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Polynomial approximation via de la Vallée Poussin means

Woula Themistoclakis

CNR - National Research Council of Italy

Institute for Computational Applications “Mauro Picone”, Naples, Italy.

▶ Lecture 1: De la Vallée Poussin means

- Approximation properties
- Basic facts on polynomial approximation
- Application to prove boundedness of some CSIO in Lipschitz type spaces

▶ Lecture 2: Discrete de la Vallée Poussin means

- Approximation properties
- Comparison with Lagrange interpolation
- Interpolating de la Vallée Poussin polynomials

▶ Lecture 3: Applications of de la Vallée Poussin type interpolation

- Part 1: A numerical method for solving the generalized airfoil equation
- Part 2: Construction of interpolating polynomial wavelets on [-1,1]

APPROXIMATION THEOREM: [*(1885) Karl Weierstrass*]

For any $f \in C[-1,1]$ and each $\epsilon > 0$, there exists an algebraic polynomial Q such that

$$\|f - Q\|_\infty < \epsilon$$

Weighted extensions: $\|(f - Q)u\|_p < \epsilon, \qquad 1 \le p \le \infty$

where $u(x) := v^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta \in L^p[-1,1]$ is a Jacobi weight and if $1 \le p < \infty$, we assume $f \in L^p_u$ with

$$L^p_u := \{f : \ \|fu\|_p < \infty\}$$

while in the case $p = \infty$, we suppose $f \in C^0_u$, with

$$C^0_u := \{f \in C(-1,1) : \ \lim_{|x|\to 1} f(x)u(x) = 0\}, \quad \text{if } \ \alpha, \beta > 0,$$
$$C^0_u := \{f \in C(-1,1] : \ \lim_{x\to -1} f(x)u(x) = 0\}, \quad \text{if } \ \alpha = 0 < \beta$$
$$C^0_u := \{f \in C[-1,1) : \ \lim_{x\to +1} f(x)u(x) = 0\}, \quad \text{if } \ \alpha > 0 = \beta$$

Fourier partial sums: $$S_n(w,f,x) := \sum_{k=0}^{n} c_k(w,f)p_k(w,x)$$

where $w(x) := v^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha,\beta > -1$, is a Jacobi weight, $\{p_k(w,x)\}_k$ denotes the corresponding system of orthonormal Jacobi polynomials and $c_k(w,f) := \int_{-1}^{1} p_k(w,y)f(y)w(y)dy$.

- **Invariance:** $S_n(w,P) = P, \qquad P \in \mathbb{P}_n := \{P : \deg(P) \le n\}$.

- **Boundedness in L^p_u:** Under some conditions on u, w, we have

$$1 < p < \infty \quad \Longrightarrow \quad \sup_n \|S_n(w)\|_{L^p_u \to L^p_u} < \infty,$$

- **Critical cases:** $p = 1, \infty \quad \Longrightarrow \quad \sup_n \|S_n(w)\|_{L^p_u \to L^p_u} = +\infty, \quad \forall u, w$

De la Vallée Poussin means:

$$V_n^m(w, f, x) := \frac{1}{m - n + 1} \sum_{k=n}^{m} S_k(w, f, x)$$

- **Quasi-projection:** $V_n^m(w, P) = P$ whenever $P \in \mathbb{P}_n$, but $n < m$.

In integral form:

$$V_n^m(w, f, x) := \int_{-1}^{1} H_n^m(w, x, y) f(y) w(y) dy$$

where

$$\begin{cases} H_n^m(w, x, y) := \dfrac{1}{m-n+1} \displaystyle\sum_{r=n}^{m} K_r(w, x, y) & \text{de la Vallée Poussin kernel} \\ K_r(w, x, y) := \displaystyle\sum_{j=0}^{r} p_j(w, x) p_j(w, y) & \text{Darboux kernel} \end{cases}$$

Theorem $[M.R.Capobianco,\ T.]$ For all Jacobi weights $w = v^{\alpha,\beta}$, for any pair of sufficiently large integers $n \sim m \sim m-n$ (i.e. $n < m < \mathcal{C}_1 n$, and $\mathcal{C}_2 m < m-n < m$ with $\mathcal{C}_1, \mathcal{C}_2 > 0$ independent of n, m) and for each $x, y \in \left[-1+\frac{c}{m^2}, 1-\frac{c}{m^2}\right]$ ($c > 0$ fixed), we have

$$|H_n^m(w,x,y)| \leq C m v^{-\frac{\alpha}{2}-\frac{1}{4},-\frac{\beta}{2}-\frac{1}{4}}(x) v^{-\frac{\alpha}{2}-\frac{1}{4},-\frac{\beta}{2}-\frac{1}{4}}(y), \tag{1}$$

If in addition $x \neq y$, then we have

$$|H_n^m(w,x,y)| \leq \quad C \frac{v^{-\frac{\alpha}{2}-\frac{1}{4},-\frac{\beta}{2}-\frac{1}{4}}(x) v^{-\frac{\alpha}{2}-\frac{1}{4},-\frac{\beta}{2}-\frac{1}{4}}(y)}{m|x-y|} E_{\pm}(x,y), \tag{2}$$

where, in the case $|x-y| \geq a > 0$, $E_{\pm}(x,y) \leq C$, and generally

$$E_{\pm}(x,y) := \frac{(\sqrt{1 \pm x} + \sqrt{1 \pm y})^2}{|x-y|} + \frac{\sqrt{1 \pm x} + \sqrt{1 \pm y}}{m\sqrt{1-x^2}\,\sqrt{1-y^2}}.$$

In all the previous estimates, C is a positive constant independent of n, x, y.

Theorem [$M.R.Capobianco,\ T.$] Let $1 \le p \le \infty$ and consider the map $V_n^m(w) : L_u^p \to L_u^p$, where $n \sim m \sim m-n$ and $w = v^{\alpha,\beta}$, $u = v^{\gamma,\delta}$ satisfy the inequalities

$$\frac{\alpha}{2} - \frac{1}{4} < \gamma + \frac{1}{p} < \frac{\alpha}{2} + \frac{5}{4}, \qquad \text{and} \qquad 0 < \gamma + \frac{1}{p} < \alpha + 1,$$

$$\frac{\beta}{2} - \frac{1}{4} < \delta + \frac{1}{p} < \frac{\beta}{2} + \frac{5}{4}, \qquad \text{and} \qquad 0 < \delta + \frac{1}{p} < \beta + 1,$$

Moreover assume $\left|\gamma - \delta - \frac{\alpha-\beta}{2}\right| \le 1$. Then for all $f \in L_u^p$, we have

$$\boxed{\|(V_n^m(w,f)u\|_p \le C\|fu\|_p, \qquad C \ne C(n,m,f).} \qquad (3)$$

OPEN PROBLEM: State necessary and sufficient conditions for (3)

Theorem [$G. Mastroianni,\ T.$] Let $1 \le p \le \infty$, and $w = v^{\alpha,\beta}$, $u = v^{\gamma,\delta}$ be such that the following bounds

$$\frac{\alpha}{2} + \frac{1}{4} - \nu \quad < \gamma + \frac{1}{p} < \quad \frac{\alpha}{2} + \frac{5}{4} - \nu, \qquad \text{and} \qquad 0 < \gamma + \frac{1}{p} < \alpha + 1,$$

$$\frac{\beta}{2} + \frac{1}{4} - \nu \quad < \delta + \frac{1}{p} < \quad \frac{\beta}{2} + \frac{5}{4} - \nu, \qquad \text{and} \qquad 0 < \delta + \frac{1}{p} < \beta + 1,$$

are satisfied for some $\nu \in [0, 1/2]$. Then for all positive integers $n \sim m \sim m - n$ and any $f \in L^p_u$, we have

$$\|(V_n^m(w, f)u\|_p \le C\|fu\|_p, \qquad C \ne C(n, m, f).$$

Limiting cases also possible: $\begin{cases} p = \infty & \text{and} \quad u = 1 \\ p = 1 & \text{and} \quad u = w \end{cases}$ (if $\alpha, \beta < \frac{1}{2}$)

Comparison with Fourier sums

$$S_n(w,f,x) = \int_{-1}^{1} K_n(w,x,y)f(y)w(y)dy,$$

$$V_n^m(w,f,x) = \int_{-1}^{1} H_n^m(w,x,y)f(y)w(y)dy, \qquad m > n$$

- **Invariance:** $\begin{cases} S_n(w,P) & = P, \quad \forall P \in \mathbb{P}_n \quad \text{(projection on } \mathbb{P}_n) \\ V_n^m(w,P) & = P, \quad \forall P \in \mathbb{P}_n \quad \text{(quasi-projection on } \mathbb{P}_m) \end{cases}$

- **Boundedness in C_u^0 and L_u^1:** Let $n \sim m \sim m-n$. Then

$$\begin{cases} \sup_n \|S_n(w)\|_{C_u^0 \to C_u^0} = \sup_n \|S_n(w)\|_{L_u^1 \to L_u^1} = \infty & \text{for any } u, w \\ \sup_n \|V_n^m(w)\|_{C_u^0 \to C_u^0} = \sup_n \|V_n^m(w)\|_{L_u^1 \to L_u^1} < \infty & \text{for suitable } u, w \end{cases}$$

▶ **Boundedness in L^p_u :** Let $n \sim m \sim m - n$ and $1 < p < \infty$. Then

$$\begin{cases} \sup_n \|S_n(w)\|_{L^p_u \to L^p_u} < \infty & \text{for suitable (more restricted) } u \\ \sup_n \|V_n^m(w)\|_{L^p_u \to L^p_u} < \infty & \text{for suitable } u \end{cases}$$

In particular if $w = v^{\alpha,\beta}$ and $u = v^{\gamma,\delta}$ satisfy the technical requirements $0 < \gamma + \frac{1}{p} < \alpha + 1$ and $0 < \delta + \frac{1}{p} < \beta + 1$, then we have

$$\sup_n \|S_n(w)\|_{L^p_u \to L^p_u} < \infty \Leftrightarrow \begin{cases} \frac{\alpha}{2} + \frac{1}{4} < \gamma + \frac{1}{p} < \frac{\alpha}{2} + \frac{3}{4}, \\ \frac{\beta}{2} + \frac{1}{4} < \delta + \frac{1}{p} < \frac{\beta}{2} + \frac{3}{4}, \end{cases}$$

$$\sup_n \|V_n^m(w)\|_{L^p_u \to L^p_u} < \infty \Leftarrow \begin{cases} \frac{\alpha}{2} + \frac{1}{4} - \nu < \gamma + \frac{1}{p} < \frac{\alpha}{2} + \frac{5}{4} - \nu, \\ \frac{\beta}{2} + \frac{1}{4} - \nu < \delta + \frac{1}{p} < \frac{\beta}{2} + \frac{5}{4} - \nu, \end{cases} \quad 0 \le \nu \le \frac{1}{2}$$

ERROR OF BEST POLYNOMIAL APPROXIMATION IN L^p_u

$$E_n(f)_{u,p} := \inf_{deg(P_n)\le n} \|(f-P_n)u\|_p \qquad 1 \le p \le \infty,$$

- Invariance on P_n implies $f - V_n^m(w,f) = (f-P_n) - V_n^m(w, f-P_n)$, i.e.

$$\|[f - V_n^m(w,f)]u\|_p \le (1 + \|V_n^m(w)\|_{L^p_u \to L^p_u})\ E_n(f)_{u,p}$$

- Boundedness in L^p_u is equivalent to:

$$E_m(f)_{u,p} \le \|[f - V_n^m(w,f)]u\|_p \le C E_n(f)_{u,p}$$

i.e. $V_n^m(w,f)$ is a **near best polynomial** approximating $f \in L^p_u$.

Ditzian–Totik moduli of smoothness:

Non weighted case: $\omega_\varphi^r(f,t)_p := \sup_{0<h\le t} \|\Delta_{h\varphi}^r f\|_{L^p[-1,1]}$

$$\Omega_\varphi^r(f,t)_{u,p} := \sup_{0<h\le t} \|(\Delta_{h\varphi}^r f)u\|_{L^p[-1+4r^2h^2,1-4r^2h^2]} \quad \leftarrow \textbf{(main–part)}$$

$$\omega_\varphi^r(f,t)_{u,p} := \Omega_\varphi^r(f,t)_{u,p} + \inf_{deg(P)<r} \|(f-P)u\|_{L^p[-1,-1+4r^2t^2]} + \inf_{deg(P)<r} \|(f-P)u\|_{L^p[1-4r^2t^2,1]}$$

where we set $\varphi(x) := \sqrt{1-x^2}$ and $\Delta_{h\varphi}^r f$ is the central rth difference of f of variable step size $h\varphi(x)$, i.e.

$$\Delta_{h\varphi} f(x) := f\left(x+\frac{h}{2}\,\varphi(x)\right) - f\left(x-\frac{h}{2}\,\varphi(x)\right), \qquad \Delta_{h\varphi}^r f = \Delta\Delta_{h\varphi}^{r-1}$$

Some basic properties: (worth for $\Omega^r_\varphi(f,t)_{u,p}$ too)

$$
\begin{array}{ll}
(i) & \lim_{t\to 0}\omega^r_\varphi(f,t)_{u,p}=0, \quad \forall f\in L^p_u \quad (f\in C^0_u \text{ when } p=\infty)\\
(ii) & t_1\le t_2 \Longrightarrow \omega^r_\varphi(f,t_1)_{u,p}\le \omega^r_\varphi(f,t_2)_{u,p}\\
(iii) & \omega^r_\varphi(f,t)_{u,p}\le C\omega^{r-1}_\varphi(f,t)_{u,p}\\
(iv) & \omega^r_\varphi(f,\lambda t)_{u,p}\le C\lambda^r\omega^r_\varphi(f,t)_{u,p}, \qquad (\lambda>1)\\
(v) & \omega^r_\varphi(f,t)_{u,p}\le C\,t\,\omega^{r-1}_\varphi(f',t)_{u\varphi,p}, \quad (f\in AC_{loc}:\ \|f'u\varphi\|_p<\infty)
\end{array}
$$

Equivalent K–functionals:

$$
\begin{array}{lcl}
\omega^r_\varphi(f,t)_{u,p} & \sim & K^r_\varphi(f,t^r)_{u,p}\\
\Omega^r_\varphi(f,t)_{u,p} & \sim & \tilde{K}^r_\varphi(f,t^r)_{u,p}
\end{array}
$$

where:

$$
\begin{cases}
K^r_\varphi(f,t)_{u,p} := \inf\limits_{g^{(r-1)}\in AC_{loc}} \{\|(f-g)u\|_p + t\|g^{(r)}\varphi^r u\|_p\}\\
\tilde{K}^r_\varphi(f,t)_{u,p} := \sup\limits_{0<h\le t}\ \inf\limits_{g^{(r-1)}\in AC_{loc}} \{\|(f-g)u\|_{L^p(I_{r,h})} + h\|g^{(r)}\varphi^r u\|_{L^p(I_{r,h})}\}\\
\qquad\qquad I_{r,h} := [-1+4r^2h^2, 1-4r^2h^2]
\end{cases}
$$

DIRECT AND CONVERSE ESTIMATES

- **Jackson–type:** $E_n(f)_{u,p} \le C\omega_\varphi^r\left(f,\frac{1}{n}\right)_{u,p}, \qquad r < n$
- **Weak Jackson–type:** $E_n(f)_{u,p} \le C\int_0^{\frac{1}{n}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t}dt, \qquad r < n$
- **Stechkin–type:** $\omega_\varphi^r(f,t)_{u,p} \le Ct^r \sum_{0\le k\le 1/t}(1+k)^{r-1}E_k(f)_{u,p}$

Exercise: $f(x) = |x|^3 \Longrightarrow E_n(f)_\infty = O(n^{-3})$.

COROLLARY: If $0 < a < r$ and $f \in L_u^p$ ($f \in C_u^0$ if $p = \infty$), we have

$$E_n(f)_{u,p} = O(n^{-a}) \Longleftrightarrow \Omega_\varphi^r(f,t)_{u,p} = O(t^a) \Longleftrightarrow \omega_\varphi^r(f,t)_{u,p} = O(t^a)$$

Hölder–Zygmund spaces: We have the norm–equivalence

$$\|fu\|_\infty + \sup_{t>0} \frac{\omega_\varphi^k(f,t)_{u,\infty}}{t^r} \sim \|fu\|_\infty + \sup_{k>0}(k+1)^r E_k(f)_{u,\infty}$$

and we can equivalently define:

$$Z_r(u) = \left\{ f \in C_u^0 : \sup_{t>0} \frac{\omega_\varphi^k(f,t)_{u,\infty}}{t^r} < \infty \right\}$$

$$Z_r(u) = \left\{ f \in C_u^0 : \sup_{k>0}(k+1)^r E_k(f)_{u,\infty} < \infty \right\}$$

Useful to state boundedness properties not true in C_u^0.

CAUCHY SINGULAR INTEGRAL OPERATOR: Let $\alpha, \beta \in]-1,1[-\{0\}$ be such that $\chi := -(\alpha+\beta) \in \{0,1,-1\}$. Define

$$Df(x) = D^{\alpha,\beta}f(x) := \cos\pi\alpha f(x) v^{\alpha,\beta}(x) - \frac{\sin\pi\alpha}{\pi}\int_{-1}^{1}\frac{f(y)}{y-x}v^{\alpha,\beta}(y)dy.$$

Theorem: The map $D : L^2_{\sqrt{v^{\alpha,\beta}}} \to L^2_{\sqrt{v^{-\alpha,-\beta}}}$ is a Fredholm operator of index χ. $\widehat{D} := D^{-\alpha,-\beta}$ is the adjoint operator of $D = D^{\alpha,\beta}$ and it satisfies

$$\widehat{D}D = I, \qquad \chi \in \{-1,0\}; \qquad D\widehat{D} = I, \qquad \chi \in \{0,1\}$$

Moreover the map $D : L^{2,\chi}_{\sqrt{v^{\alpha,\beta}}} \to L^{2,-\chi}_{\sqrt{v^{-\alpha,-\beta}}}$ is invertible, where

$$\chi = 0,-1 \implies L^{2,\chi}_u = L^2_u$$

$$\chi = 1 \implies L^{2,\chi}_u = \left\{ f \in L^2_u : \int_{-1}^{1} f(x)u(x)dx = 0 \right\}$$

Mapping properties w.r.t. the weighted uniform norm

Notations: $$\begin{cases} v := v^{\alpha,\beta}, & v = v_+/v_- \\ v_+ := v^{\max\{\alpha,0\},\max\{\beta,0\}}, & v_- := v^{-\min\{\alpha,0\},-\min\{\beta,0\}} \end{cases}$$

Theorem: $$D^{\alpha,\beta} f \in C^0_{v_-}, \quad \forall f \in \mathcal{A} := \left\{ f \in C^0_{v_+} : \int_0^1 \frac{\omega_\varphi(f,t)_{v_+,\infty}}{t} dt < \infty \right\}$$

Problem: The map $D^{\alpha,\beta} : \mathcal{A} \subset C^0_{v_+} \to C^0_{v_-}$ is not bounded. What about the boundedness of $D^{\alpha,\beta}$ on some suitable subspaces?

- **P.Junghanns, U.Luther** proved boundedness by defining

$$C^{r,s}_u := \left\{ f : \|fu\|_\infty + \sup_{k>0} (k+1)^r \ln^s(k+2) E_k(f)_{u,\infty} \right\}, \ r > 0, s \in \mathbb{R}.$$

- But **de la V.P. mean** $V_n^{2n-1}(v^{-1}, f) = \frac{1}{n} \sum_{k=n}^{2n-1} S_k(v^{-1}, f)$ gives:

Theorem: The map $D^{\alpha,\beta} : Z_r(v_+) \to Z_r(v_-)$ is bounded.

Theorem: For all $f \in \mathcal{A}$ and $k < n$, set $V_n f := V_n^{2n-1}(v^{-1}, f)$, we have

$$E_n(Df)_{v_-,\infty} \le \|[Df - V_{[n/2]}(Df)]v_-\|_\infty \le C \int_0^{\frac{1}{n}} \frac{\omega_\varphi^k(f,t)_{v_+,\infty}}{t}\, dt$$

Proof. By $Dp_m(v) = p_{m-\chi}(v^{-1})$, we get

$$V_n(Df)(x) = \int_{-1}^{1} \left[\frac{1}{n} \sum_{r=n}^{2n-1} \tilde{K}_r(x,y)\right] f(y)v(y)dy,$$

where $\tilde{K}_n(x,y) := \sum_{j=0}^{n} p_j(v^{-1},x)p_{j+\chi}(v,y)$ can be written as

$$\tilde{K}_n(x,y) = \gamma_n \frac{p_{n+\chi+1}(v,y)p_n(v^{-1},x) - p_{n+\chi}(v,y)p_{n+1}(v^{-1},x)}{y-x} - \frac{\sin\pi\alpha}{\pi}\frac{1}{y-x}, \quad \gamma_n \sim 1$$

Then using $Df - V_n(Df) = \sum_{i=0}^{\infty}[V_{2^{i+1}n}(Df) - V_{2^i n}(Df)]$, we estimate the difference kernels and obtain the statement by classical arguments.$\square$

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