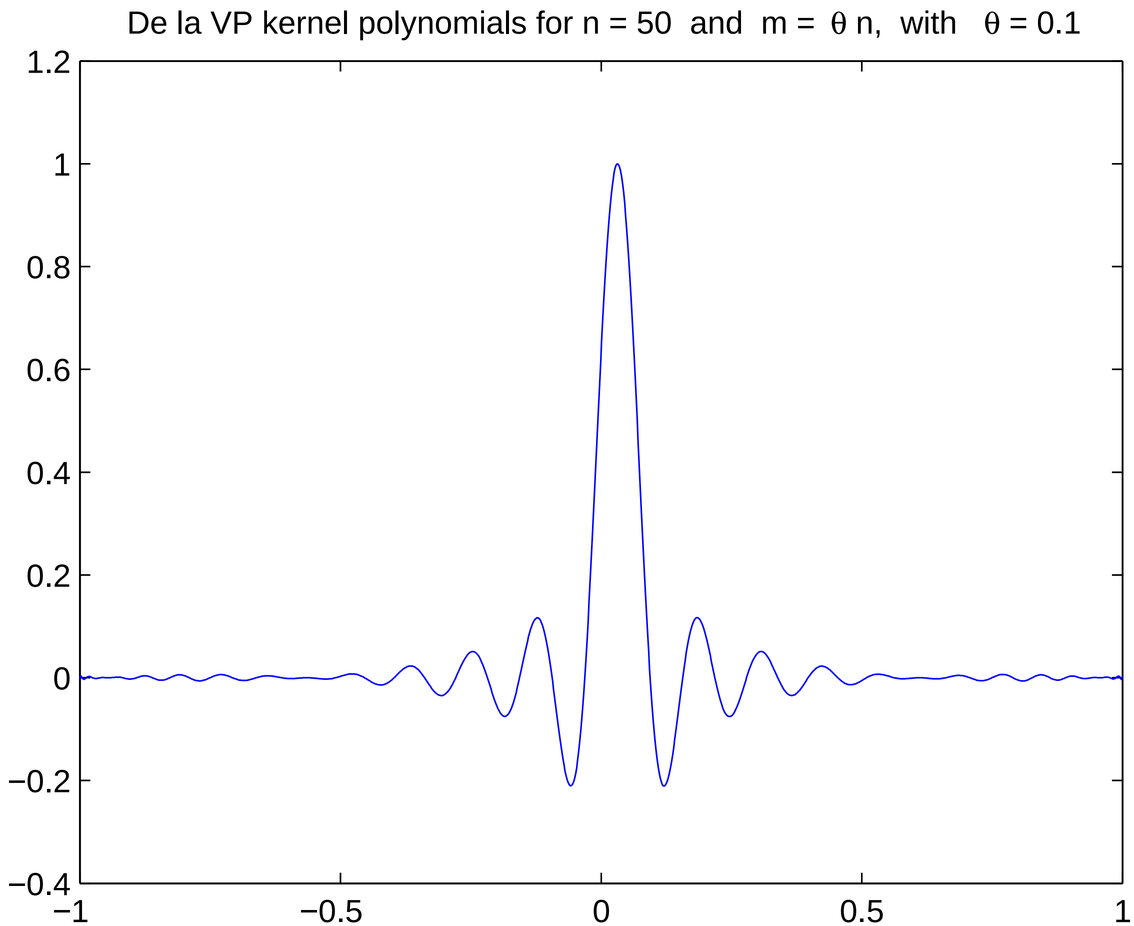
Error curves by Lagrange (in red) and de la V.P. interpolation for $\theta = 0.8$

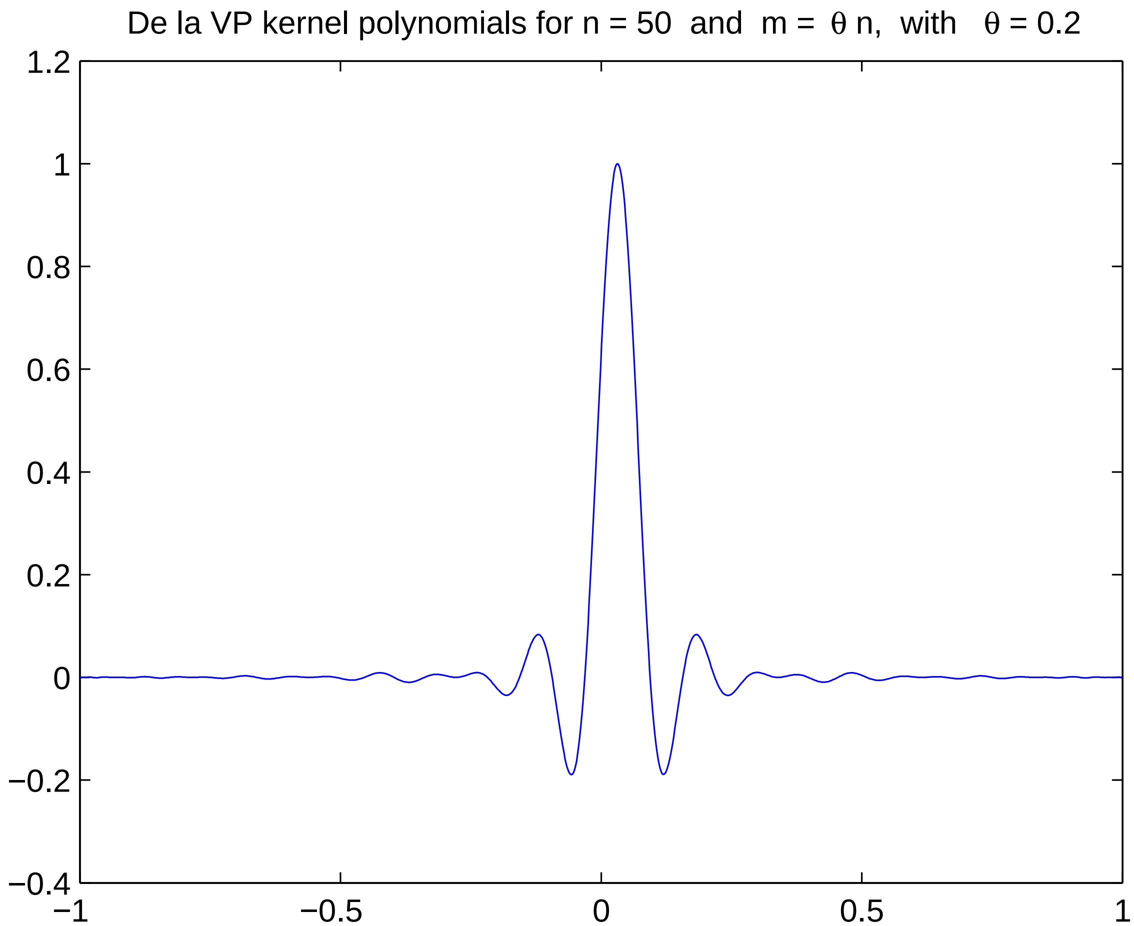


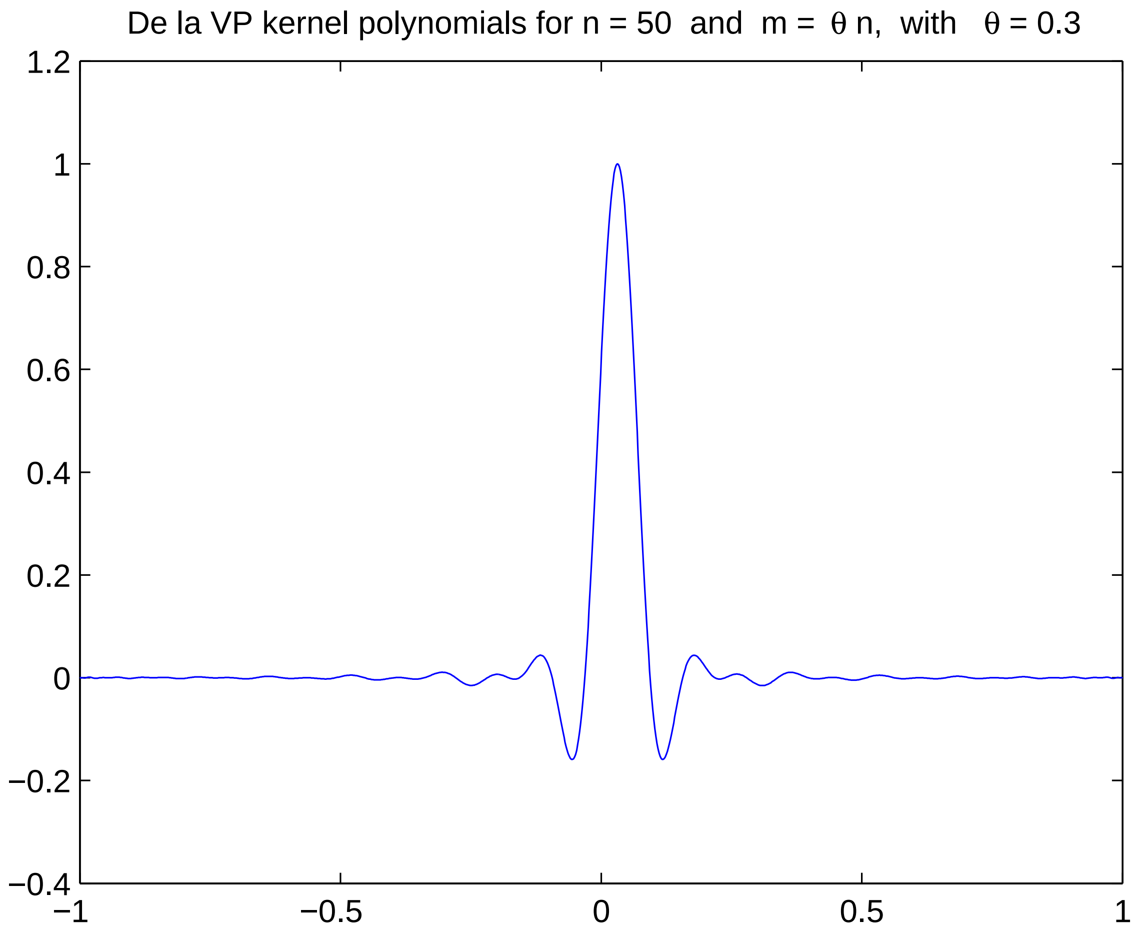

Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.9
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−1
−0.5
0
0.5
1

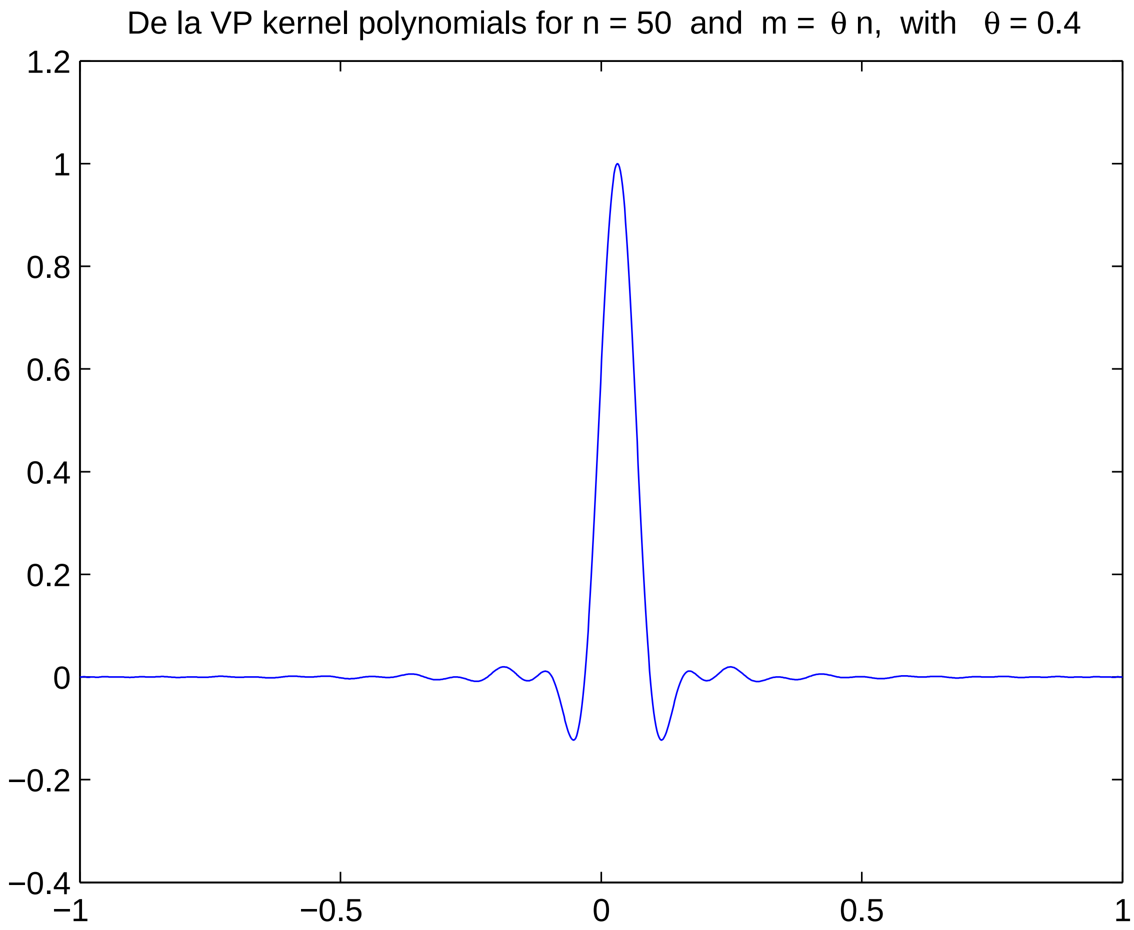
Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 1
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−1
−0.5
0
0.5
1


Fundamental Lagrange polynomial for n = 50
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−1
−0.5
0
0.5
1

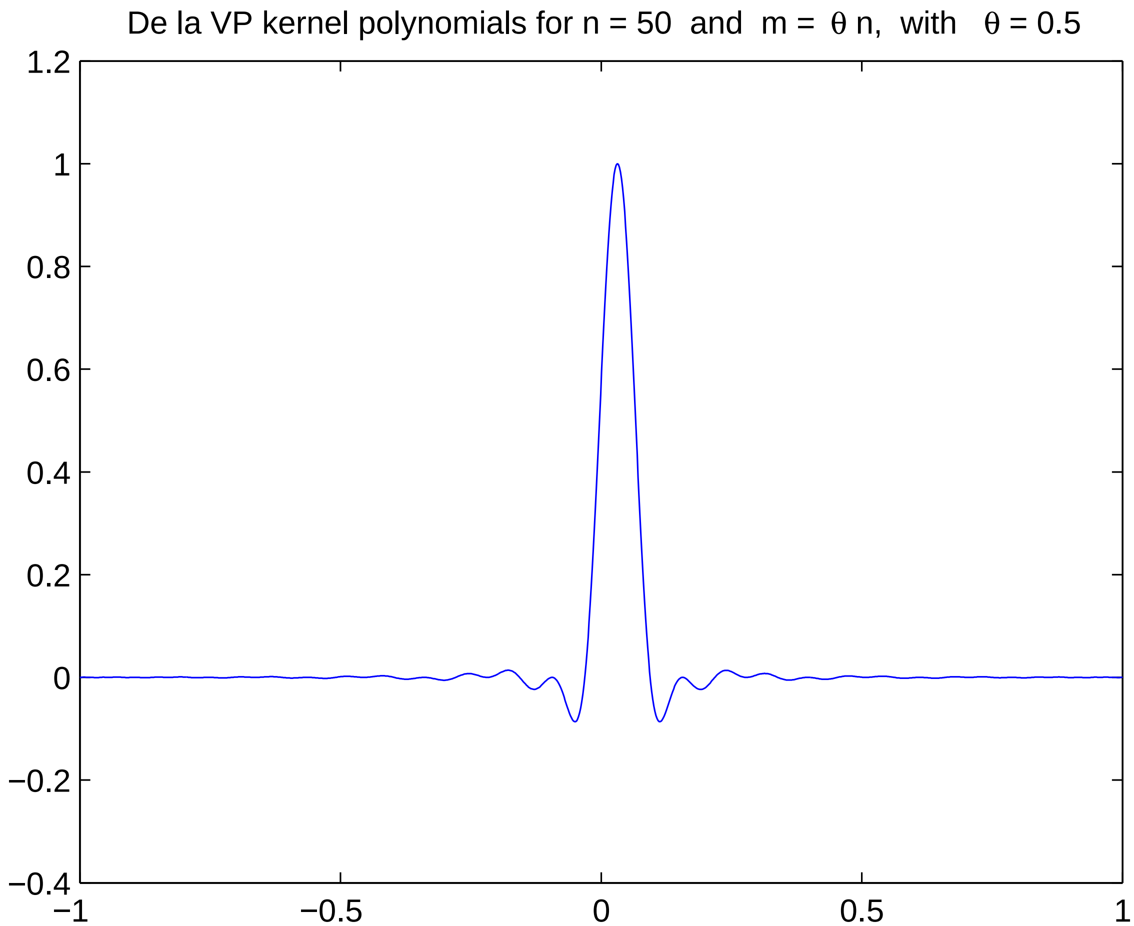

De la VP kernel polynomials for n = 50 and m = θ n, with θ = 0.1
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−1
−0.5
0
0.5
1

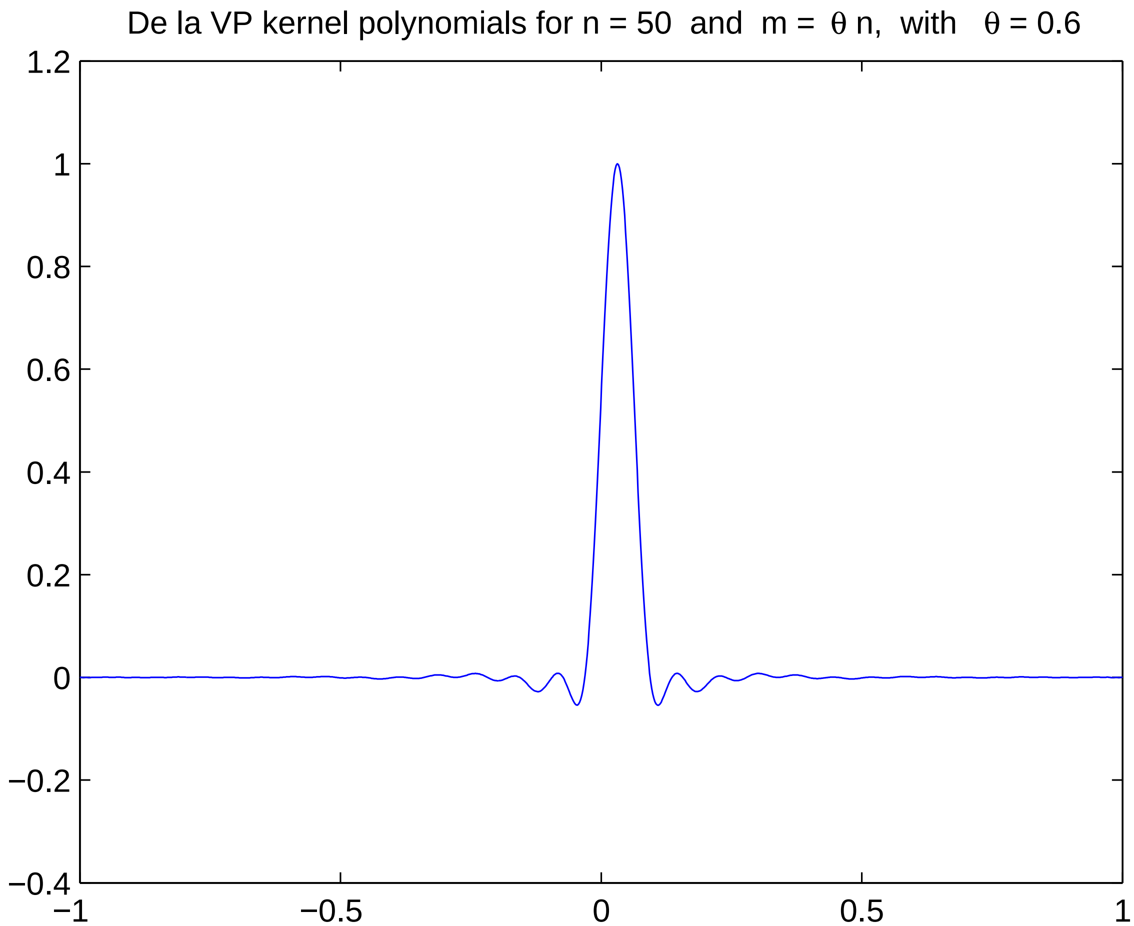

De la VP kernel polynomials for n = 50 and m = θ n, with θ = 0.2
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−1
−0.5
0
0.5
1

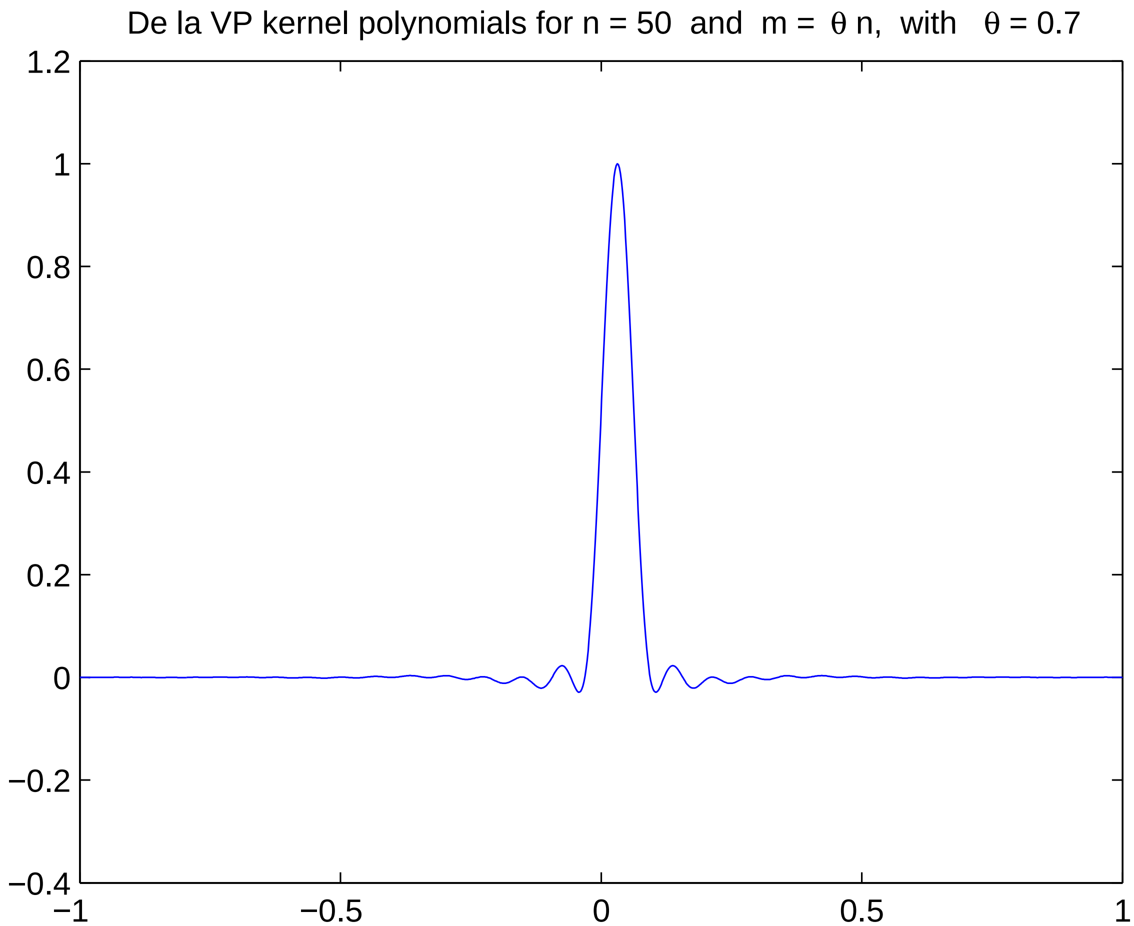

De la VP kernel polynomials for n = 50 and m = θ n, with θ = 0.3
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−1
−0.5
0
0.5
1

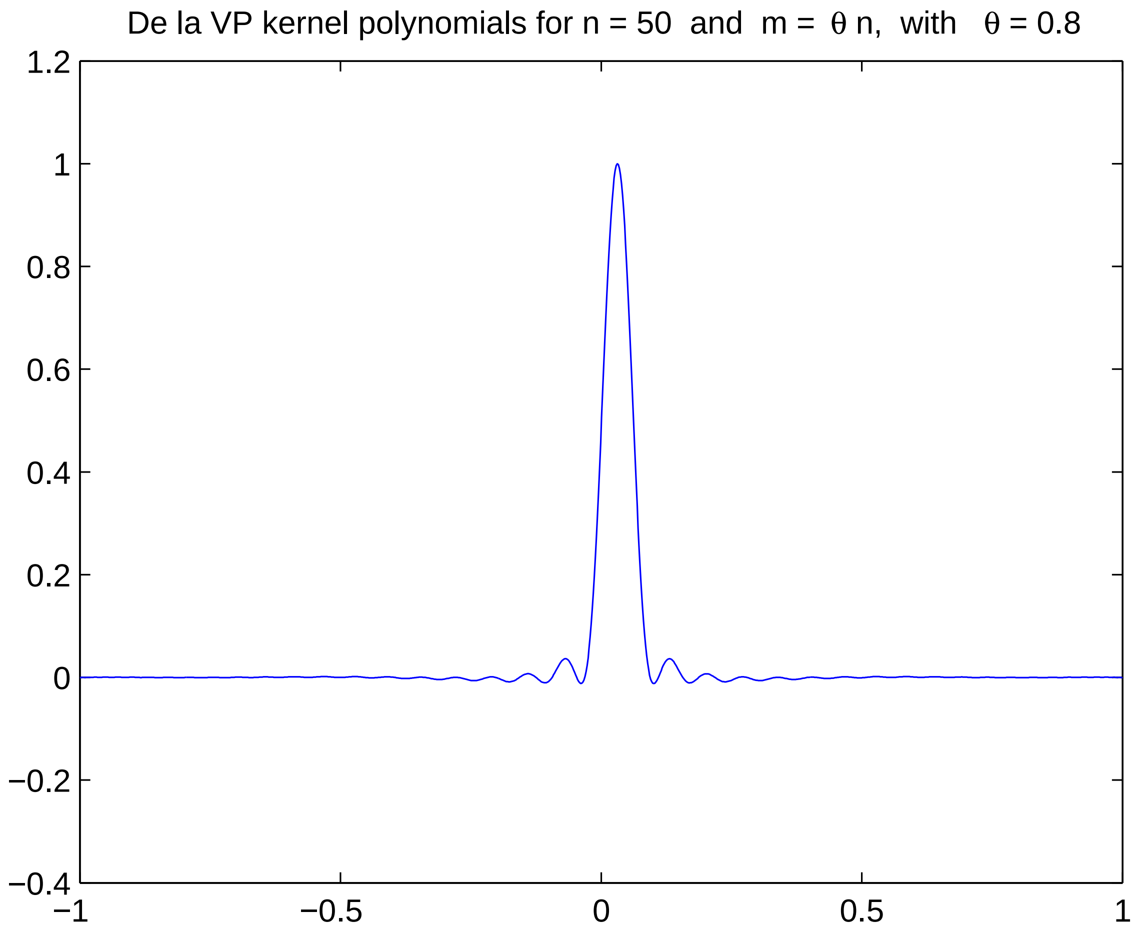


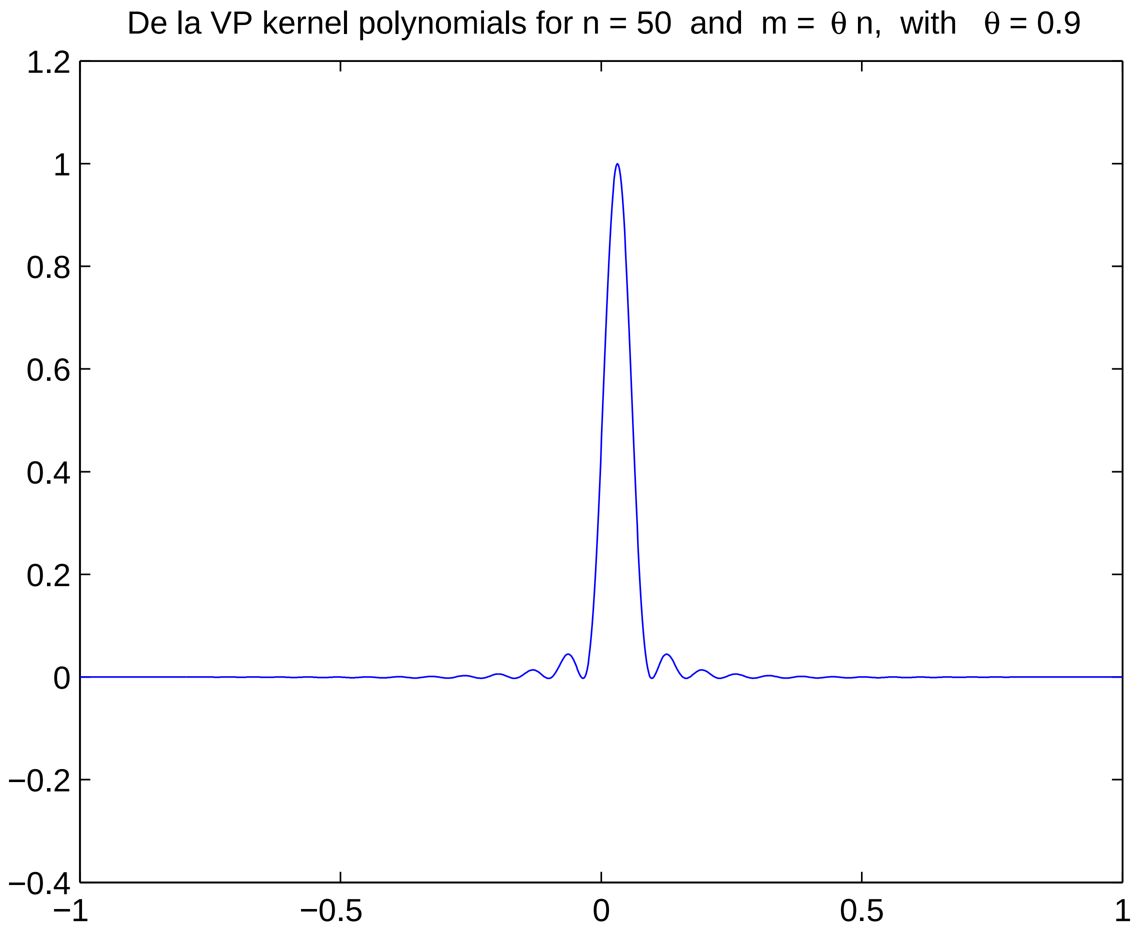
De la VP kernel polynomials for n = 50 and m = θ n, with θ = 0.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−1
−0.5
0
0.5
1

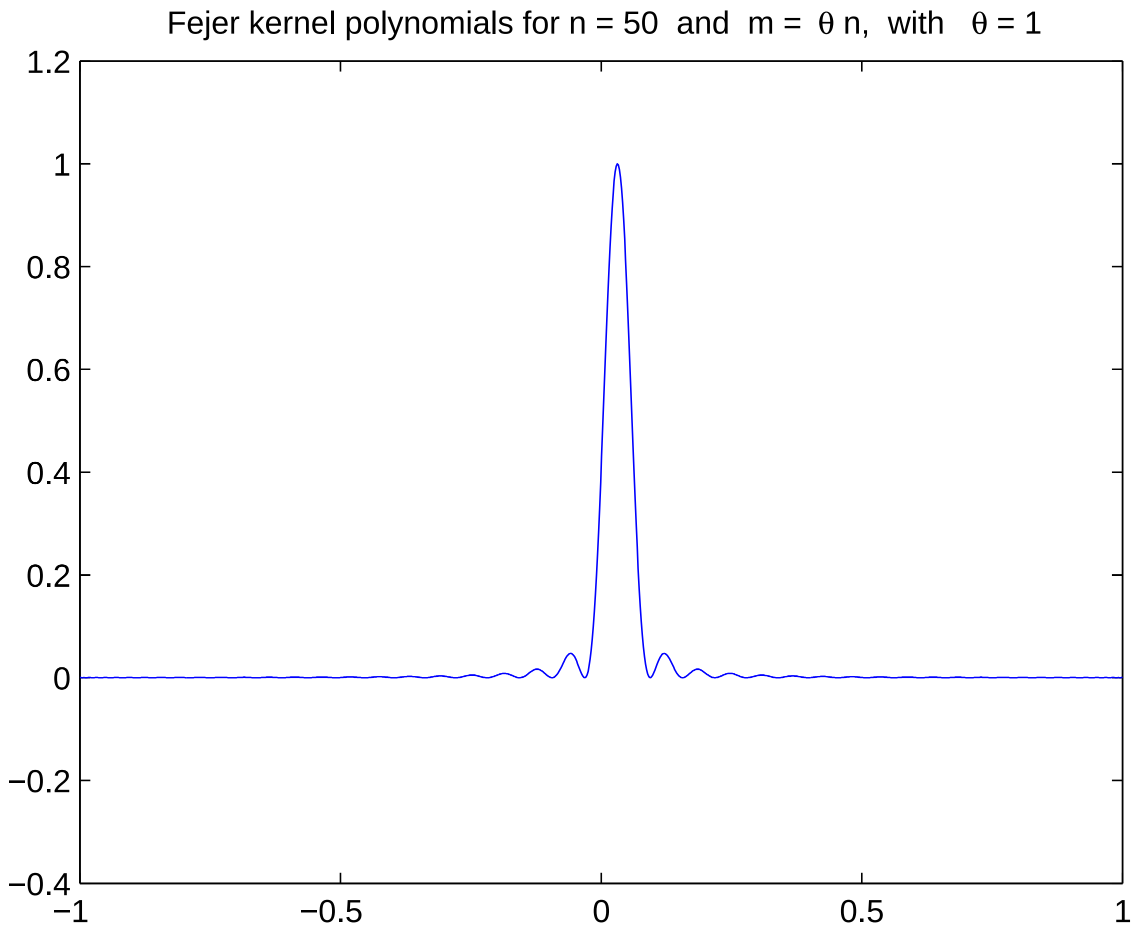

De la VP kernel polynomials for n = 50 and m = θ n, with θ = 0.5
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−1
−0.5
0
0.5
1

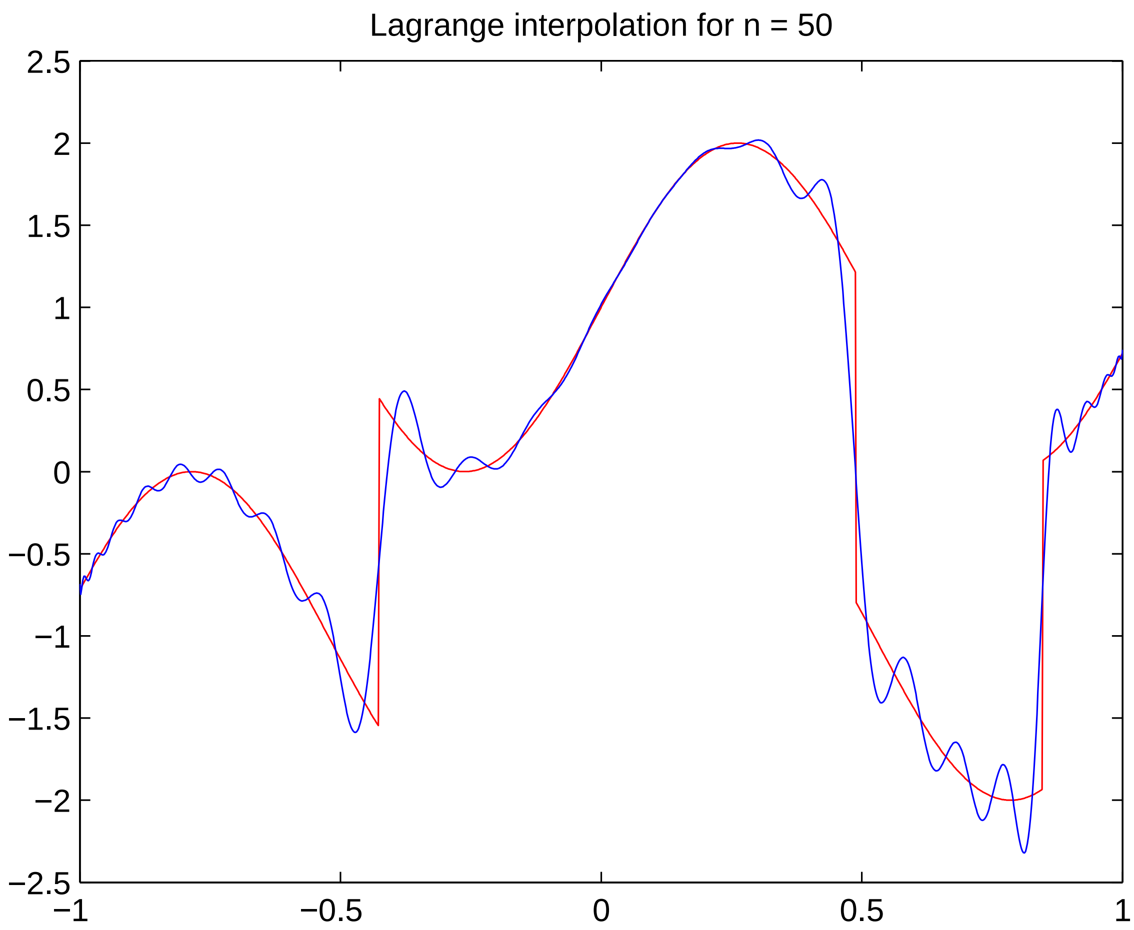

De la VP kernel polynomials for n = 50 and m = θ n, with θ = 0.6
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−1
−0.5
0
0.5
1

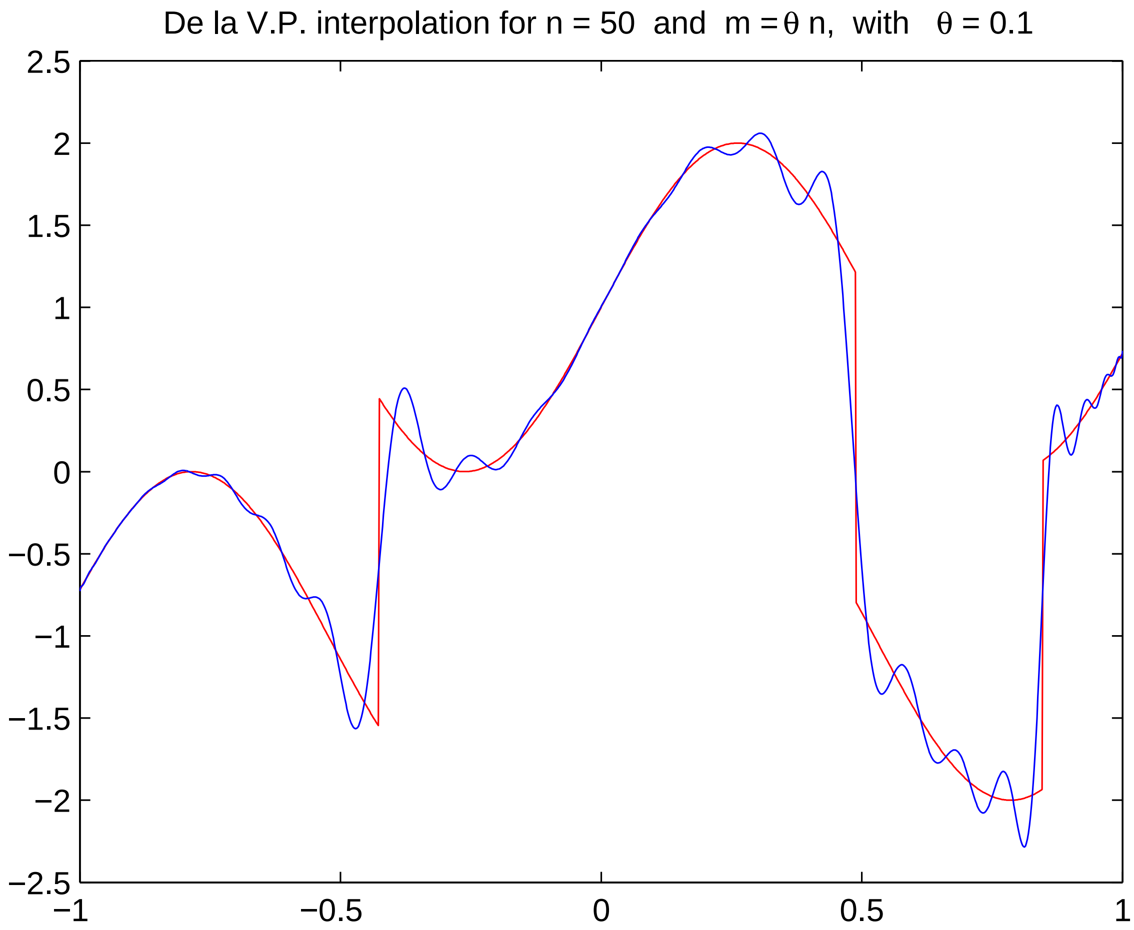

De la VP kernel polynomials for n = 50 and m = θ n, with θ = 0.7
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−1
−0.5
0
0.5
1

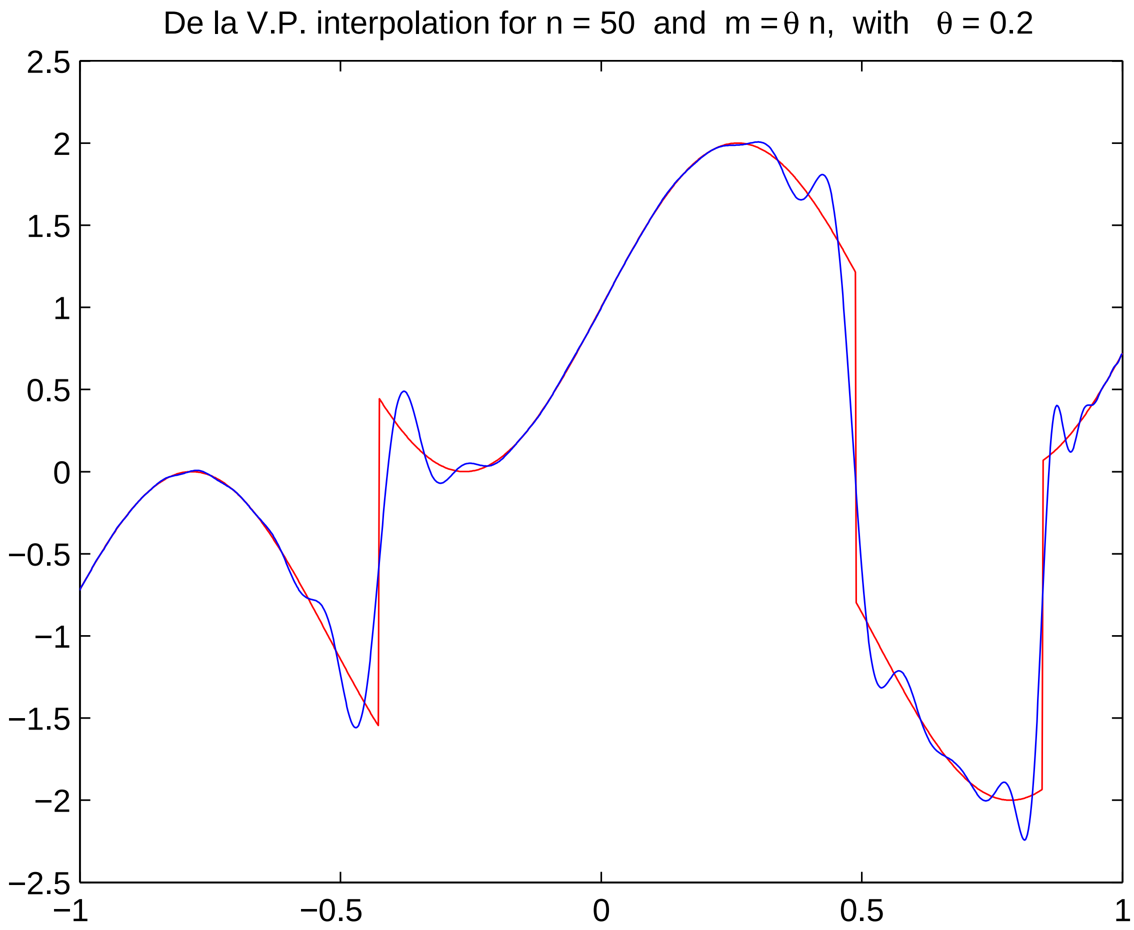

De la VP kernel polynomials for n = 50 and m = θ n, with θ = 0.8
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−1
−0.5
0
0.5
1

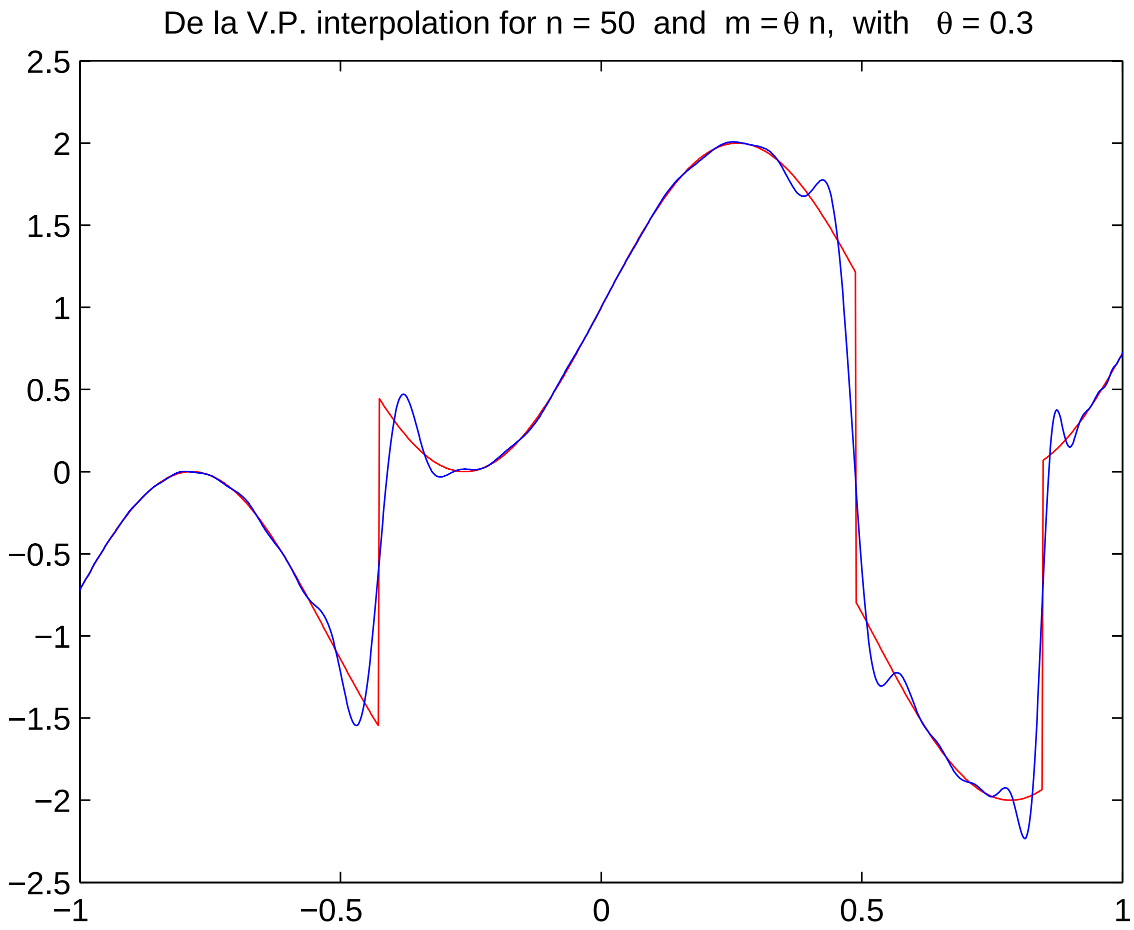

De la VP kernel polynomials for n = 50 and m = θ n, with θ = 0.9
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−1
−0.5
0
0.5
1

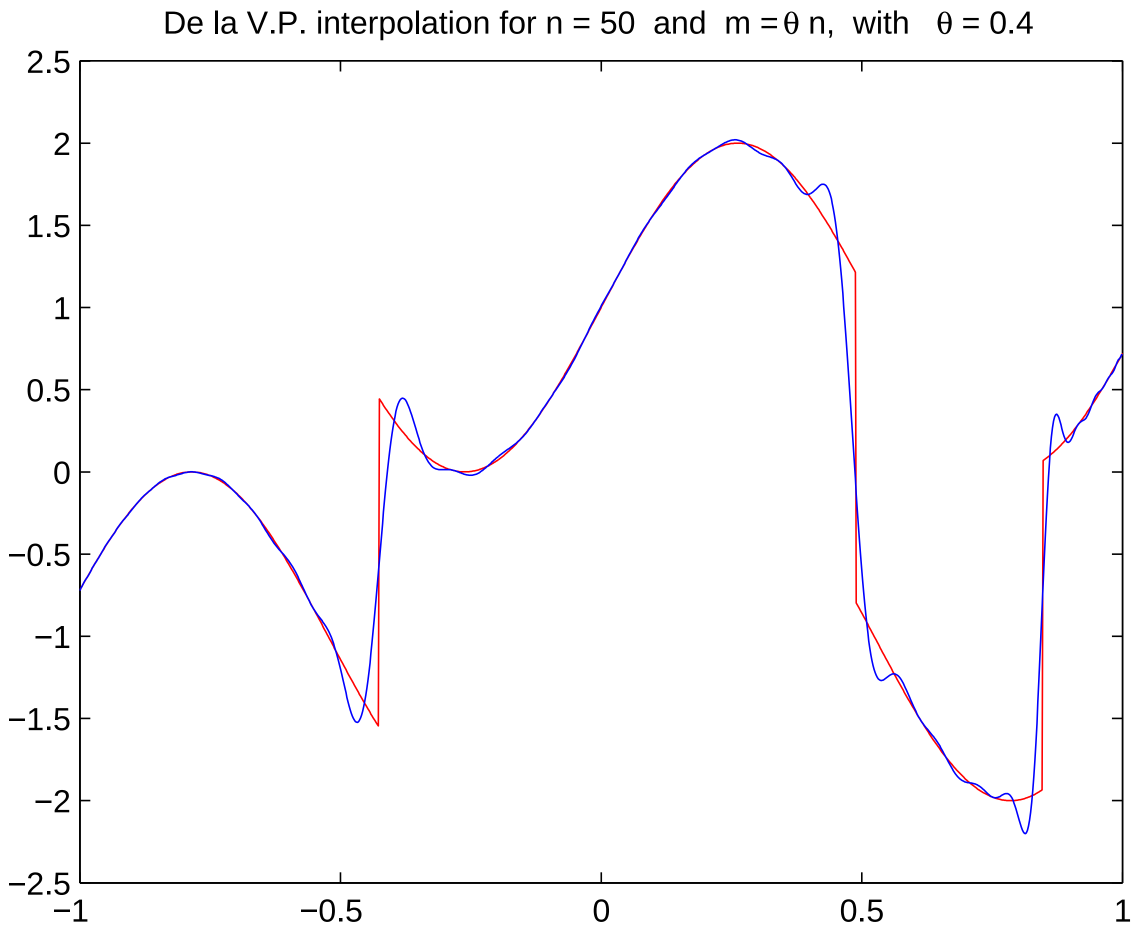

Fejer kernel polynomials for n = 50 and m = θ n, with θ = 1
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−1
−0.5
0
0.5
1

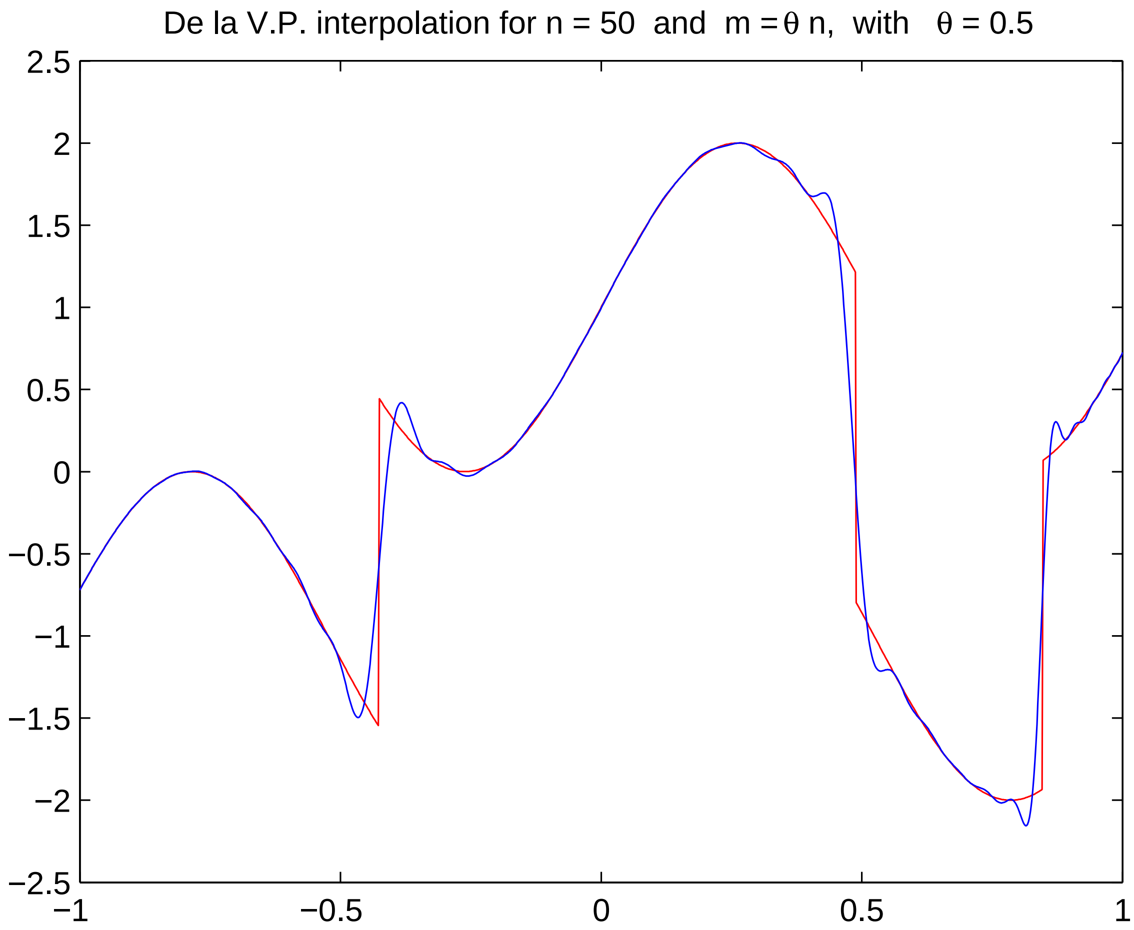

Lagrange interpolation for n = 50
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−1
−0.5
0
0.5
1

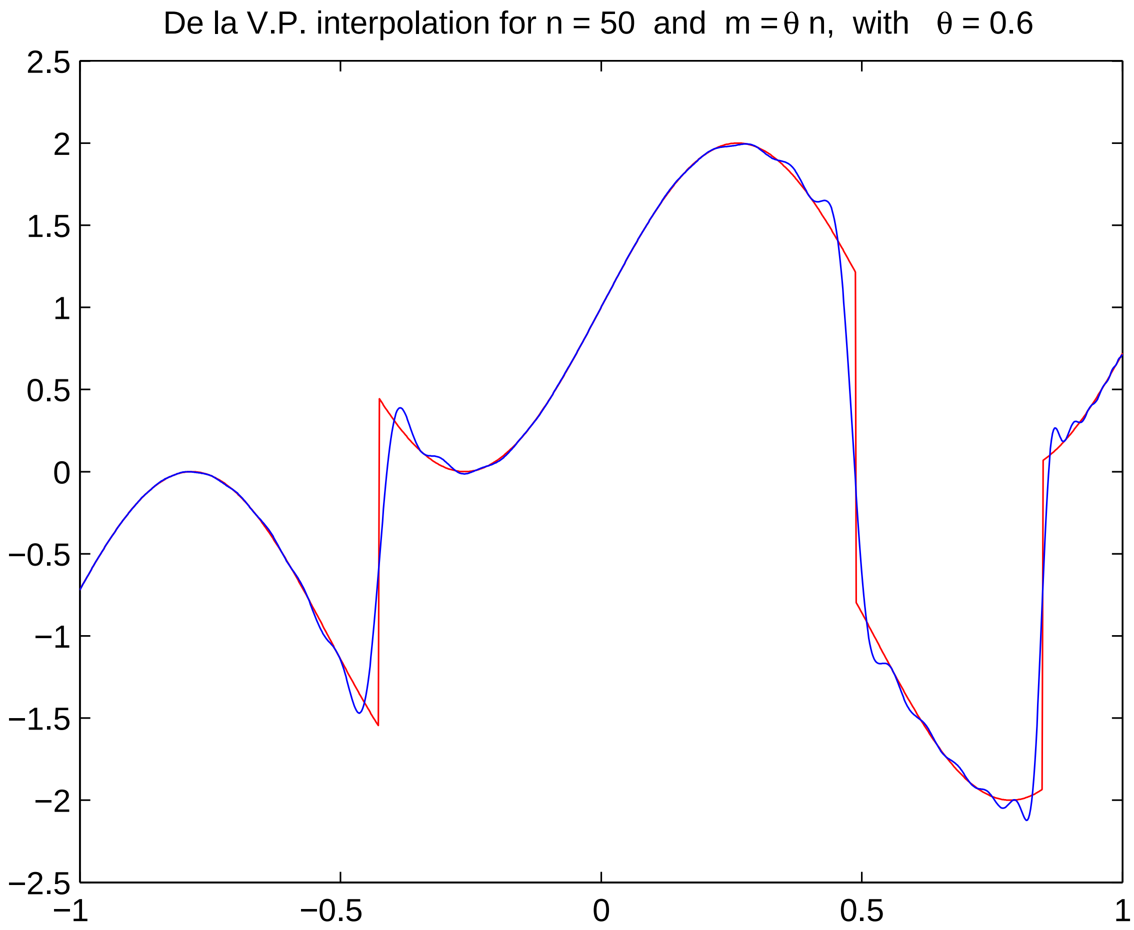

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.1
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−1
−0.5
0
0.5
1

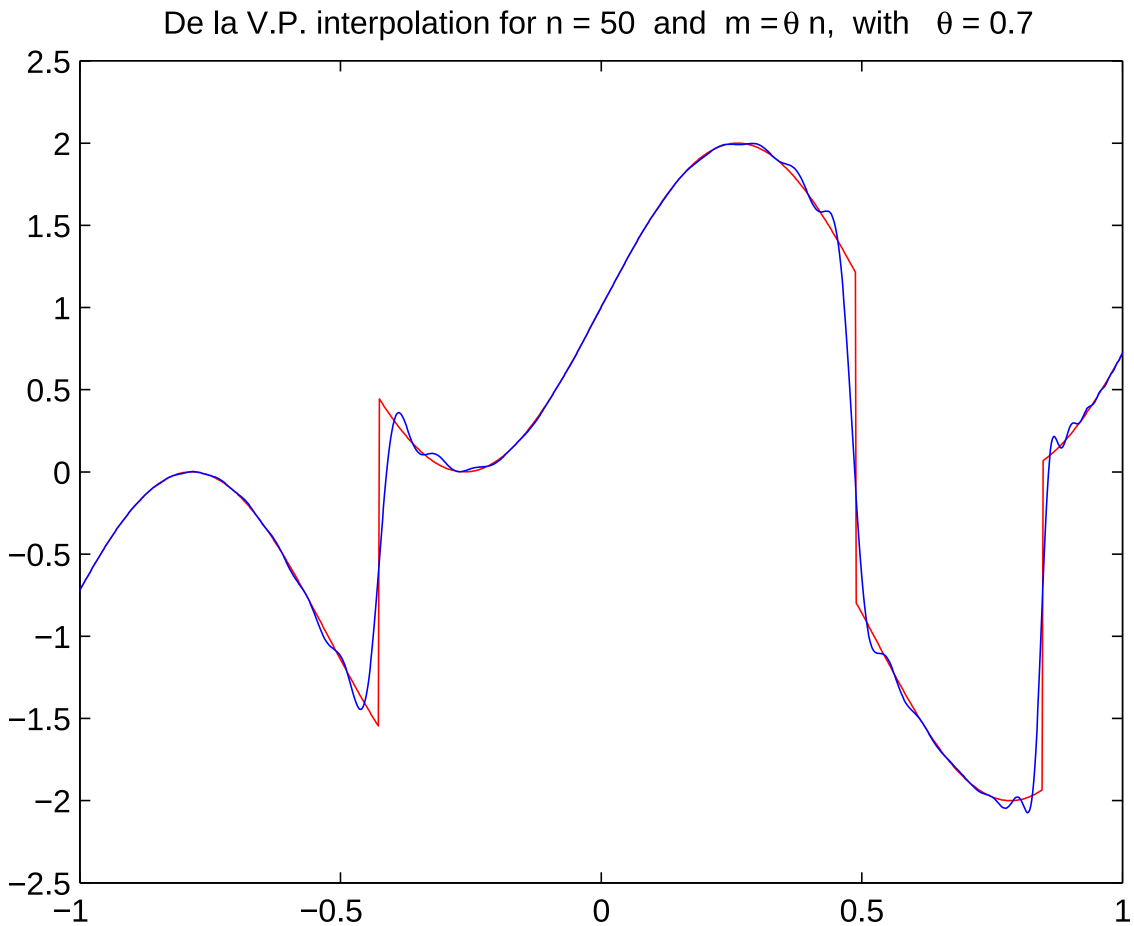

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.2
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−1
−0.5
0
0.5
1

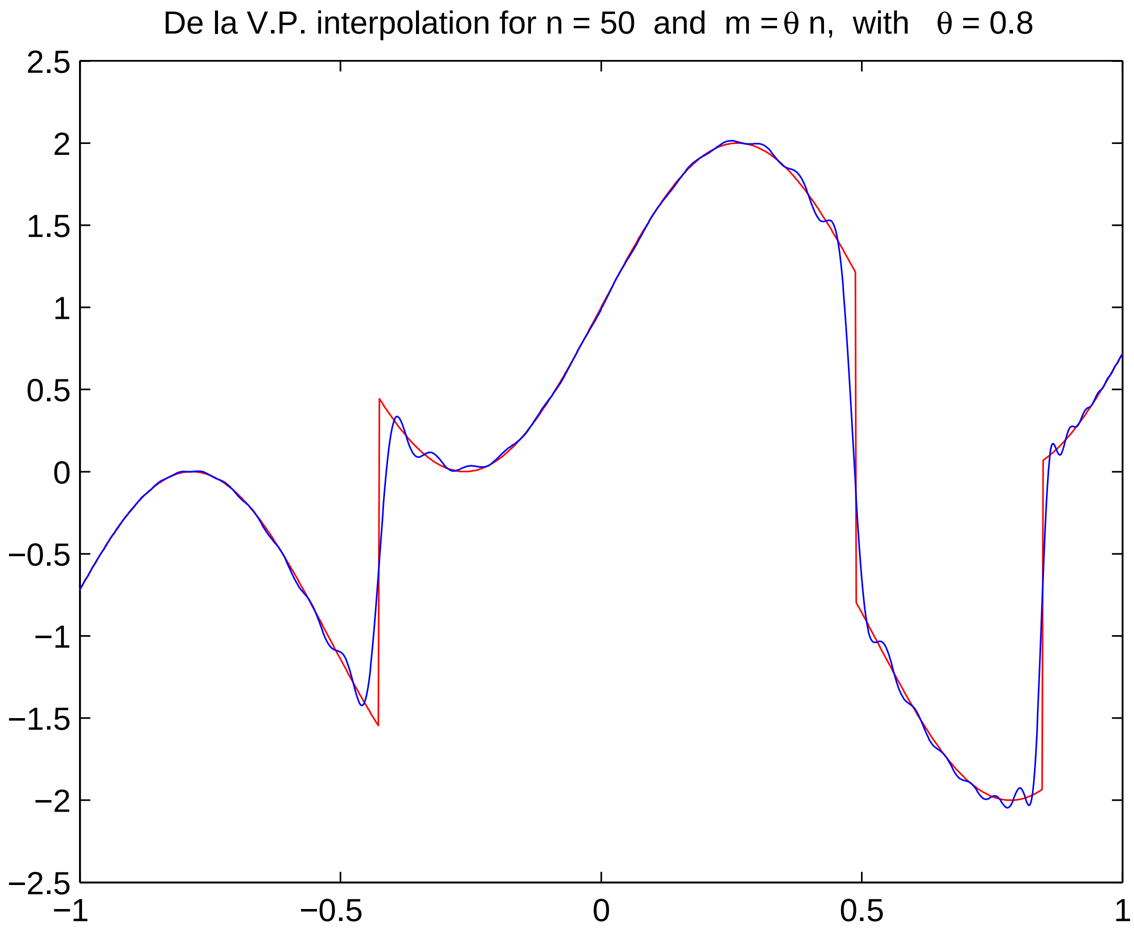

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−1
−0.5
0
0.5
1

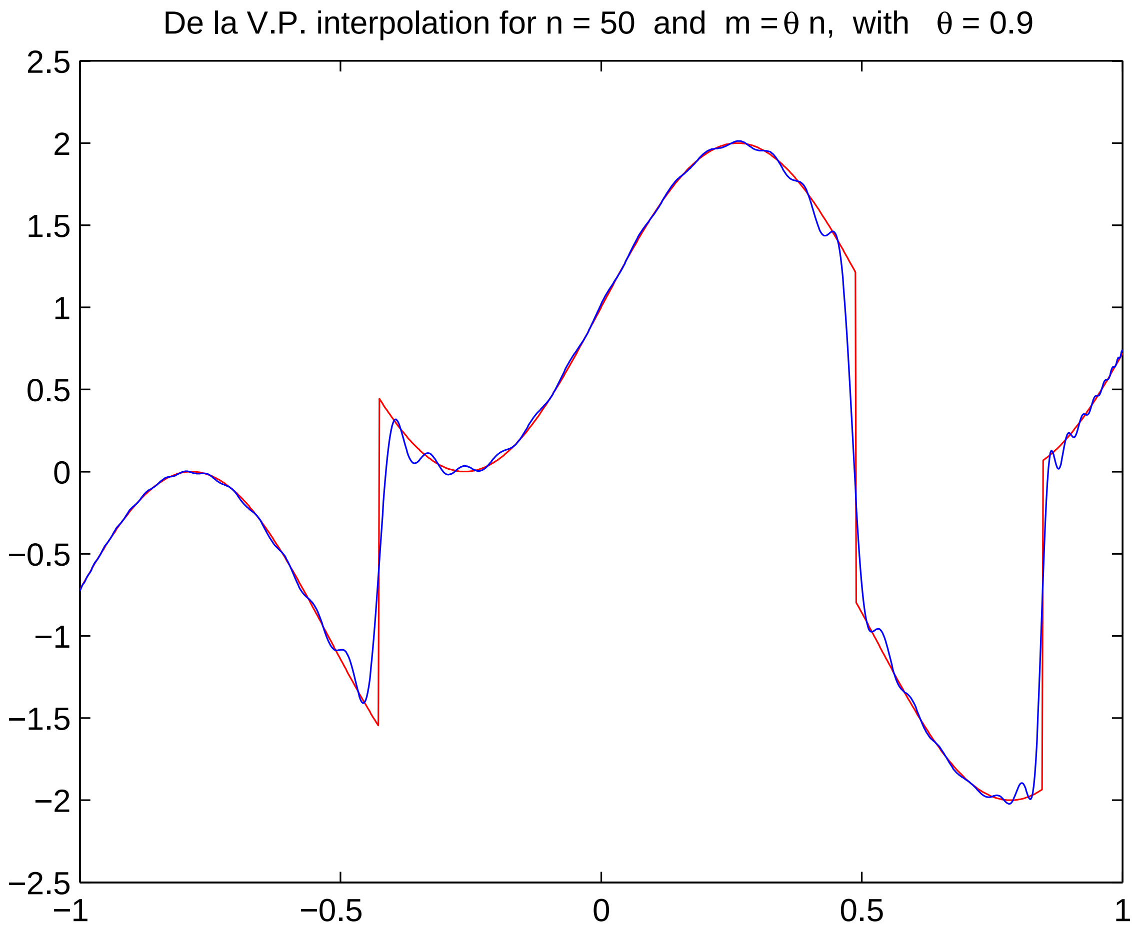

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.4
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−1
−0.5
0
0.5
1

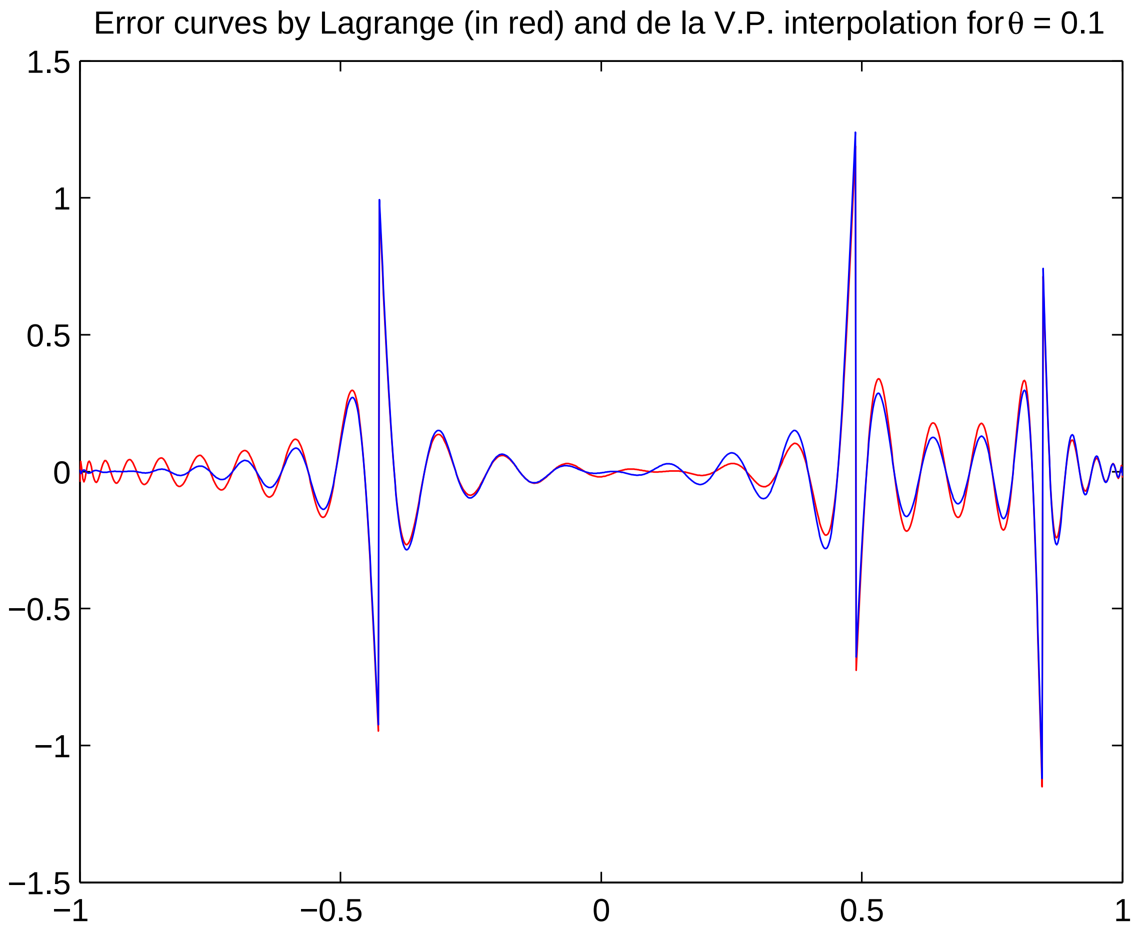

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.5
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−1
−0.5
0
0.5
1

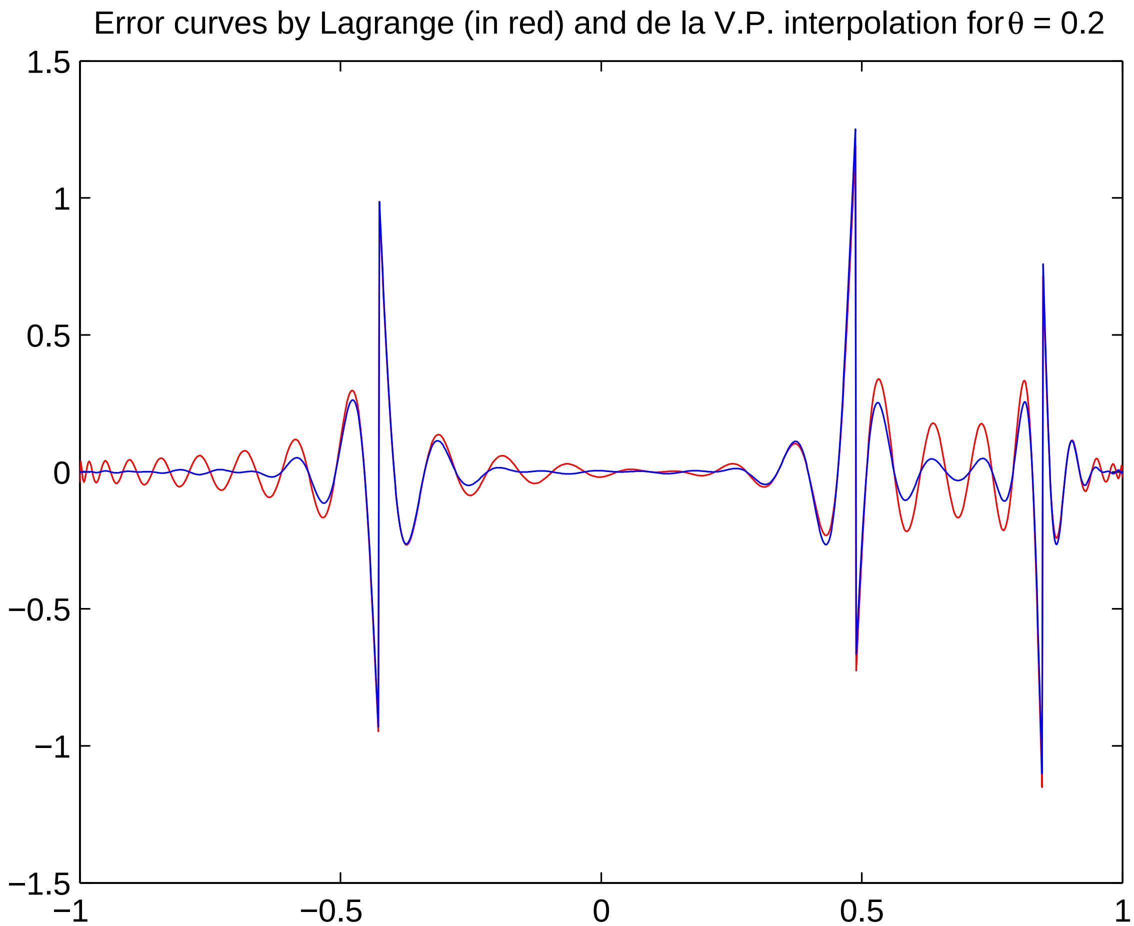

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.6
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−1
−0.5
0
0.5
1

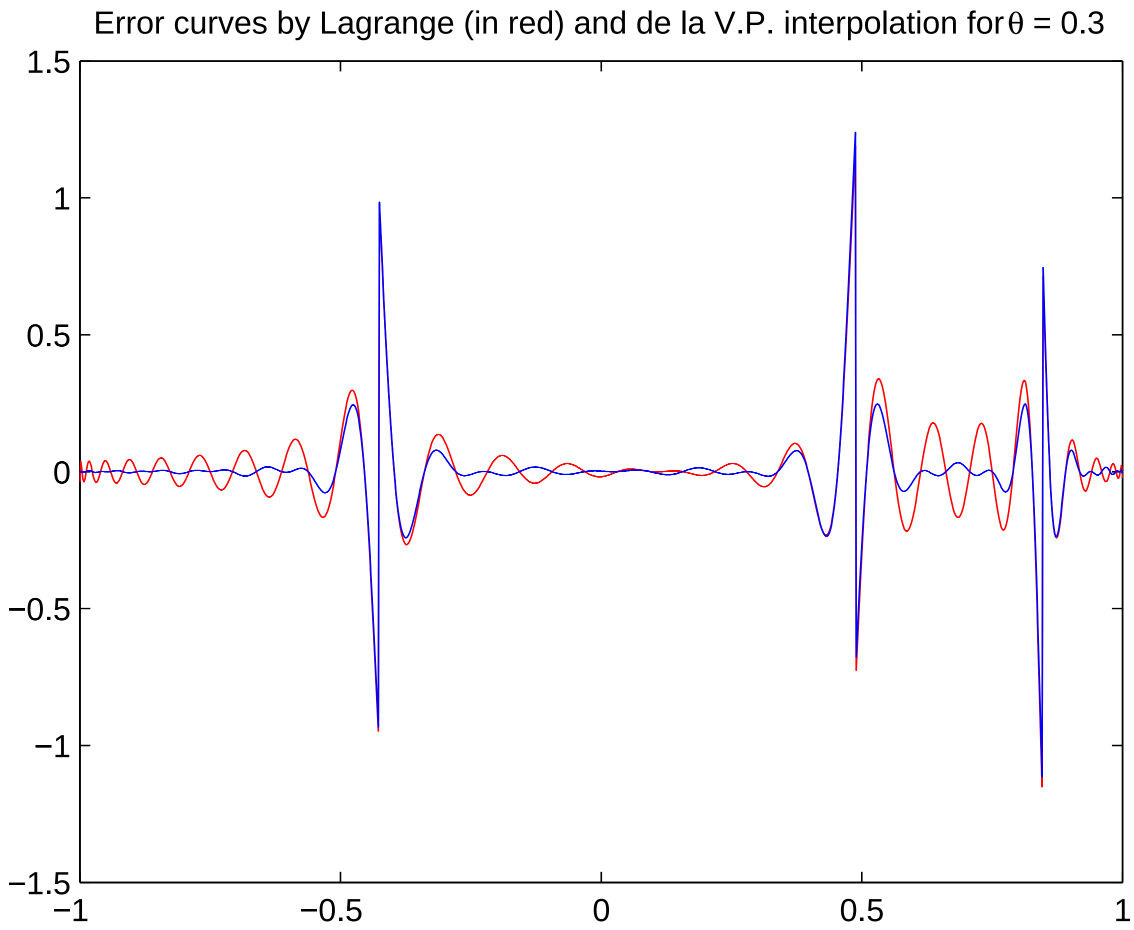

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.7
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−1
−0.5
0
0.5
1

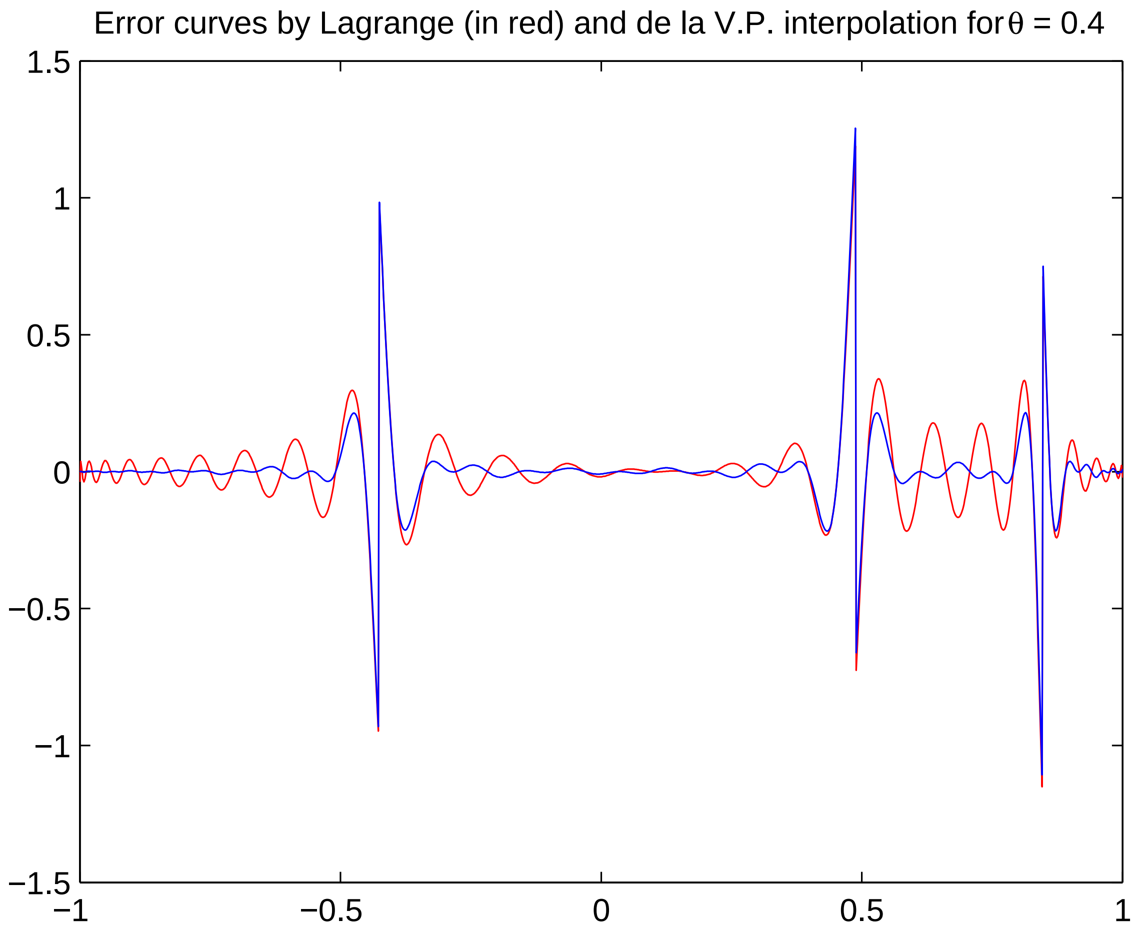

De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.8
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−1
−0.5
0
0.5
1


De la V.P. interpolation for n = 50 and m = θ n, with θ = 0.9
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−1
−0.5
0
0.5
1


Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.1
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

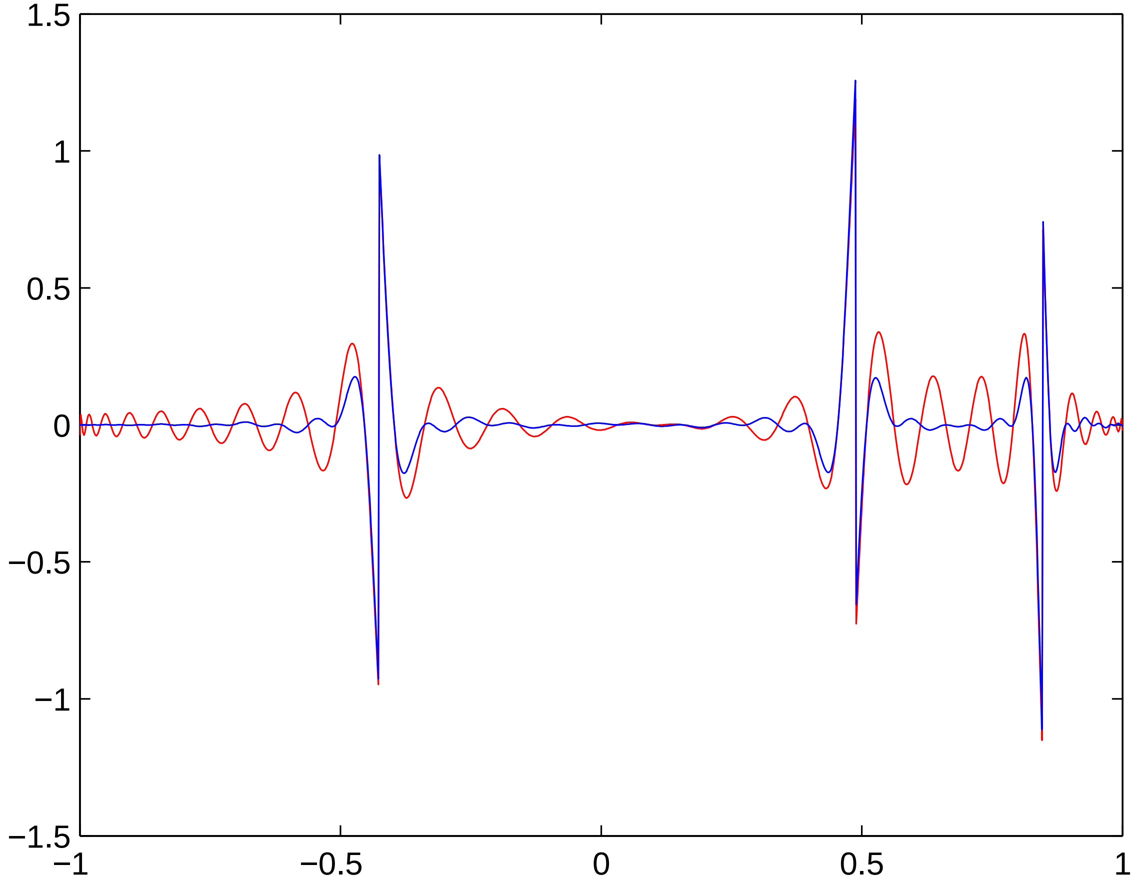

Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.2
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

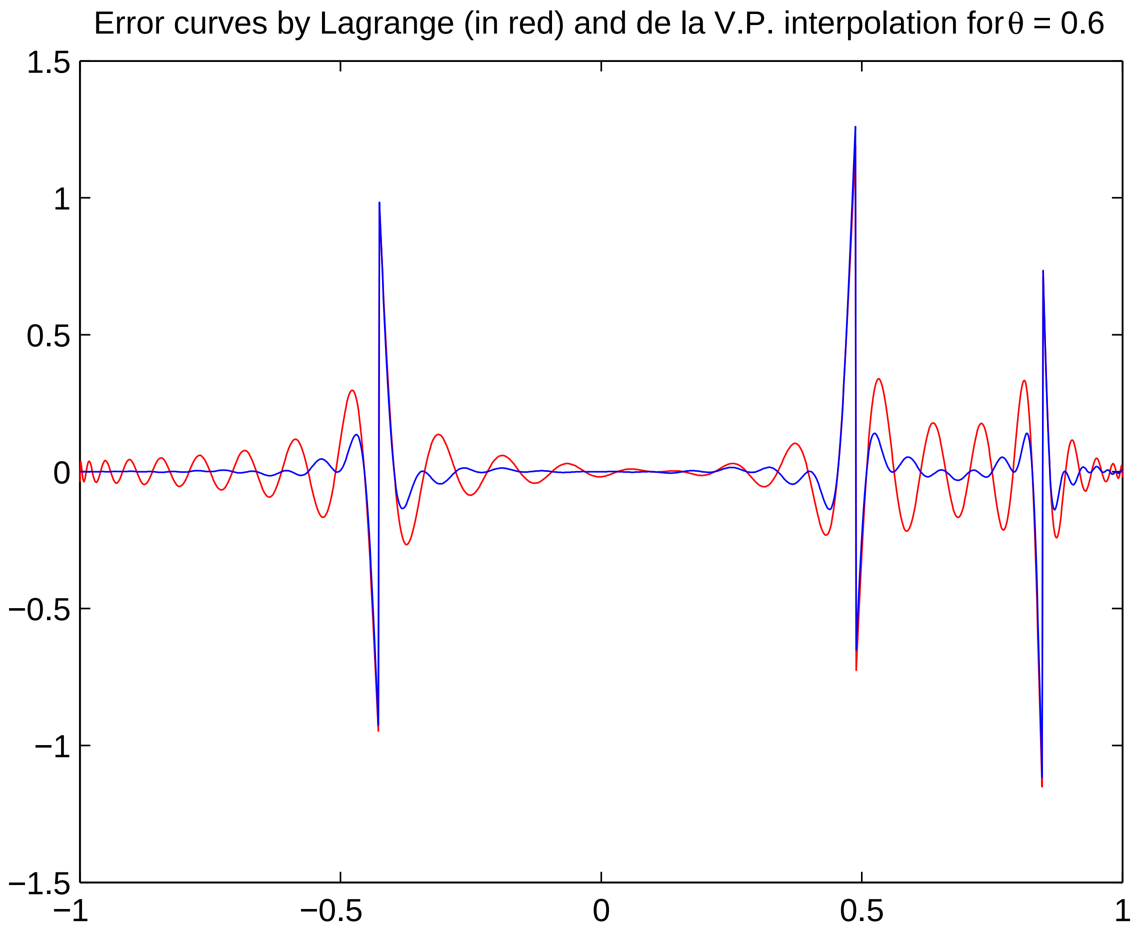

Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.3
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1



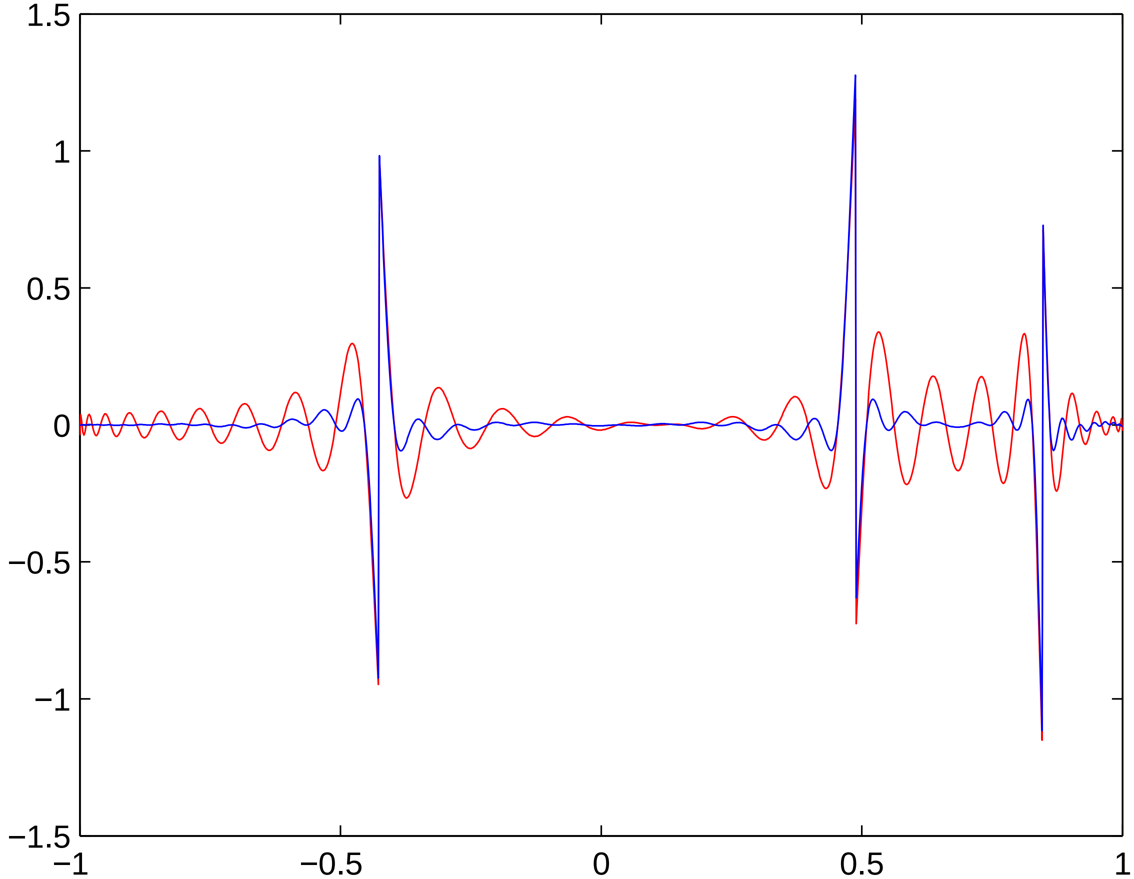
Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.4
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

Error curves by Lagrange (in red) and de la V.P. interpolation for $\theta = 0.5$

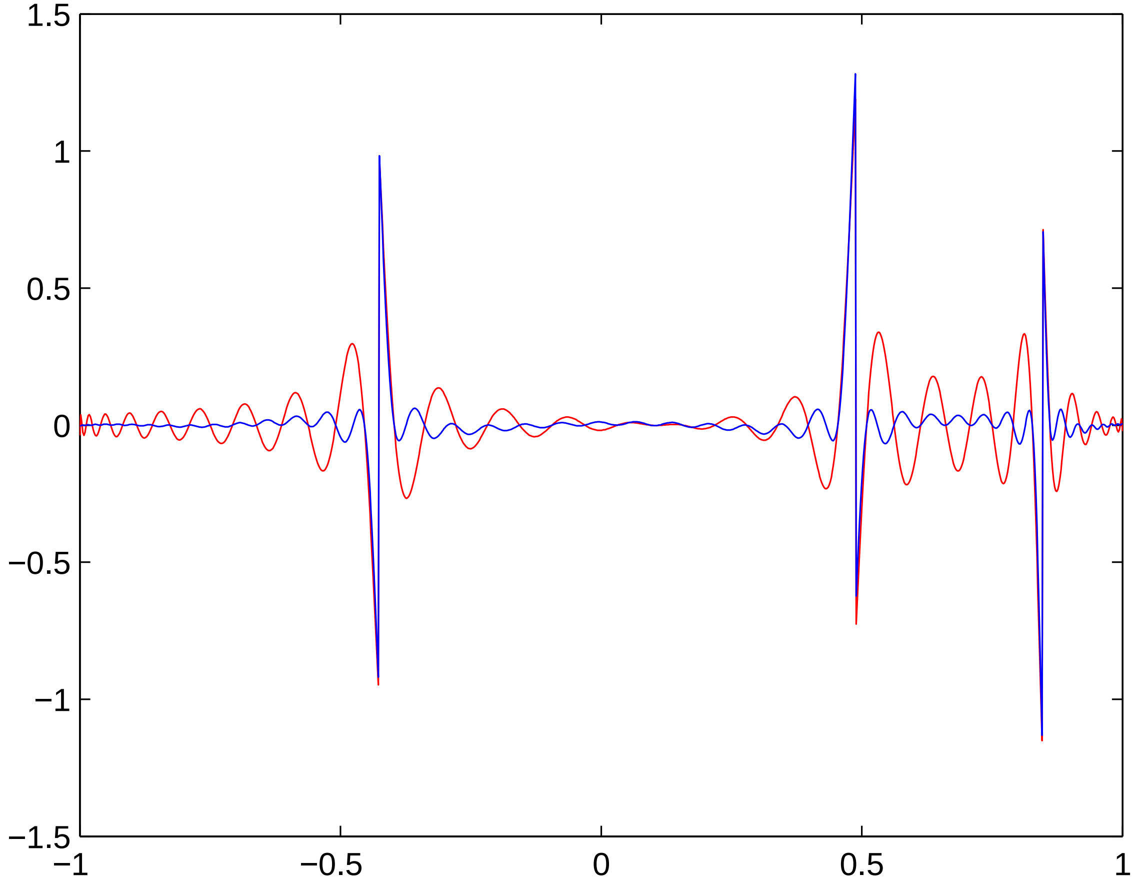


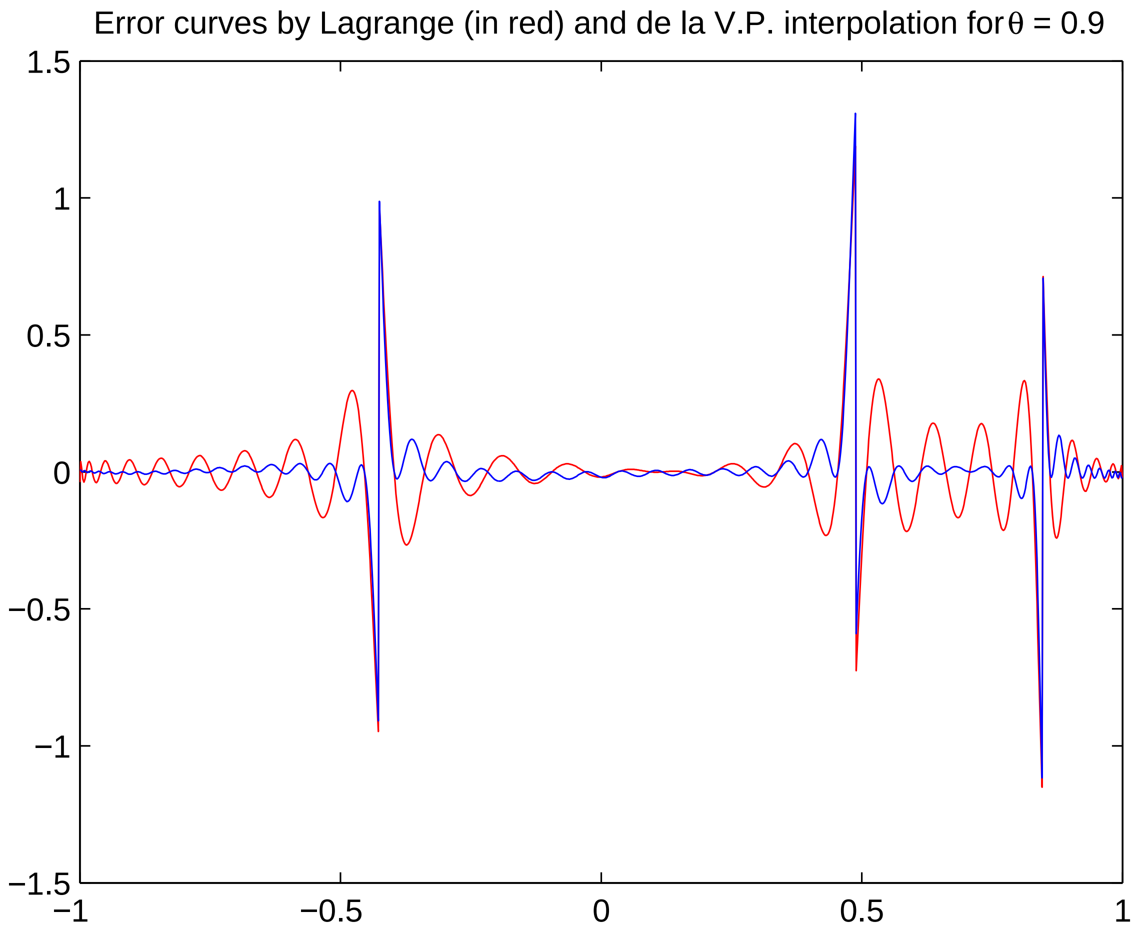

Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.6
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

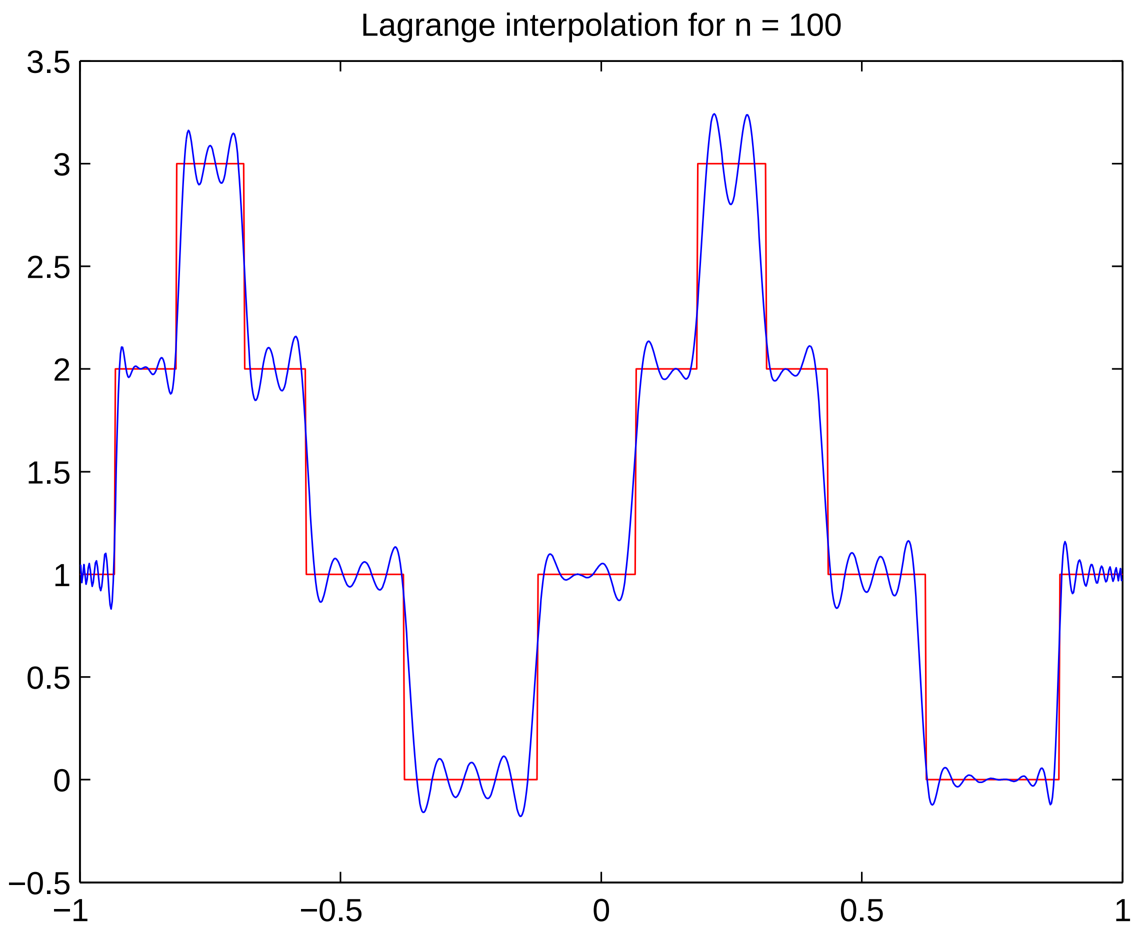
Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.7



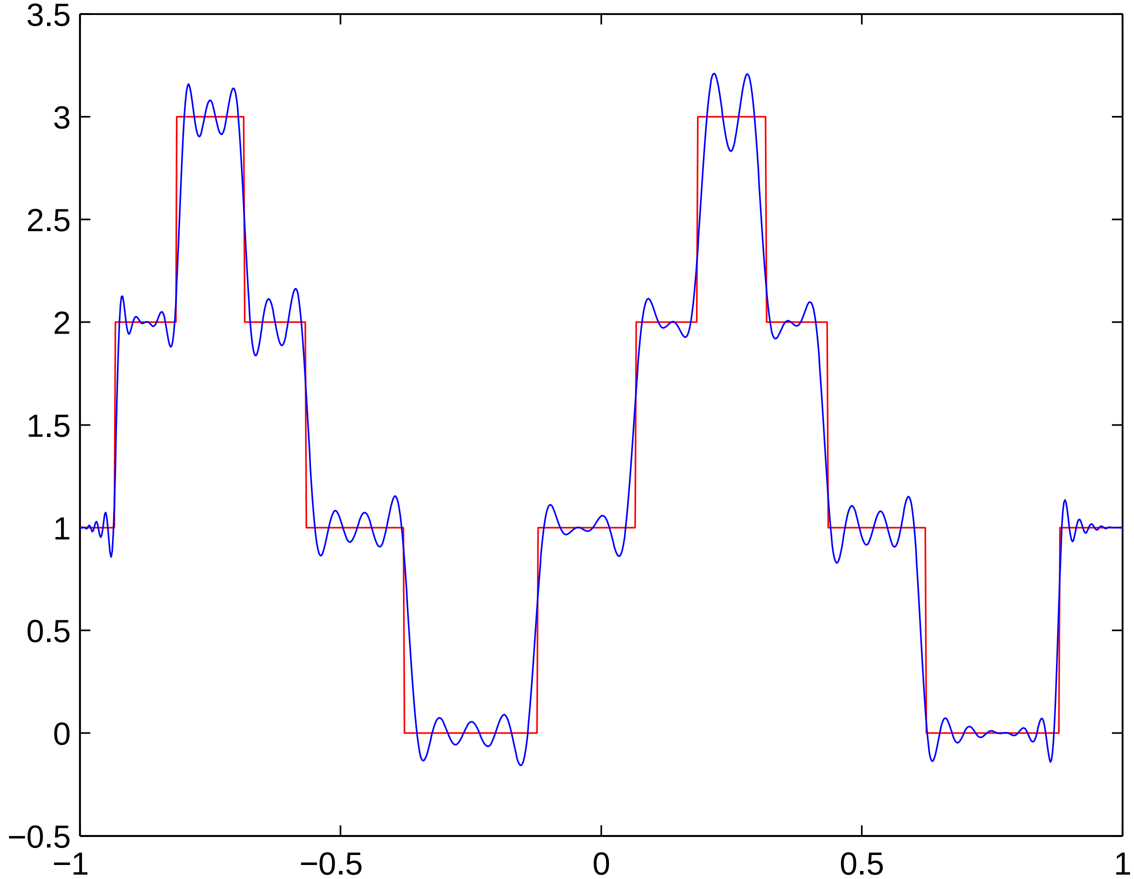
Error curves by Lagrange (in red) and de la V.P. interpolation for $\theta = 0.8$

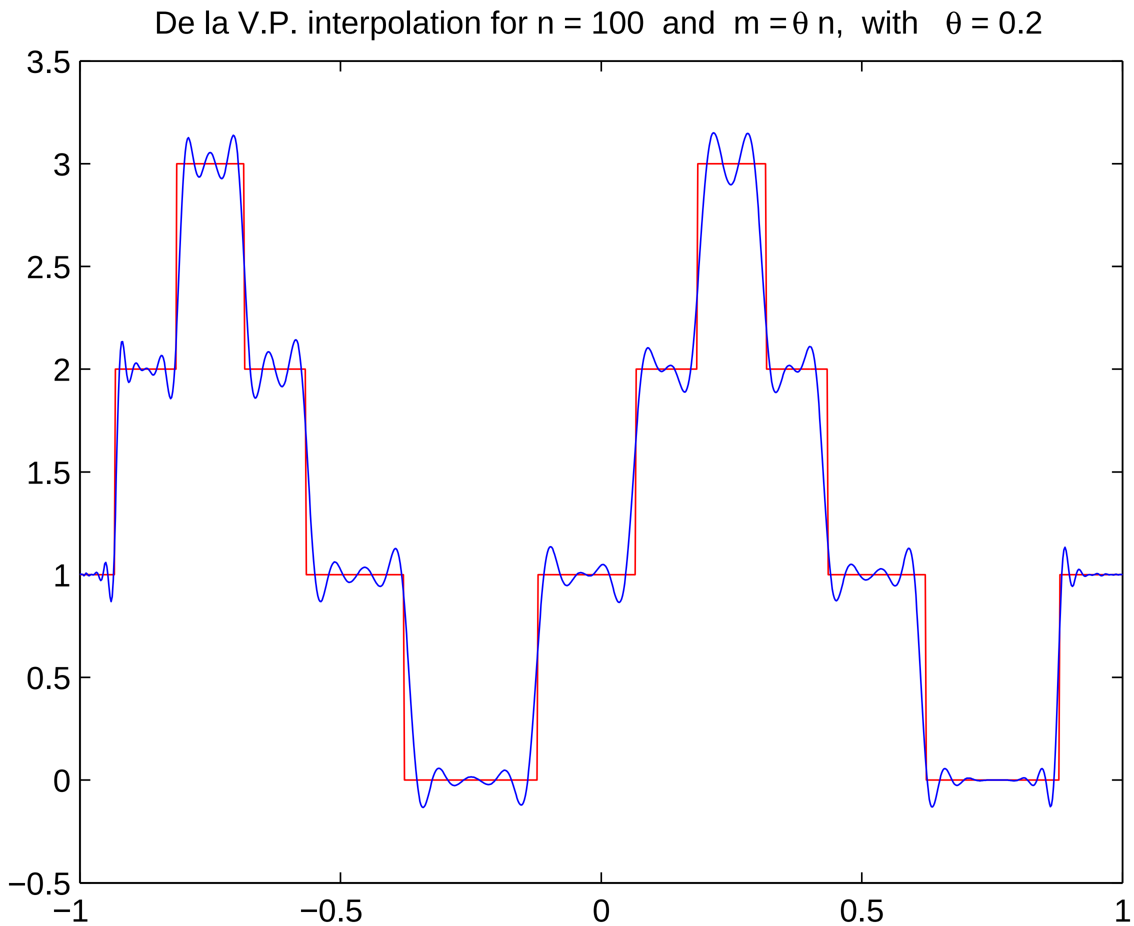



Error curves by Lagrange (in red) and de la V.P. interpolation for θ = 0.9
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1

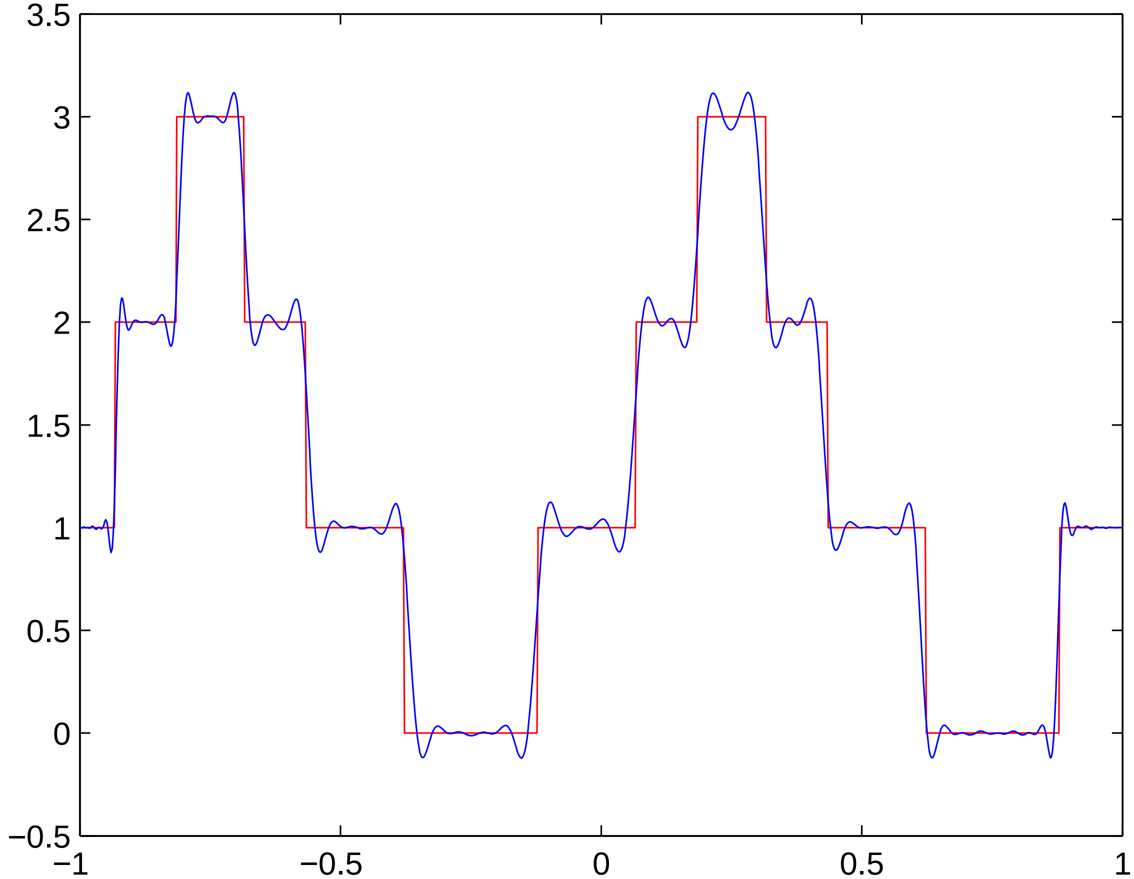

Lagrange interpolation for n = 100
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−0.5
0
0.5
1

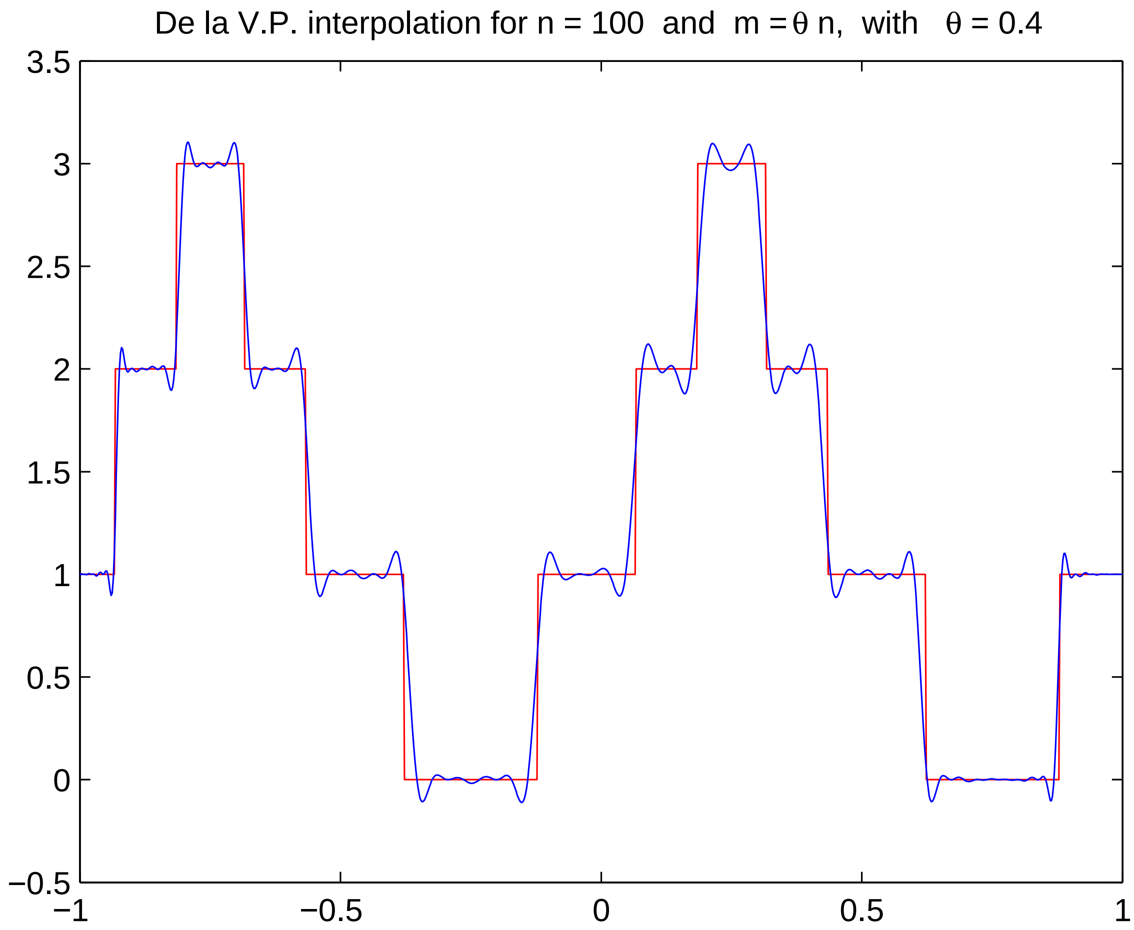
De la V.P. interpolation for $n = 100$ and $m = \theta n$, with $\theta = 0.1$

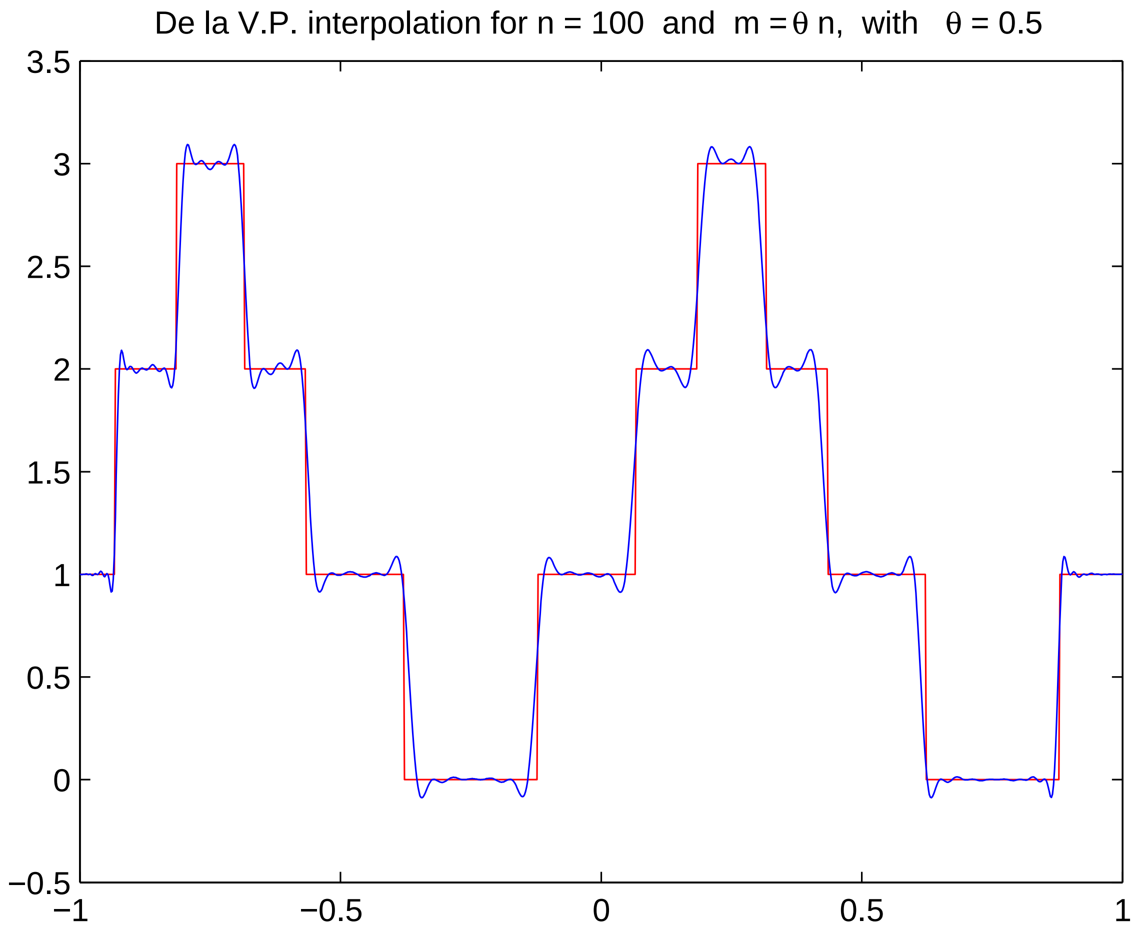



De la V.P. interpolation for n = 100 and m = θ n, with θ = 0.2
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−0.5
0
0.5
1

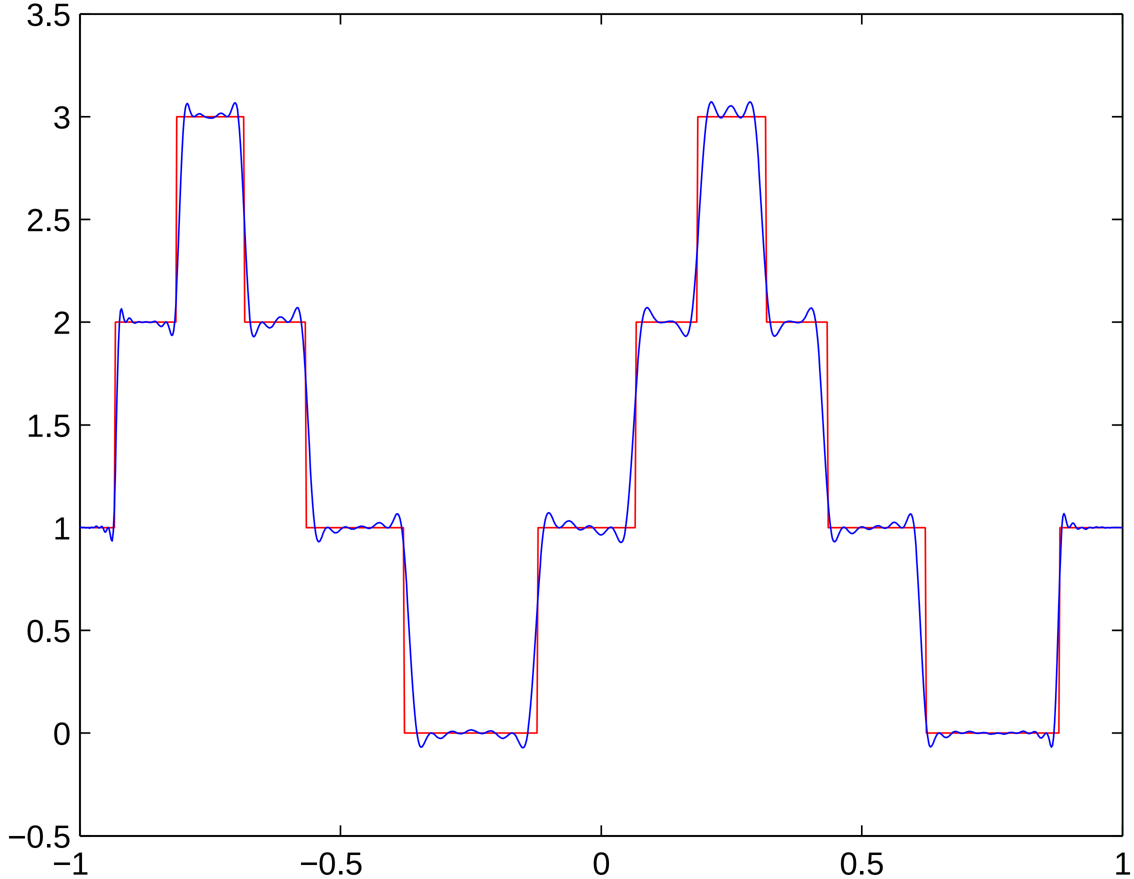
De la V.P. interpolation for n = 100 and m = θ n, with θ = 0.3

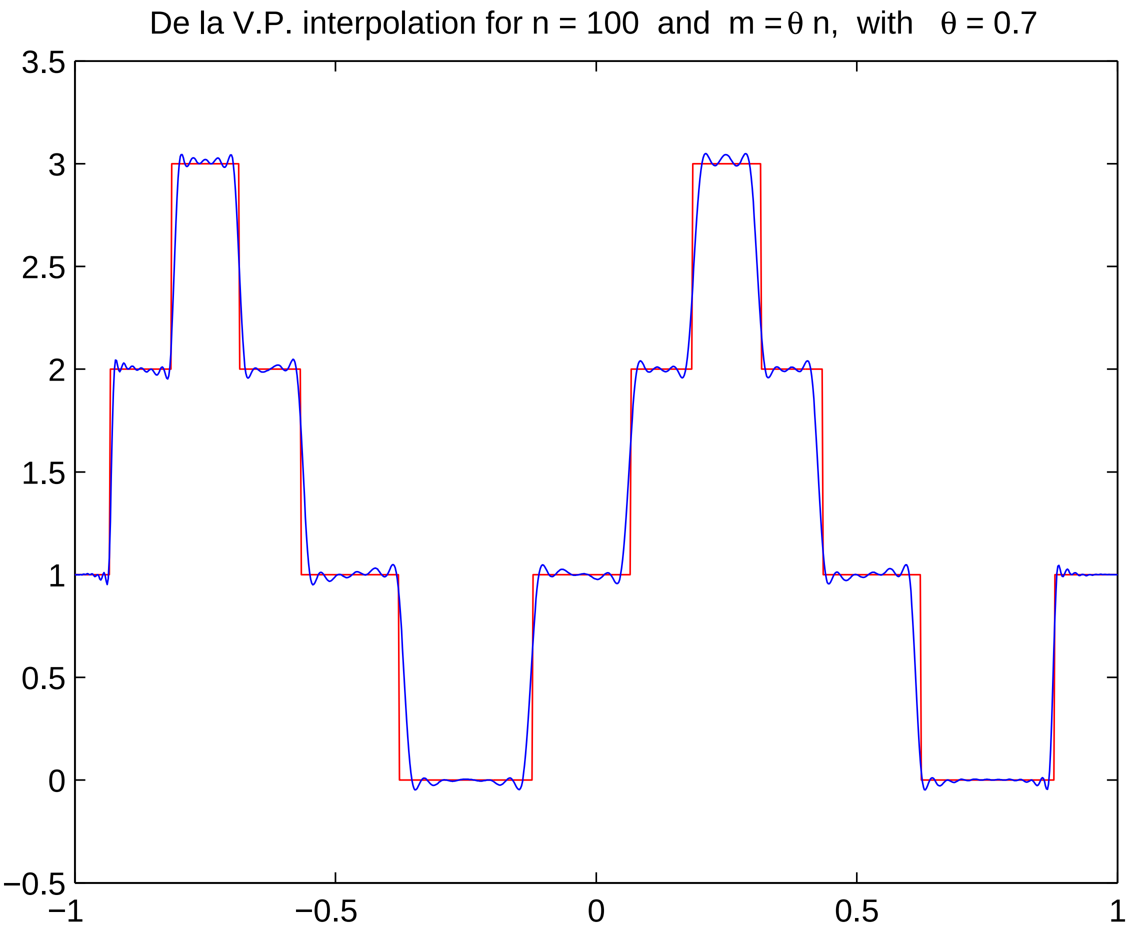


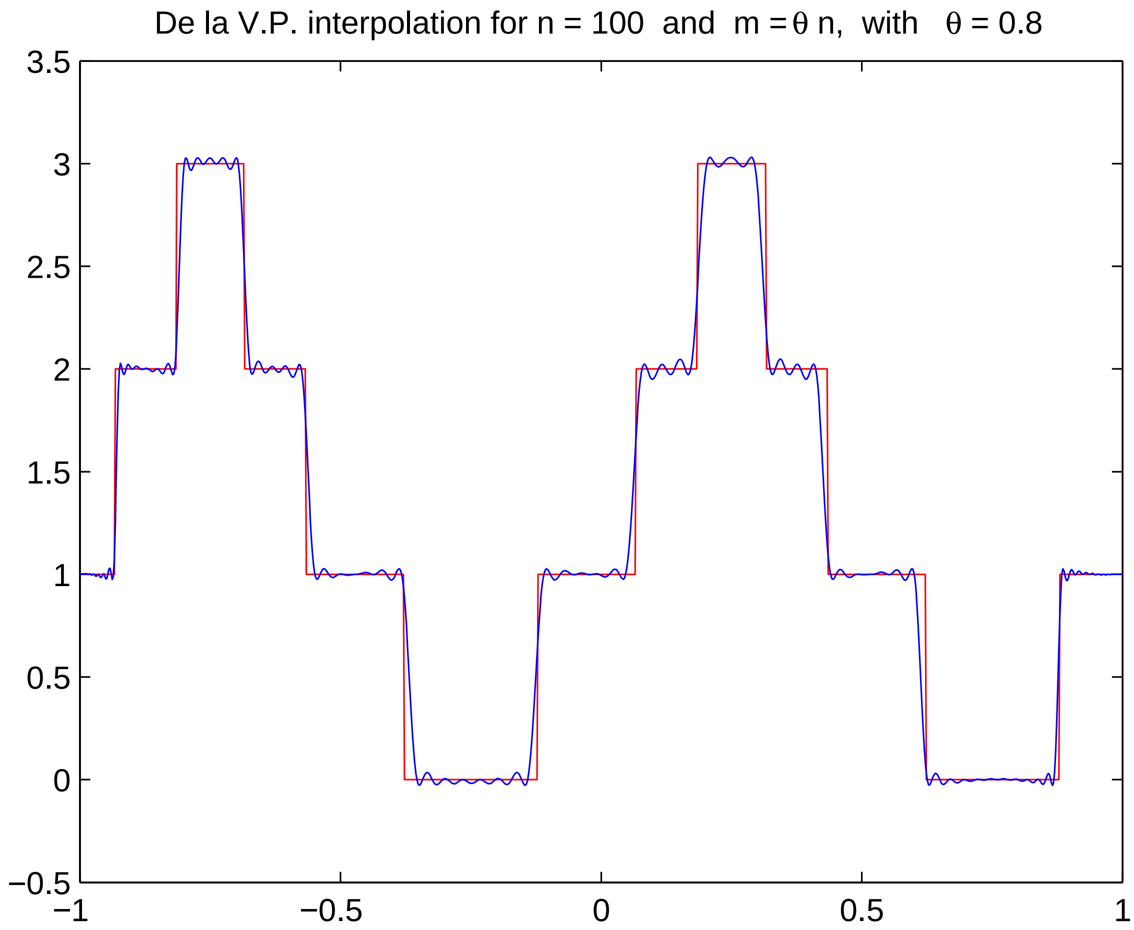

De la V.P. interpolation for n = 100 and m = θ n, with θ = 0.4
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−0.5
0
0.5
1

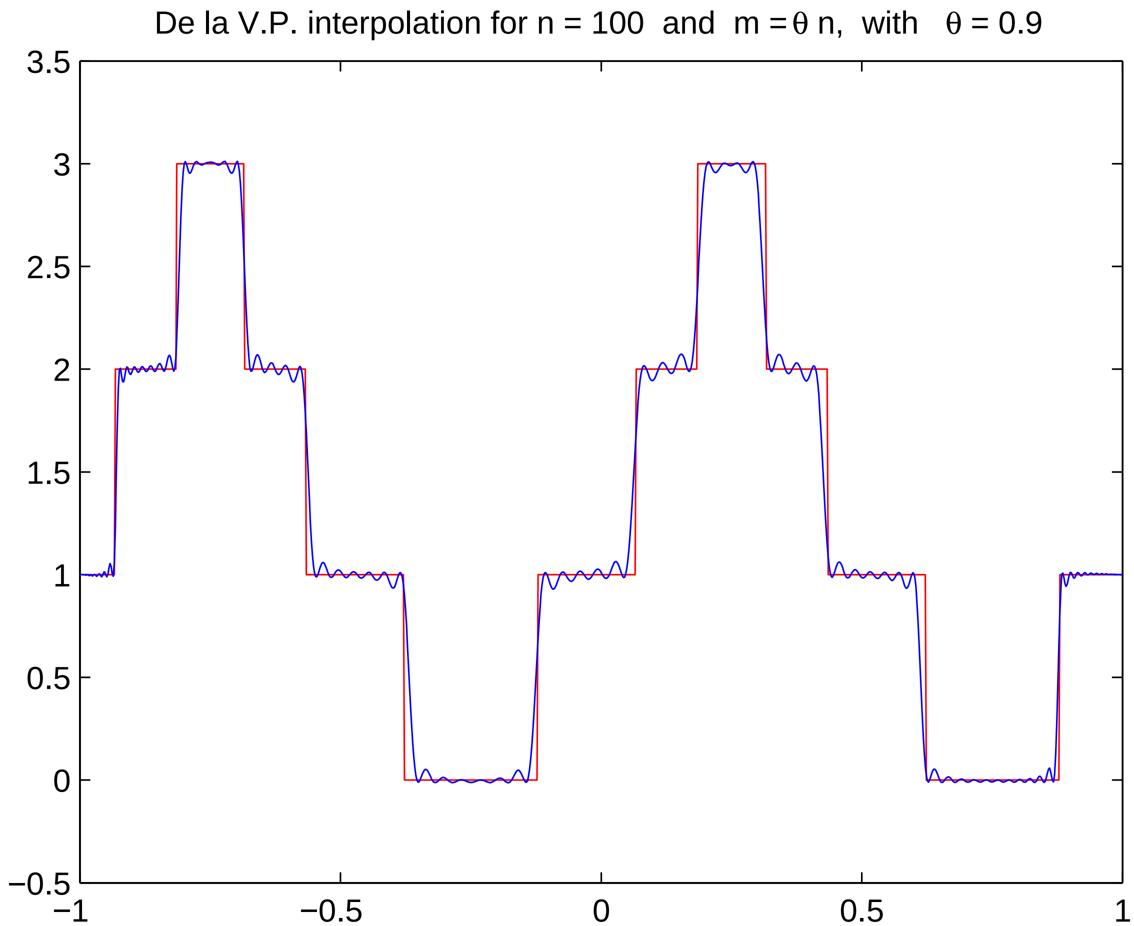

De la V.P. interpolation for n = 100 and m = θ n, with θ = 0.5
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−0.5
0
0.5
1

De la V.P. interpolation for n = 100 and m = θ n, with θ = 0.6




De la V.P. interpolation for n = 100 and m = θ n, with θ = 0.7
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−0.5
0
0.5
1


De la V.P. interpolation for n = 100 and m = θ n, with θ = 0.8
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−0.5
0
0.5
1



De la V.P. interpolation for n = 100 and m = θ n, with θ = 0.9
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−0.5
0
0.5
1

SOME REFERENCES:

- M.R.Capobianco, G.Criscuolo, G.Mastroianni, **Special Lagrange and Hermite interpolation processes** in: *Approximation Theory and Applications*, T.M.Rassias Editor, Hadronic Press, Palm Harbor, USA, ISBN 1-57485-041-5, 1998, pp.37–62
- M.R.Capobianco, W.Themistoclakis, **On the boundedness of some de la Vallée Poussin operators**, *East J. Approx.*, **7**, n. 4 (2001), 417–444. Corrigendum in *East J. Approx.*, **13**, n. 2 (2007), 223–226.
- G.Criscuolo, G.Mastroianni, **Fourier and Lagrange operators in some Sobolev–type spaces**, *Acta Sci. Math. (Szeged)*, **60** (1995), 131–148.
- G.Mastroianni, G.Milovanovic, **Interpolation processes. Basic theory and applications.** Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2008.
- G. Mastroianni, M.G.Russo, **Lagrange interpolation in weighted Besov spaces**, Constr. Approx., 15 (1999), 257–289.
- G. Mastroianni, W. Themistoclakis, **De la Vallée Poussin means and Jackson theorem**, Acta Sci. Math. (Szeged), 74 (2008), 147–170.
- W.Themistoclakis, **Some interpolating operators of de la Vallée Poussin type**, Acta Math. Hungar. 84 No.3 (1999), 221-235
- N.Trefethen, **Approximation Theory and Approximation Practice** (In preparation) http://www2.maths.ox.ac.uk/chebfun/ATAP/