Spectra and Finite Sections of Band Operators

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26.-30. September 2011
What is this course about?

- spectral theory
- Fredholm theory
- stable approximation

of infinite matrices \((a_{ij})\), understood as bounded linear operators on a sequence space \(E\).
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of infinite matrices \((a_{ij})\), understood as bounded linear operators on a sequence space \(E\).

Simplest example: \(E = \ell^2(\mathbb{Z}, \mathbb{C})\)

\[
\begin{pmatrix}
\vdots & \vdots & \vdots \\
\cdots & a_{ij} & \cdots \\
\vdots & \vdots & \vdots
\end{pmatrix} : \begin{pmatrix}
\vdots \\
x_j \\
\vdots
\end{pmatrix} \mapsto \begin{pmatrix}
\vdots \\
b_i \\
\vdots
\end{pmatrix}
\]

with indices \(i, j \in \mathbb{Z}\) and entries \(a_{ij}, x_j, b_i \in \mathbb{C}\)
1 Classes of Infinite Matrices
1 Classes of Infinite Matrices

2 The Finite Section Method, Part I

3 Limit Operators
1. Classes of Infinite Matrices
2. The Finite Section Method, Part I
3. Limit Operators
4. The Spectrum: Formulas and Bounds
1. Classes of Infinite Matrices
2. The Finite Section Method, Part I
3. Limit Operators
4. The Spectrum: Formulas and Bounds
5. Spectral Bounds: An Example
6. The Finite Section Method, Part II
In what follows, we consider sequence spaces such as

$$E = \ell^2(\mathbb{Z}, \mathbb{C})$$
In what follows, we consider sequence spaces such as

\[ E = \ell^p(\mathbb{Z}, \mathbb{C}) \]

with

- \( p \in [1, \infty] \)
In what follows, we consider sequence spaces such as

$$E = \ell^p(\mathbb{Z}^N, \mathbb{C})$$

with

- $p \in [1, \infty]$ 
- $N \in \mathbb{N}$
In what follows, we consider sequence spaces such as

\[ E = \ell^p(\mathbb{Z}^N, X) \]

with

- \( p \in [1, \infty] \)
- \( N \in \mathbb{N} \)
- \( X \) ... complex Banach space

Clearly, the entries of a matrix \((a_{ij})\) acting on \( E \) also need to be indexed by \( i, j \in \mathbb{Z}^N \), and the entries \( a_{ij} \) are themselves bounded linear operators \( X \to X \).
Vector-valued $\ell^p$-spaces

$x \in E = \ell^p(\mathbb{Z}^N, X)$ iff $x = (x_k)_{k \in \mathbb{Z}^N}$ with $x_k \in X$ for $k \in \mathbb{Z}^N$ and

\[ \|x\|_E := \sqrt[p]{\sum_{k \in \mathbb{Z}^N} \|x_k\|^p_X} < \infty, \quad p < \infty, \]

\[ \|x\|_E := \sup_{k \in \mathbb{Z}^N} \|x_k\|_X < \infty, \quad p = \infty. \]
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\]

Simplest case

$E = \ell^p := \ell^p(\mathbb{Z}, \mathbb{C}), \quad N = 1, \ X = \mathbb{C}$
Vector-valued $\ell^p$-spaces

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$$

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$$

Somewhat more complicated case

$$
E = L^p(\mathbb{R}^N) \cong \ell^p(\mathbb{Z}^N, X), \quad X = L^p([0,1]^N)
$$

via identification of $f \in L^p(\mathbb{R}^N)$ with $(f\,|_{\alpha+[0,1]^N})_{\alpha \in \mathbb{Z}^N}$
Vector-valued \( \ell^p \) -spaces

\[ x \in E = \ell^p(\mathbb{Z}^N, X) \text{ iff } x = (x_k)_{k \in \mathbb{Z}^N} \text{ with } x_k \in X \text{ for } k \in \mathbb{Z}^N \text{ and } \]

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via identification of \( f \in L^p(\mathbb{R}^N) \) with \( (f|_{\alpha+[0,1]^N})_{\alpha \in \mathbb{Z}^N} \)
Let \( E = \ell^p(\mathbb{Z}^N, X) \) be one of our sequence spaces.
Then we denote by

\[
L(E) \quad \text{... space of all \textbf{bounded linear} operators } E \to E,
\]
\[
K(E) \quad \text{... space of all \textbf{compact} operators } E \to E.
\]

Important:

\( L(E) \) is a Banach algebra and
\( K(E) \) is a closed two-sided ideal in \( L(E) \).
Two basic types of operators

- **Shift operators**, $V_k$

\[ V := V_1 = \begin{pmatrix} \cdots & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \rightarrow \begin{pmatrix} x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \]

In general: $(V_k x)_i = x_{i-k}$ with $i, k \in \mathbb{Z}^N$

Shift operators are isometric and invertible
Two basic types of operators

- **shift operators**, $V_k$
- **multiplication operators**, $M_b$

\[
\begin{pmatrix}
\ddots \\
& b_{-2} \\
& & b_{-1} \\
& & & b_0 \\
& & & & b_1 \\
& & & & & b_2 \\
& & & & & & \ddots
\end{pmatrix}
\begin{pmatrix}
\vdots \\
x_{-2} \\
x_{-1} \\
x_0 \\
x_1 \\
x_2 \\
\vdots
\end{pmatrix}
\mapsto
\begin{pmatrix}
\vdots \\
b_{-2}x_{-2} \\
b_{-1}x_{-1} \\
b_0x_0 \\
b_1x_1 \\
b_2x_2 \\
\vdots
\end{pmatrix}
\]

$M_b$ is bounded if $b = (b_k)$ is so; $\|M_b\| = \|b\|_\infty$
Two basic types of operators

- shift operators, $V_k$
- multiplication operators, $M_b$

Now let them mingle: Take

- scalar multiples
- sums
- products

or combinations of those.

$\Rightarrow$ an operator algebra
An algebra of shifts and multiplications

Typical elements of the algebra look like this:

\[ A = V^2 M_b + 3M_b + M_a V^{-1} \]

i.e.

\[
A = \begin{pmatrix}
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & 3b_2 & a_2 & \ldots & \ldots \\
\ldots & 0 & 3b_1 & a_1 & \ldots \\
 b_2 & 0 & 3b_0 & a_0 & \ldots \\
b_1 & 0 & 3b_1 & a_1 & \ldots \\
b_0 & 0 & 3b_2 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\]

Note: \( M_a M_b = M_{a \cdot b} \), \( V M_b = M_{Vb} V \)
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\ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\]

Note: \( M_a M_b = M_{a \cdot b}, \ VM_b = M_{Vb} V \)
It is easy to see that

\[ A \text{ is a finite sum-product of shifts and multiplications} \]

\[ \iff \]

\[ A \text{ acts as a band matrix} \]

The set of these operators is denoted by \( BO(E) \), where \( BO \) is short for \textbf{band operator}. 
Algebras of shifts and multiplications

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**Band operators**

\[
A \in BO(E) : \quad A = \sum_{k=-w}^{w} M_{b(k)} V^k
\]

The number \( w \) is called the \textbf{band-width} of \( A \).
Band operators

\[ A \in BO(E) : \quad A = \sum_{k=-w}^{w} M_{b(k)} V^k \]

The number \( w \) is called the **band-width** of \( A \).

It is nice to have an algebra of operators (i.e. a set that is closed under addition, multiplication and taking scalar multiples) but it is even nicer to have a **Banach algebra** (also closed w.r.t. \( \| \cdot \| \)).
Algebras of shifts and multiplications

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Band-dominated operators

\[ BDO(E) := \text{clos}_{L(E)} BO(E) \]

The matrices have a certain off-diagonal decay.
The norm under which $BDO(E)$ is closed is the usual operator norm

$$\|A\| := \sup_{\|x\|=1} \|Ax\|.$$ 

Here is another norm: For $A = \sum_{k=-w}^{w} M_{b(k)} V^k \in BO(E)$, we have

$$\|A\| \leq \sum_{k=-w}^{w} \|M_{b(k)}\| \|V^k\| = \sum_{k=-w}^{w} \|b^{(k)}\|_\infty =: \|A\|_\infty.$$
Now let $\mathcal{W}(E)$ denote the completion of $BO(E)$ w.r.t. $[\cdot]$. 

\[ [A] := \sum_{k=-w}^{w} \| b^{(k)} \|_\infty \]
Wiener norm

\[ [A] := \sum_{k=-w}^{w} \| b(k) \|_\infty \]

Now let \( \mathcal{W}(E) \) denote the completion of \( BO(E) \) w.r.t. \([ \cdot , \cdot ]\). This also gives a Banach algebra (w.r.t. \([ \cdot , \cdot ]\)):

Wiener algebra

\[ \mathcal{W}(E) = \left\{ \sum_{k=-\infty}^{+\infty} M_{b(k)} V^k : \sum_{k=-\infty}^{+\infty} \| b(k) \|_\infty < \infty \right\} \]

Clearly: \( BO(E) \subset \mathcal{W}(E) \subset BDO(E) \subset L(E) \)
Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$ and fix a function $a \in L^\infty(\mathbb{T})$.

Fourier series of $a$:
$$
\sum_{k=-\infty}^{+\infty} a_k t^k, \quad t \in \mathbb{T}.
$$

This function is closely related to the associated operator

$$
L(a) := \sum_{k=-\infty}^{+\infty} a_k V^k,
$$

which is a so-called **Laurent operator** (constant matrix diagonals):

$$
L(a) = \\
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots \\
\ddots & a_0 & a_{-1} & a_{-2} \\
\ddots & a_1 & a_0 & a_{-1} \\
a_2 & a_1 & a_0 & \ddots \\
\ddots & \ddots & \ddots & \ddots
\end{pmatrix}
$$
The function $a$ is called the **symbol** of the operator $L(a)$.

$L(a) = \text{discrete convolution by } (a_k)_{k \in \mathbb{Z}} \quad \text{Fourier} \quad \cong \text{multiplication by } a$

For simplicity, suppose $E = \ell^2$. Then $E \quad \text{Fourier} \quad \cong L^2(\mathbb{T})$. 
Example: Laurent operator

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Correspondence between \( L(a) \) and its symbol \( a \) on \( \mathbb{T} \):

<table>
<thead>
<tr>
<th>Laurent operator ( L(a) )</th>
<th>symbol ( a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>bounded operator</td>
<td>bounded function</td>
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<tr>
<td>( |L(a)| ) = ( |a|_\infty )</td>
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| bounded operator \[
\| L(a) \| = \| a \|_\infty
\] | bounded function |
| invertible operator \[
L(a)^{-1} = L(a^{-1})
\] \spec L(a) = a(\mathbb{T}) | no zeros on \( \mathbb{T} \) |
Example: Laurent operator

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\[
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<td>in ( BO(E) )</td>
<td>trig. polynomial</td>
</tr>
<tr>
<td>in ( \mathcal{W}(E) )</td>
<td>Wiener function</td>
</tr>
<tr>
<td>in ( BDO(E) )</td>
<td>continuous function</td>
</tr>
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</table>
Example: Laurent operator

Here we call \( a : \mathbb{T} \to \mathbb{C} \) a Wiener function and write \( a \in W(\mathbb{T}) \) if its Fourier coefficients are summable, i.e. \( (a_k) \in \ell^1 \). Note that

\[
\|a\|_W := \sum_{k=-\infty}^{+\infty} |a_k| = \|[L(a)]\|.
\]

**Wiener’s theorem**

If \( a \in W(\mathbb{T}) \) has no zeros then also \( a^{-1} \in W(\mathbb{T}) \).

**Wiener’s theorem in Laurent operator language**

If \( L(a) \in \mathcal{W}(E) \) is invertible then \( L(a)^{-1} = L(a^{-1}) \in \mathcal{W}(E) \).
This theorem

Wiener’s theorem in Laurent operator language
If \( L(a) \in \mathcal{W}(E) \) is invertible then \( L(a)^{-1} = L(a^{-1}) \in \mathcal{W}(E) \).

has an amazing generalisation:

The Wiener algebra is inverse closed
If \( A \in \mathcal{W}(E) \) is an invertible operator then \( A^{-1} \in \mathcal{W}(E) \).

And now that we’re at it:

Also \( BDO(E) \) is inverse closed
If \( A \in BDO(E) \) is an invertible operator then \( A^{-1} \in BDO(E) \).

The last two theorems hold in the general case \( E = \ell^p(\mathbb{Z}^N, X) \).
1. Classes of Infinite Matrices

2. The Finite Section Method, Part I

3. Limit Operators

4. The Spectrum: Formulas and Bounds

5. Spectral Bounds: An Example

6. The Finite Section Method, Part II
We look at linear equations $Ax = b$ in infinitely many variables:

$$
\begin{pmatrix}
... & : & : & : & \\
\vdots & a_{-1, -1} & a_{-1, 0} & a_{-1, 1} & \cdots \\
\vdots & a_{0, -1} & a_{0, 0} & a_{0, 1} & \cdots \\
\vdots & a_{1, -1} & a_{1, 0} & a_{1, 1} & \cdots \\
... & : & : & : & \\
\end{pmatrix}
\begin{pmatrix}
\vdots \\
x_{-1} \\
x_0 \\
x_1 \\
\vdots \\
\end{pmatrix}
= 
\begin{pmatrix}
\vdots \\
b_{-1} \\
b_0 \\
b_1 \\
\vdots \\
\end{pmatrix}
$$

**Assumption:** $A \in BDO(E)$ with $E = \ell^p$, i.e. $N = 1$ and $X = \mathbb{C}$.

**Task:** Given such an $A$ and a RHS $b \in E$, find $x \in E$. 
Let $A$ be invertible (bijective) as a map $E \rightarrow E$, so that $Ax = b$ is uniquely solvable for every RHS $b$.

How do we compute this unique solution $x$ of $Ax = b$, i.e.

\[ \sum_{j=-\infty}^{+\infty} a_{ij} x_j = b_i, \quad i \in \mathbb{Z} \quad ? \quad (1) \]

Replace the infinite system (1) by the sequence of finite systems

\[ \sum_{j=-n}^{n} a_{ij} x_j = b_i, \quad i = -n, \ldots, n \]

for $n = 1, 2, \ldots$
The Finite Section Method

Let $A$ be invertible (bijective) as a map $E \to E$, so that $Ax = b$ is uniquely solvable for every RHS $b$.

How do we compute this unique solution $x$ of $Ax = b$, i.e.

$$
\sum_{j=-\infty}^{+\infty} a_{ij}x_j = b_i, \quad i \in \mathbb{Z} \quad ?
$$

Or, more flexible: Take two monotonous sequences of integers

$$
-\infty \leftarrow \cdots < l_2 < l_1 < r_1 < r_2 < \cdots \rightarrow +\infty
$$

and replace the infinite system (1) by the sequence of finite systems

$$
\sum_{j=l_n}^{r_n} a_{ij}x_j = b_i, \quad i = l_n, \ldots, r_n
$$

for $n = 1, 2, \ldots$
Graphically, (2) means

We say the **finite section method** is **applicable** to $A$ if the truncated equations (2) are uniquely solvable for all $n > n_0$ and their solutions converge componentwise to the unique solution $x$ of (1).
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All of this should happen **independently** of the right-hand side $b$. So applicability of the method only depends on $A$. 

\[
\begin{pmatrix}
(l_{n, l_n})
\end{pmatrix}
\begin{pmatrix}
(?)
\end{pmatrix} =
\begin{pmatrix}
(l_n) \\
(r_n)
\end{pmatrix},
\quad n = 1, 2, \ldots
\]
Precisely: The finite section method (FSM) is applicable to $A$ iff $A$ is invertible and its so-called finite sections form a stable sequence.

Here we call a sequence $(A_n)$ stable if there exists $n_0$ such that

$$\sup_{n > n_0} \|A_n^{-1}\| < \infty.$$
When the Finite Section Method goes wrong

There are very simple examples where the FSM fails to apply.

**Example 1: a block-flip**

\[
A = \begin{pmatrix}
\ddots & 0 & 1 \\
1 & 0 & 1 \\
1 & 0 & \ddots \\
\end{pmatrix}
: \begin{pmatrix}
\ddots \\
x_{-2} \\
x_{-1} \\
0 \\
x_0 \\
x_1 \\
x_2 \\
\ddots \\
\end{pmatrix} \rightarrow \begin{pmatrix}
\ddots \\
x_{-1} \\
x_{-2} \\
x_0 \\
x_1 \\
x_2 \\
\ddots \\
\end{pmatrix}
\]
There are very simple examples where the FSM fails to apply.

**Example 1: a block-flip**

$$A = \begin{pmatrix} \ldots & 0 & 1 & 1 & 0 & \ldots \\ & 0 & 1 & 1 & 0 & \ldots \\ 0 & 1 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} : \begin{pmatrix} x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ \ldots \end{pmatrix} \mapsto \begin{pmatrix} x_{-1} \\ x_{-2} \\ x_0 \\ x_1 \\ x_2 \\ \ldots \end{pmatrix}$$
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\ldots \\
x_{-2} \\
x_{-1} \\
x_0 \\
x_1 \\
x_2 \\
\ldots \\
\end{pmatrix} \mapsto \begin{pmatrix}
\ldots \\
x_{-1} \\
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\ldots \\
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When the Finite Section Method goes wrong

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  \vdots & 0 & 1 \\
  1 & 0 & 1 \\
  1 & 0 & 1 \\
  \vdots & \vdots & \vdots 
\end{pmatrix}
\begin{pmatrix}
  x_{-2} \\
  x_{-1} \\
  x_0 \\
  x_1 \\
  x_2 \\
  \vdots 
\end{pmatrix} \rightarrow \begin{pmatrix}
  x_{-1} \\
  x_0 \\
  0 \\
  \vdots 
\end{pmatrix}
\]

Only 50\% of the finite systems are uniquely solvable.
There are very simple examples where the FSM fails to apply.

Example 1: a block-flip

\[
A = \begin{pmatrix}
\ldots & \end{pmatrix} = \begin{pmatrix}
0 & 1 & \square & \ldots \\
1 & 0 & 1 & \ldots \\
1 & 0 & \ldots \\
\ldots & \end{pmatrix} \quad \begin{pmatrix}
\ldots \\
x_{-2} \\
x_{-1} \\
x_0 \\
x_1 \\
x_2 \\
\ldots \\
\end{pmatrix} \rightarrow \begin{pmatrix}
\ldots \\
x_{-1} \\
x_{-1} \\
x_0 \\
x_0 \\
x_0 \\
\ldots \\
\end{pmatrix}
\]

Only 50% of the finite systems are uniquely solvable.

Choosing good cut-off intervals \([l_n, r_n]\) will solve the problem!
Example 2: the shift

\[ A = \begin{pmatrix} \vdots & 0 & & & & \\ \vdots & 1 & 0 & & & \\ & 1 & 0 & & & \\ & 1 & 0 & & & \\ & 1 & 0 & & & \\ & \vdots & & & & \end{pmatrix} \begin{pmatrix} \vdots \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} \rightarrow \begin{pmatrix} \vdots \\ x_{-3} \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{pmatrix} \]
Example 2: the shift

\[ A = \begin{pmatrix} \vdots & & & \vdots \\ & 0 & & & \\ & 1 & 0 & & \\ & 1 & 0 & 1 & 0 \\ & & & & \vdots \end{pmatrix} : \begin{pmatrix} x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{pmatrix} \rightarrow \begin{pmatrix} x_{-3} \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{pmatrix} \]
Example 2: the shift

\[
A = \begin{pmatrix}
\vdots \\
\vdots \\
0 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 1 \\
\vdots \\
\vdots 
\end{pmatrix}
\begin{pmatrix}
\vdots \\
\vdots \\
x_{-2} \\
x_{-1} \\
x_0 \\
x_1 \\
x_2 \\
\vdots \\
\vdots 
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\vdots \\
\vdots \\
0 \\
x_{-2} \\
x_{-1} \\
x_0 \\
x_1 \\
\vdots \\
\vdots 
\end{pmatrix}
\]
When the Finite Section Method goes wrong

Example 2: the shift

\[
A = \begin{pmatrix}
\ddots & 0 & 0 & 0 & 0 & \ldots \\
\ddots & 1 & 0 & 0 & 0 & \ldots \\
& 1 & 0 & 1 & 0 & \ldots \\
& & 1 & 0 & 1 & \ldots \\
& & & 1 & 0 & \ldots \\
& & & & \ddots & \ddots \\
\end{pmatrix}
\begin{pmatrix}
x_{-2} \\
x_{-1} \\
x_0 \\
x_1 \\
x_2 \\
\ddots \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\ddots & 0 & 0 & 0 & 0 & \ldots \\
\ddots & 0 & 0 & 0 & 0 & \ldots \\
& 0 & 0 & 1 & 0 & \ldots \\
& & 0 & 1 & 0 & \ldots \\
& & & 0 & 1 & \ldots \\
& & & & \ddots & \ddots \\
\end{pmatrix}
\begin{pmatrix}
x_{-2} \\
x_{-1} \\
x_0 \\
x_1 \\
x_2 \\
\ddots \\
\end{pmatrix}
\]

No one of the finite systems is uniquely solvable.
When the Finite Section Method goes wrong

Example 2: the shift

\[ A = \begin{pmatrix} \ddots & & & & & & & \\ & 0 & 1 & 0 & & & & \\ & 1 & 0 & 1 & 0 & & & \\ & & 1 & 0 & \ddots & & & \\ & & & \ddots & & \end{pmatrix} \begin{pmatrix} \cdots \\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ \cdots \end{pmatrix} \rightarrow \begin{pmatrix} \cdots \\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ 0 \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ \cdots \end{pmatrix} \]

No one of the finite systems is uniquely solvable.

Adapting the cut-off points \([l_n, r_n]\) will not help here! Instead, place the corners of \(A_n\) along another diagonal!
Clearly, there is a lot of room in a bi-infinite matrix and therefore a lot of \textbf{freedom} to place the finite sections.

The previous examples have shown that sometimes one \textbf{needs} to make use of that freedom by

- picking the appropriate “main” diagonal (which one is it?),
- choosing good cut-off sequences \((l_n)\) and \((r_n)\)

We will learn how to do both of that.
We need theorems that tell us when the FSM works and when not.

One can show that applicability of the finite section method
The Finite Section Method

We need theorems that tell us when the FSM works and when not. One can show that applicability of the finite section method is controlled by certain limits of the upper left and lower right corners of the finite sections $A_n$ as $n \to \infty$. 
So we have to follow the two “corners” (semi-infinite matrices)

$$
\begin{pmatrix}
  a_{l_n,l_n} & \cdots \\
  \vdots & \ddots
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
  \vdots & \cdots \\
  a_{r_n,r_n}
\end{pmatrix}
$$

of $A_n$ as $n \to \infty$ and find (partial) limits of these matrix sequences:
Following the corners as they move out to infinity

So we have to follow the two “corners” (semi-infinite matrices)

\[
\begin{pmatrix}
a_{l_n,l_n} & \cdots \\
\vdots & \ddots & \ddots
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\vdots & \cdots & a_{r_n,r_n}
\end{pmatrix}
\]

of \(A_n\) as \(n \to \infty\) and find (partial) limits of these matrix sequences:
Following the corners as they move out to infinity

So we have to follow the two “corners” (semi-infinite matrices)

\[
\begin{pmatrix}
  a_{l_n,l_n} & \cdots \\
  \vdots & \ddots \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  \cdots & \cdots \\
  \cdots & a_{r_n,r_n} \\
\end{pmatrix}
\]

of \( A_n \) as \( n \to \infty \) and find (partial) limits of these matrix sequences:
Following the corners as they move out to infinity

So we have to follow the two “corners” (semi-infinite matrices)

\[
\begin{pmatrix}
    a_{l_n,l_n} & \cdots \\
    \vdots & \ddots \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
    \cdots & \cdots \\
    \cdots & a_{r_n,r_n} \\
\end{pmatrix}
\]

of $A_n$ as $n \to \infty$ and find (partial) limits of these matrix sequences:
This leads to the study of so-called limit operators.

**Definition: Limit Operator**

For a given sequence \( h_1, h_2, ... \in \mathbb{Z} \) with \( |h_n| \to \infty \) and a matrix \( A = (a_{ij})_{i,j \in \mathbb{Z}} \), we call \( B = (b_{ij})_{i,j \in \mathbb{Z}} \) a **limit operator** of \( A \) with respect to that sequence \( h = (h_1, h_2, ...) \) if for all \( i, j \in \mathbb{Z} \),

\[
    a_{i+h_n, j+h_n} \to b_{ij} \quad \text{as} \quad n \to \infty.
\]

We write \( A_h \) instead of \( B \), where \( h = (h_1, h_2, ...) \).

For our FSM, we will use \( l = (l_1, l_2, ...) \) and \( r = (r_1, r_2, ...) \) (or subsequences of those) in place of \( h = (h_1, h_2, ...) \).
One more notation: $A_+$ and $A_-$

Think of a bi-infinite band matrix $A$ as $2 \times 2$ block matrix:

So $A_+$ and $A_-$ are one-sided infinite submatrices of $A$. 
The finite section method (2) is applicable to $A$ iff the following operators are invertible:

$$A, \quad B_-, \quad C_+$$

for all limit operators $B$ of $A$ w.r.t. a subsequence of $r$ and all limit operators $C$ of $A$ w.r.t. a subsequence of $l$.

**Strategy:**
Choose the sequences
$$r = (r_1, r_2, \cdots)$$
and
$$l = (l_1, l_2, \cdots)$$
so that $B_-$ and $C_+$ are invertible!
The FSM for one-sided infinite matrices

For a banded and one-sided infinite matrix \( A = A_+ = (a_{ij})_{i,j \in \mathbb{N}} \), acting boundedly on \( \ell^p(\mathbb{N}) \), the situation is similar:

Now \( l_n \equiv 1 \) is fixed and \( r_1 < r_2 < \cdots \to +\infty \).

**Theorem**

The FSM (2) is applicable \( \text{iff} \)

\[
A \quad \text{and} \quad B_-
\]

are invertible for all limops \( B \) of \( A \) w.r.t. a subsequence of \( r \).

**Strategy:** Again, make sure \( B_- \) is/are invertible – by choosing the sequence \( r = (r_1, r_2, \ldots) \).
Ok, limit operators seem to be useful.

Time to learn more about them!

(We come back to the FSM at some later point.)
1. Classes of Infinite Matrices

2. The Finite Section Method, Part I

3. Limit Operators

4. The Spectrum: Formulas and Bounds

5. Spectral Bounds: An Example

6. The Finite Section Method, Part II
Let $A \in BDO(E)$. Recall:

**Definition: Limit Operator**

For a given sequence $h_1, h_2, \ldots \in \mathbb{Z}$ with $|h_n| \to \infty$ and a matrix $A = (a_{ij})_{i,j \in \mathbb{Z}}$, we call $B = (b_{ij})_{i,j \in \mathbb{Z}}$ a limit operator of $A$ with respect to that sequence $h = (h_1, h_2, \ldots)$ if for all $i, j \in \mathbb{Z}$,

$$a_{i+h_n,j+h_n} \to b_{ij} \quad \text{as} \quad n \to \infty.$$

We write $A_h$ instead of $B$, where $h = (h_1, h_2, \ldots)$.

In short: $A_h$ is the entrywise limit of $V_{-h_n}AV_{h_n}$ as $n \to \infty$.

By $\sigma^{\text{op}}(A)$ we denote the set of all limit operators of $A$. 
Example: Discrete Schrödinger operator

The Schrödinger operator $-\Delta + M_b$ with a bounded potential $b \in L^\infty(\mathbb{R})$ is usually discretized as

$$A = V_{-1} + M_c + V_1 = \begin{pmatrix} \cdots & \cdots \\ \cdots & c_{-2} & 1 \\ & 1 & c_{-1} & 1 \\ & 1 & c_0 & 1 \\ & 1 & c_1 & 1 \\ & 1 & c_2 & \cdots \\ & \cdots & \cdots & \cdots \end{pmatrix}$$

where $c = (\ldots, c_{-1}, c_0, c_1, \ldots) \in \ell^\infty(\mathbb{Z})$. 

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Example: Discrete Schrödinger operator

The Schrödinger operator $-\Delta + M_b$ with a bounded potential $b \in L^\infty(\mathbb{R})$ is usually discretized as

$$A = V_{-1} + M_c + V_1 = \begin{pmatrix}
  \ldots & \ldots & \ldots & \ldots \\
  \ldots & c_{-2} & 1 & \ldots \\
  \ldots & 1 & c_{-1} & 1 \\
  1 & c_0 & 1 & \ldots \\
  1 & c_1 & 1 & \ldots \\
  1 & c_2 & \ldots & \ldots \\
\end{pmatrix}$$

where $c = (\ldots, c_{-1}, c_0, c_1, \ldots) \in \ell^\infty(\mathbb{Z})$. Clearly,

$$A_h = (V_{-1})_h + (M_c)_h + (V_1)_h = V_{-1} + (M_c)_h + V_1,$$

so that everything depends on the limit operators of $M_c$ only.
Example 1: Periodic potential

If there is a $P \in \mathbb{N}$ such that

$$c_{k+P} = c_k \quad \text{for every} \quad k \in \mathbb{Z},$$

then $\sigma^{\text{op}}(M_c) = \left\{ M_{V_{kc}} : k \in \{0, 1, \ldots, P-1\} \right\}$. 
Example 1: Periodic potential

If there is a $P \in \mathbb{N}$ such that

$$c_{k+P} = c_k$$

for every $k \in \mathbb{Z}$,

then $\sigma^{op}(M_c) = \left\{ M_{V_k c} : k \in \{0, 1, \ldots, P-1\} \right\}$. 
Example: Discrete Schrödinger operator

Example 1: Periodic potential

If there is a $P \in \mathbb{N}$ such that

$$c_{k+P} = c_k$$

for every $k \in \mathbb{Z}$,

then $\sigma^{op}(M_c) = \left\{ M_{V_k}c : k \in \{0, 1, \ldots, P - 1\} \right\}$. 

But then $\sigma^{op}(A) = \left\{ V_{-k}AV_k : k \in \{0, 1, \ldots, P - 1\} \right\}$. 

---

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Spectra and Finite Sections of Band Operators
**Example: Discrete Schrödinger operator**

**Example 2: Almost-periodic potential**

**Note:** $c \in \ell^\infty$ is **periodic** iff the set $\{V_k c : k \in \mathbb{Z}\}$ of all its translates is **finite**.

E.g. if $c = (\cdots, c_1, c_2, c_3, c_1, c_2, c_3, \cdots)$ then

$$\{V_k c : k \in \mathbb{Z}\} = \{V_0 c = (\cdots, c_1, c_2, c_3, c_1, c_2, c_3, \cdots), V_1 c = (\cdots, c_3, c_1, c_2, c_3, c_1, c_2, \cdots), V_2 c = (\cdots, c_2, c_3, c_1, c_2, c_3, c_1, \cdots) \}$$
Example: Discrete Schrödinger operator

Example 2: Almost-periodic potential

Note: $c \in \ell^\infty$ is periodic iff the set $\{V_k c : k \in \mathbb{Z}\}$ of all its translates is finite.

Definition: $c \in \ell^\infty$ is almost-periodic iff the set $\{V_k c : k \in \mathbb{Z}\}$ of all its translates is relatively compact in $\ell^\infty$.

![](image)

The set $h(c) := \text{clos}\{V_k c : k \in \mathbb{Z}\} \subset \ell^\infty$ is called the hull of $c$.

In that case $\sigma^{\text{op}}(M_c) = \{M_d : d \in h(c)\} = \text{clos}\{M_{V_k c} : k \in \mathbb{Z}\}$, $\sigma^{\text{op}}(A) = \{V_{-1} + M_d + V_1 : d \in h(c)\} = \text{clos}\{V_{-1} A V_k : k \in \mathbb{Z}\}$. 
Example 2: Almost-periodic potential (continued)

Fix $\lambda \in \mathbb{R}$ and look at $c = (c_k)_{k \in \mathbb{Z}} \in \ell^\infty$ with entries

$$c_k := e^{i\lambda k}, \quad k \in \mathbb{Z}.$$ 

If $\lambda/\pi$ is rational then $c$ is periodic; otherwise it is almost-periodic and its hull is

$$h(c) = \left\{ d = \left( e^{i(\lambda k + \gamma)} \right)_{k \in \mathbb{Z}} : \gamma \in [0, 2\pi) \right\}.$$
Example 2: Almost-periodic potential (continued)

Fix $\lambda \in \mathbb{R}$ and look at $c = (c_k)_{k \in \mathbb{Z}} \in \ell^\infty$ with entries

$$c_k := e^{i\lambda k}, \quad k \in \mathbb{Z}.$$ 

If $\lambda/\pi$ is rational then $c$ is periodic; otherwise it is almost-periodic and its hull is

$$h(c) = \left\{ d = (e^{i(\lambda k+\gamma)})_{k \in \mathbb{Z}} : \gamma \in [0, 2\pi) \right\}.$$

Slightly more advanced: If

$$c_k := \alpha e^{i\lambda k} + \beta e^{i\mu k}, \quad k \in \mathbb{Z}$$

with $\lambda/\pi$, $\mu/\pi$ and $\lambda/\mu$ all irrational then the hull is

$$h(c) = \left\{ d = \left(\alpha e^{i(\lambda k+\gamma)} + \beta e^{i(\mu k+\delta)}\right)_{k \in \mathbb{Z}} : \gamma, \delta \in [0, 2\pi) \right\}.$$
Example 2: Almost-periodic potential (continued 2)

For the so-called Almost-Mathieu operator, the potential $c$ is the real part of what we studied earlier:

Fix $\alpha, \lambda \in \mathbb{R}$ and take $c = (c_k)_{k \in \mathbb{Z}} \in \ell^\infty$ with entries

$$c_k := \alpha \cos(\lambda k), \quad k \in \mathbb{Z}.$$ 

If $\lambda/\pi$ is irrational then $c$ is almost-periodic (but non-periodic) and its hull is

$$h(c) = \left\{ d = (\alpha \cos(\lambda k + \gamma))_{k \in \mathbb{Z}} : \gamma \in [0, 2\pi) \right\}.$$ 

**Ten-Martini problem:** Show that $\text{spec} (V_{-1} + M_c + V_1)$ is a Cantor set. (Puig 2003, Avila&Jitomiskaya 2005)
Example 3: Slowly oscillating potential

If

\[ c_{k+1} - c_k \to 0 \quad \text{as} \quad k \to \pm \infty, \]

then \( \sigma^{\text{op}}(M_c) = \{aI : a \in c(\infty)\} \).

Now all limit operators \( A_h \) are Laurent operators (i.e., they have constant diagonals).
Example 3: Slowly oscillating potential

If

\[ c_{k+1} - c_k \to 0 \quad \text{as} \quad k \to \pm \infty, \]

then \( \sigma^{\text{op}}(M_c) = \{ aI : a \in c(\infty) \} \).

Now all limit operators \( A_h \) are Laurent operators (i.e., they have constant diagonals).

For example, let \( c = (c_k)_{k \in \mathbb{Z}} \) with

\[ c_k = \sin \sqrt{|k|}, \quad k \in \mathbb{Z}. \]

Then \( c \) is slowly oscillating and \( c(\infty) = [-1, 1] \).
Example 4: Pseudo-ergodic potential

Let $\Sigma$ be a compact subset of $\mathbb{C}$.

Definition

A sequence $c = (c_k)_{k \in \mathbb{Z}}$ over $\Sigma$ is called \textbf{pseudoergodic} over $\Sigma$ if every finite vector $f = (f_i)_{i \in F}$ with values $f_i \in \Sigma$ can be found, up to arbitrary precision $\varepsilon > 0$, somewhere inside the infinite sequence $c$, i.e.

$$\forall \varepsilon > 0 \; \exists m \in \mathbb{Z} : \max_{i \in F} |c_{i+m} - f_i| < \varepsilon.$$  

Idea: Model random behaviour by a purely deterministic concept.
Example 4: Pseudo-ergodic potential

Let \( \Sigma \) be a compact subset of \( \mathbb{C} \).

**Definition**

A sequence \( c = (c_k)_{k \in \mathbb{Z}} \) over \( \Sigma \) is called **pseudoergodic** over \( \Sigma \) if every finite vector \( f = (f_i)_{i \in F} \) with values \( f_i \in \Sigma \) can be found, up to arbitrary precision \( \varepsilon > 0 \), somewhere inside the infinite sequence \( c \), i.e.

\[
\forall \varepsilon > 0 \quad \exists m \in \mathbb{Z} : \quad \max_{i \in F} |c_{i+m} - f_i| < \varepsilon.
\]

**Idea:** Model random behaviour by a purely deterministic concept.

**Example:** \( dec(\pi) = 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, \ldots \) is pseudo-ergodic over \( \Sigma = \{0, 1, 2, \ldots, 9\} \) (conjecture)

**Example:** \( bin(1), bin(2), bin(3), \ldots \) is pseudo-erg. over \( \Sigma = \{0, 1\} \).
Example 4: Pseudo-ergodic potential (continued)

If \( c \) is pseudo-ergodic over \( \Sigma \) then \textbf{every} multiplication operator \( M_d \) with a sequence \( d = (\ldots, d_{-1}, d_0, d_1, \ldots) \) over \( \Sigma \) is a limit operator of \( M_c \):

\[
\sigma^{\text{op}}(M_c) = \{ M_d : d : \mathbb{Z} \to \Sigma \} \tag{3}
\]

The statement also holds the other way round!

So \( c \) is pseudo-ergodic \textbf{iff} (3) holds.
Example 5: A locally constant potential

\[ c = (\ldots, \beta, \beta, \beta, \beta, \alpha, \alpha, \alpha, \beta, \beta, \alpha, \beta, \beta, \alpha, \alpha, \alpha, \beta, \beta, \beta, \beta, \beta, \ldots). \]
Example 5: A locally constant potential

\[ c = (\ldots, \beta, \beta, \beta, \beta, \alpha, \alpha, \alpha, \beta, \beta, \alpha, \beta, \beta, \alpha, \alpha, \alpha, \beta, \beta, \beta, \beta, \ldots) \].

Then all limit operators of \( A \) are of the form

\[
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \beta & 1 & \cdots & \cdots \\
\cdots & 1 & \beta & \cdots & \cdots \\
\cdots & \cdots & \cdots & \ddots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \alpha & 1 \\
\cdots & \cdots & \cdots & \cdots & 1 & \alpha \\
\cdots & \cdots & \cdots & \cdots & \cdots & \ddots \\
\end{pmatrix},
\]

or they are translates of the latter two matrices.
Example 5: A locally constant potential

\[ c = (\ldots, \beta, \beta, \beta, \beta, \alpha, \alpha, \alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \alpha, \beta, \beta, \beta, \beta, \ldots). \]

Then all limit operators of \( A \) are of the form

\[
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots \\
\ddots & \beta & 1 & \ddots \\
1 & \beta & \ddots & \ddots \\
\end{pmatrix}, \\
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots \\
1 & \alpha & 1 & \ddots \\
\ddots & \ddots & \alpha & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}, \\
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots \\
\ddots & \beta & 1 & \ddots \\
1 & \alpha & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}, \\
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots \\
1 & \beta & \ddots & \ddots \\
\ddots & \ddots & \beta & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\]
Example 5: A locally constant potential

\[ c = (\ldots, \beta, \beta, \beta, \beta, \alpha, \alpha, \alpha, \beta, \beta, \alpha, \alpha, \alpha, \beta, \beta, \beta, \beta, \beta, \ldots). \]

Then all limit operators of \( A \) are of the form

\[
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots \\
\beta & 1 & & \\
1 & \beta & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix},
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots \\
& \alpha & 1 & \\
1 & \alpha & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix},
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots \\
\beta & 1 & & \\
& & & \\
1 & \alpha & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix},
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots \\
& \alpha & 1 & \\
1 & \beta & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix},
\]

or they are translates of the latter two matrices.
Let $A, B \in BDO(E)$ and $h = (h_1, h_2, \cdots)$ be a sequence of integers going to $\pm \infty$.

Recall: limit op $A_h$ is the entrywise limit of $V_{-h_n}AV_{h_n}$ as $n \to \infty$.

**Basic rules**

If the right-hand side exists then also the left-hand side exists and equality holds:

\[
(A + B)_h = A_h + B_h \\
(AB)_h = A_h B_h \\
(\alpha A)_h = \alpha A_h \\
(\lim A_n)_h = \lim (A_n)_h \quad (w.r.t. \ op-\| \cdot \|)
\]

$\Rightarrow$ Compute limit operators of elements of an operator algebra in terms of limit operators of generators of the algebra.

Moreover, $\|A_h\| \leq \|A\|$.
Besides studying the FSM, limit operators are also important for determining the essential spectrum of $A$.

Here we define

**Definition: Essential Spectrum**

For a (not necessarily self-adjoint) operator $A$, we denote by

$$\operatorname{spec}_{\text{ess}} A := \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm} \}$$

the essential spectrum of $A$. 
Fredholm operators
Fredholm operators

E

A

E

{0}_{\text{im A}}
Fredholm operators

Let $A$ be an operator. The kernel of $A$, denoted $\ker A$, is a subset of the domain space $E$. The image of $A$, denoted $\text{im} A$, is a subset of the codomain space $E$. The operator $A$ maps elements from $\ker A$ to $\text{im} A$, with $\text{im} A$ containing the zero vector $0$. This diagram illustrates the relationship between the kernel, image, and the operator $A$. 
A \colon E \to E \text{ is a Fredholm operator if } \alpha := \dim(\ker A) \text{ and } \beta := \operatorname{codim}(\operatorname{im} A) \text{ are both finite. The difference } \alpha - \beta \text{ is then called the index of } A.
Fredholm operators

Definition

A : E → E is a **Fredholm operator** if \( \alpha := \dim(\ker A) \) and \( \beta := \operatorname{codim}(\operatorname{im} A) \) are both finite. The difference \( \alpha - \beta \) is then called the **index of A**.
Fredholm operators

**Definition**

A : $E \to E$ is a **Fredholm operator** if $\alpha := \dim(\ker A)$ and $\beta := \operatorname{codim}(\operatorname{im} A)$ are both finite. The difference $\alpha - \beta$ is then called the **index of** $A$.

$A \in L(E)$ is Fredholm iff $A + K(E)$ is invertible in $L(E)/K(E)$. 

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Take $A \in \mathcal{W}(E)$.

Then it is not hard to see that

$$A \text{ Fredholm} \implies \text{all limit operators of } A \text{ are invertible.}$$
Limit Operators vs. Fredholmness

Take $A \in \mathcal{W}(E)$.

Then it is not hard to see that

$A$ Fredholm $\implies$ all limit operators of $A$ are invertible.

In fact, also the reverse implication holds:

**Theorem**

Rabinovich, Roch, Silbermann 1998; ML 2003

The following are equivalent for all $p \in [1, \infty]$:

- $A$ is Fredholm on $\ell^p$,
- all limit operators of $A$ are invertible on $\ell^p$, 

...moreover, the Fredholm index of $A$ does not depend on $p$.

If we repeat the same argument with $A - I$ in place of $A$, we get:

**Essential Spectrum**

Rabinovich, Roch, Silbermann 1998; ML 2003; Chandler-Wilde, ML 2007

$$\text{spec}_{\text{ess}}(A) = \left[ \text{spec}_{\text{ess}}(Ah) \right] = \left[ \text{spec}_{\text{ess}}(Ah) \right]$$

$p \in [1, \infty]$.
Take \( A \in \mathcal{W}(E) \).

Then it is not hard to see that

\[
A \text{ Fredholm} \implies \text{all limit operators of } A \text{ are invertible.}
\]

In fact, also the reverse implication holds:

**Theorem** \( \text{Rabinovich, Roch, Silbermann 1998; ML 2003; Chandler-Wilde, ML 2007} \)

The following are equivalent for all \( p \in [1, \infty] \):

- \( A \) is Fredholm on \( \ell^p \),
- all limit operators of \( A \) are invertible on \( \ell^p \),
- all limit operators of \( A \) are injective on \( \ell^\infty \).

...moreover, the **Fredholm index** of \( A \) does not depend on \( p \).
The following are equivalent for all \( p \in [1, \infty] \):

- \( A \) is **Fredholm** on \( \ell^p \),
- all limit operators of \( A \) are **invertible** on \( \ell^p \),
- all limit operators of \( A \) are **injective** on \( \ell^\infty \).

...moreover, the **Fredholm index** of \( A \) does not depend on \( p \).

If we repeat the same argument with \( A - \lambda I \) in place of \( A \), we get:

<table>
<thead>
<tr>
<th>Essential Spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{spec}^p_{\text{ess}}(A) = \bigcup_{h} \text{spec}^p(A_h) = \bigcup_{h} \text{spec}^\infty_{\text{point}}(A_h), \quad p \in [1, \infty] )</td>
</tr>
</tbody>
</table>

---

**Marko Lindner**

Spectra and Finite Sections of Band Operators
Think of a bi-infinite band matrix $A$ as $2 \times 2$ block matrix:

\[
A = \begin{pmatrix}
A_-
&
A
\end{pmatrix}
\]

Then

- $A$ is Fredholm iff $A_+$ and $A_-$ are Fredholm
- $\text{spec}_{\text{ess}} A = \text{spec}_{\text{ess}} A_+ \cup \text{spec}_{\text{ess}} A_-$
- $\text{ind} A = \text{ind} A_+ + \text{ind} A_-$
Limit operators vs. Fredholm index

\[ C = \begin{pmatrix} \text{invertible} \end{pmatrix} \]

\[ A = \begin{pmatrix} \text{Fredholm} \end{pmatrix} \]

\[ B = \begin{pmatrix} \text{invertible} \end{pmatrix} \]
Limit operators vs. Fredholm index

\[ C = \begin{pmatrix} + \end{pmatrix} \text{ invertible} \]

\[ A = \begin{pmatrix} & \cr & \cr \end{pmatrix} \text{ Fredholm} \]

\[ B = \begin{pmatrix} + \end{pmatrix} \text{ invertible} \]
Limit operators vs. Fredholm index

\[ \begin{align*}
  C &= \left( \begin{array}{c}
    \text{invertible}
  \end{array} \right) \\
  A &= \left( \begin{array}{c}
    \text{Fredholm}
  \end{array} \right) \\
  B &= \left( \begin{array}{c}
    \text{invertible}
  \end{array} \right)
\end{align*} \]
Limit operators vs. Fredholm index

\[ C = \begin{pmatrix} k_- \\ + \end{pmatrix} \text{ invertible} \]

\[ A = \begin{pmatrix} \begin{pmatrix} k_- \\ \end{pmatrix} \\ \begin{pmatrix} k_+ \\ \end{pmatrix} \end{pmatrix} \text{ Fredholm} \]

\[ B = \begin{pmatrix} + \\ k_+ \end{pmatrix} \text{ invertible} \]
Limit operators vs. Fredholm index

\[ C = \begin{pmatrix} k_- & 1 \\ 1 & -k_- \end{pmatrix} \quad \text{invertible} \]

\[ A = \begin{pmatrix} k_- & \text{Fredholm} \\ \text{Fredholm} & k_+ \end{pmatrix} \]

\[ B = \begin{pmatrix} -k_+ & 1 \\ 1 & k_+ \end{pmatrix} \quad \text{invertible} \]
Limit operators vs. Fredholm index

\[ \text{ind } A = \text{ind } A_+ + \text{ind } A_- \]

Theorem | Rabinovich, Roch, Roe 2004
--- | ---

\[ \text{ind } A_+ = \text{ind } B_+ \quad \text{for all limops } B \text{ of } A \text{ at } +\infty \]

\[ \text{ind } A_- = \text{ind } C_- \quad \text{for all limops } C \text{ of } A \text{ at } -\infty \]
Example: Random Jacobi Operator

Let $U$, $V$ and $W$ be compact sets in $\mathbb{C}$ and

$$
A = \begin{pmatrix}
\vdots & \vdots & \vdots \\
\vdots & v_{-2} & w_{-1} \\
u_{-2} & v_{-1} & w_{0} \\
u_{-1} & v_{0} & w_{1} \\
u_{0} & v_{1} & w_{2} \\
u_{1} & v_{2} & \ddots \\
\vdots & \ddots & \vdots
\end{pmatrix}
$$

with iid entries

$$u_i \in U, \ v_i \in V \text{ and } w_i \in W.$$
Example: Random Jacobi Operator

Let $U$, $V$ and $W$ be compact sets in $\mathbb{C}$ and

$$A = \begin{pmatrix}
\ddots & \ddots & & \\
\ddots & \ddots & \ddots & \\
& v_{-2} & w_{-1} & \\
u_{-2} & v_{-1} & w_0 & \\
u_{-1} & v_0 & w_1 & \\
u_0 & v_1 & w_2 & \\
u_1 & v_2 & \ddots & \\
& & & \ddots
\end{pmatrix} \quad \text{(4)}$$

with iid entries

$$u_i \in U, \ v_i \in V \text{ and } w_i \in W. \quad \text{(5)}$$

Then, almost surely, $A$ is pseudoergodic (i.e. it contains all finite tridiagonal matrices with diagonals in $U$, $V$ and $W$).
Example: Random Jacobi Operator

Let $U$, $V$ and $W$ be compact sets in $\mathbb{C}$ and

$$
A = \begin{pmatrix}
\ddots & \ddots & \ddots \\
\ddots & v_{-2} & w_{-1} \\
& u_{-2} & v_{-1} & w_0 \\
& & u_{-1} & v_0 & w_1 \\
& & & u_0 & v_1 & w_2 \\
& & & & u_1 & v_2 & \ddots \\
& & & & & \ddots & \ddots
\end{pmatrix}
$$

(4)

with iid entries

$$
u_i \in U, \ v_i \in V \text{ and } w_i \in W.
$$

(5)

$\Rightarrow$ The set of all limops of $A$ equals the set of all operators of the form (4) with entries (5). That includes all Laurent operators.
So, if the random (meaning pseudoergodic) operator $A$ is Fredholm then, for all choices $u \in U$, $v \in V$ and $w \in W$: 

\[
\text{ind } A_- = \text{ind } \begin{pmatrix} \ldots & \ldots & w \\ \vdots & u & v \\ v & w \end{pmatrix} \\
\text{ind } A_+ = \text{ind } \begin{pmatrix} \vdots & \vdots & \vdots \\ u & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix}
\]
So, if the random (meaning pseudoergodic) operator $A$ is Fredholm then, for all choices $u \in U$, $v \in V$ and $w \in \mathcal{W}$:

$$\text{ind } A_\neq = \text{ind } \begin{pmatrix} 
\ddots & \ddots & \ddots \\
\ddots & \ddots & w \\
u & v & \ddots
\end{pmatrix} = \text{wind}(E_{u,w}, v)$$

$$\text{ind } A_\neq = \text{ind } \begin{pmatrix} 
v & w \\
u & \ddots & \ddots \\
\ddots & \ddots
\end{pmatrix} = -\text{wind}(E_{u,w}, v)$$
So, if the random (meaning pseudoergodic) operator $A$ is Fredholm then, for all choices $u \in U$, $v \in V$ and $w \in W$:

$$\text{ind } A_- = \text{wind}(E_{u,w}, v), \quad \text{ind } A_+ = -\text{wind}(E_{u,w}, v)$$
The previous slide has shown how ‘hard’ it is for a pseudoergodic Jacobi operator to be Fredholm. Let us underline this. Put

\[
J(U, V, W) := \{ \text{Jacobi ops (4)} : u_i \in U, v_i \in V, w_i \in W \},
\]

\[
\Psi E(U, V, W) := \{ A \in J(U, V, W) : A \text{ pseudoergodic} \}.
\]

\[
A = \begin{pmatrix}
\cdots & \cdots & & & & \\
\cdot & \cdot & v_{-2} & w_{-1} & & \\
\cdot & v_{-1} & & w_0 & & \\
u_{-1} & v_0 & w_1 & & & \\
u_0 & v_1 & w_2 & & & \\
u_1 & v_2 & & & & \\
\cdots & \cdots & & & & \\
\end{pmatrix}
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\Psi E(U, V, W) := \{ A \in J(U, V, W) : A \text{ pseudoergodic} \}.
\]

For \( A \in \Psi E(U, V, W) \), the set of limops is \textbf{all of} \( J(U, V, W) \). Hence, the following are equivalent:

- \( A \) is Fredholm,
- \( A \) is invertible,
- all \( B \in J(U, V, W) \) are Fredholm,
- all \( B \in J(U, V, W) \) are invertible.
Example: Random Jacobi Operator

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In other words:

\[
\text{spec}_{\text{ess}} A = \text{spec} A = \bigcup \text{spec}_{\text{ess}} B = \bigcup \text{spec} B, 
\]

with the unions taken over all \( B \in J(U, V, W) \).
Example: Random Jacobi Operator

\[ J(U, V, W) := \{ \text{Jacobi ops (4)} : u_i \in U, v_i \in V, w_i \in W \}, \]
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In other words:

\[ \specess A = \spec A = \bigcup \specess B = \bigcup \spec B, \]

with the unions taken over all \( B \in J(U, V, W) \).

In particular,

\[ \spec A \supseteq \bigcup_{\text{Laurent}} \spec L = \bigcup_{u,v,w} (v + E_{u,w}). \]
We come back to discrete Schrödinger operators (i.e. discretisations of $-\Delta + M_b$):

$$A = V_{-1} + M_c + V_1 = \begin{pmatrix}
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
\end{pmatrix}$$

where $c = (...) \in \ell^\infty(\mathbb{Z})$. Clearly,

$$A_h = (V_{-1})_h + (M_c)_h + (V_1)_h = V_{-1} + (M_c)_h + V_1,$$

so that everything depends on the limit operators of $M_c$ only.
Example 1: Periodic potential

If

\[ c_{k+P} = c_k \quad \text{for every} \quad k \in \mathbb{Z}, \]

then \( \sigma^{\text{op}}(M_c) = \left\{ M_{V_k c} : k \in \{0, 1, \ldots, P - 1\} \right\}. \)
Example 1: Periodic potential

If

\[ c_{k+P} = c_k \quad \text{for every} \quad k \in \mathbb{Z}, \]

then \( \sigma^{\text{op}}(M_c) = \left\{ M_{V_k c} : k \in \{0, 1, \ldots, P - 1\} \right\}. \)

But then \( \sigma^{\text{op}}(A) = \left\{ V_{-k} A V_k : k \in \{0, 1, \ldots, P - 1\} \right\}. \)

Consequently, \( A \) is invertible iff any/all of its limit operators are invertible. So in this case, \( A \) is \textbf{Fredholm} iff it is \textbf{invertible}, and

\[ \text{spec}_{\text{ess}} A = \text{spec} A = \text{spec}_{\text{point}} A \]
Example 2: Almost-periodic potential

Let $c$ be almost-periodic with hull $h(c) = \text{clos}\{ V_k c : k \in \mathbb{Z} \}$.

\[
\sigma^\text{op}(M_c) = \{ M_d : d \in h(c) \} = \text{clos}\{ M_{V_k c} : k \in \mathbb{Z} \},
\]
\[
\sigma^\text{op}(A) = \{ V_{-1} + M_d + V_1 : d \in h(c) \} = \text{clos}\{ V_{-k} A V_k : k \in \mathbb{Z} \}.
\]
Example 2: Almost-periodic potential

Let \( c \) be almost-periodic with hull \( h(c) = \text{clos}\{ V_k c : k \in \mathbb{Z} \} \).

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\]
\[
\sigma^{\text{op}}(A) = \{V_{-1} + M_d + V_1 : d \in h(c)\} = \text{clos}\{V_{-k} A V_k : k \in \mathbb{Z}\}.
\]

If any \( A_h = \lim_{n \to \infty} V_{-h_n} A V_{h_n} \) is invertible then \( V_{-h_n} A V_{h_n} \) is invertible for large \( n \), so that \( A \) itself is invertible!

Hence

\[ A \text{ Fredholm} \iff A \text{ invertible} \iff \text{any } A_h \text{ invertible} \]

\[
\text{spec}_{\text{ess}} A = \text{spec } A = \text{spec } A_h = \bigcup_{h} \text{spec}^{\infty}_{\text{point}} A_h
\]
Example 3: Slowly oscillating potential

If
\[ c_{i+1} - c_i \to 0 \quad \text{as} \quad i \to \pm \infty, \]
then \( \sigma^{\text{op}}(M_c) = \{ \alpha I : \alpha \in c(\infty) \} \).
Example 3: Slowly oscillating potential

If

\[ c_{i+1} - c_i \to 0 \quad \text{as} \quad i \to \pm \infty, \]

then \( \sigma^{\text{op}}(M_c) = \{ \alpha I : \alpha \in c(\infty) \} \).

But then all limit operators \( A_h \) are Laurent operators (i.e., they have constant diagonals), for which invertibility & spectrum are well-understood.

\[ \text{spec}_{\text{ess}} A = \bigcup \text{spec } A_h = c(\infty) + [-2, 2] \]
Summary: Limit operators help us to determine the essential spectrum. So they give us lower bounds on the spectrum. It would be good to also have upper bounds!

Classical upper bounds

- Gershgorin circles
- Numerical range

We will discuss another approach.
We study bi-infinite matrices of the form

\[
A = \begin{pmatrix}
\vdots & \vdots & \cdots & & \beta_{-2} & \gamma_{-1} \\
\vdots & \alpha_{-2} & \beta_{-1} & \gamma_{0} \\
\alpha_{-1} & \beta_{0} & \gamma_{1} \\
\alpha_{0} & \beta_{1} & \gamma_{2} \\
\alpha_{1} & \beta_{2} & \ddots & \ddots & \ddots
\end{pmatrix},
\]

where \( \alpha = (\alpha_i) \), \( \beta = (\beta_i) \) and \( \gamma = (\gamma_i) \) are bounded sequences of complex numbers (more general: operators on a Banach space).
Upper spectral bounds

\[ A = \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & & \beta_{-2} & \gamma_{-1} & & \\ & \alpha_{-2} & \beta_{-1} & \gamma_{0} & & \\ & & \alpha_{-1} & \beta_{0} & \gamma_{1} & \\ & & & \alpha_{0} & \beta_{1} & \gamma_{2} \\ & & & & \alpha_{1} & \beta_{2} & \ddots \end{pmatrix} \]

Task

Compute upper bounds on spectrum and pseudospectra of \( A \), understood as a bounded linear operator \( \ell^2 \to \ell^2 \).
One more notion: Pseudospectrum

For $A \in L(E)$ and $\varepsilon > 0$, we put

$$\text{spec } A = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ not invertible on } E \}.$$

The sets $\text{spec } A$, $0$, are the so-called $\varepsilon$-pseudospectra of $A$. It holds that $\text{spec } A =: \text{spec } 0 A \subset \text{spec } 1 A \subset \text{spec } 2 A$, $0 < 1 < 2$. 
One more notion: Pseudospectrum

For $A \in L(E)$ and $\varepsilon > 0$, we put

$$\begin{align*}
\text{spec } A &= \{ \lambda \in \mathbb{C} : A - \lambda I \text{ not invertible on } E \}, \\
\text{spec}_{\text{ess}} A &= \{ \lambda \in \mathbb{C} : A - \lambda I \text{ not Fredholm on } E \},
\end{align*}$$

The sets $\text{spec } A$, $\text{spec}_{\text{ess}} A$, $\text{spec}_{\text{0}} A$, $\text{spec}_{\text{1}} A$, and $\text{spec}_{\text{2}} A$ are the so-called "pseudo-spectra" of $A$. It holds that

$$\text{spec } A =: \text{spec}_{\text{0}} A \subset \text{spec}_{\text{1}} A \subset \text{spec}_{\text{2}} A,$$

$=$ if $A$ is normal.
For $A \in L(E)$ and $\varepsilon > 0$, we put

\[
\begin{align*}
\text{spec} \ A &= \{ \lambda \in \mathbb{C} : A - \lambda I \text{ not invertible on } E \}, \\
\text{spec}_{\text{ess}} \ A &= \{ \lambda \in \mathbb{C} : A - \lambda I \text{ not Fredholm on } E \}, \\
\text{spec}_{\varepsilon} \ A &= \text{spec} \ A \cup \{ \lambda \in \mathbb{C} : \| (A - \lambda I)^{-1} \| > 1/\varepsilon \}
\end{align*}
\]
For $A \in L(E)$ and $\varepsilon > 0$, we put

\[
\begin{align*}
\text{spec } A &= \{ \lambda \in \mathbb{C} : A - \lambda I \text{ not invertible on } E \}, \\
\text{spec}_{\text{ess}} A &= \{ \lambda \in \mathbb{C} : A - \lambda I \text{ not Fredholm on } E \}, \\
\text{spec } \varepsilon A &= \text{spec } A \cup \{ \lambda \in \mathbb{C} : \| (A - \lambda I)^{-1} \| > 1/\varepsilon \} \\
&= \bigcup_{\| T \| < \varepsilon} \text{spec } (A + T)
\end{align*}
\]
For $A \in L(E)$ and $\varepsilon > 0$, we put

$$\text{spec } A \ = \ \{ \lambda \in \mathbb{C} : A - \lambda I \text{ not invertible on } E \}$$

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$$\text{spec}_{\varepsilon} A \ = \ \text{spec } A \cup \{ \lambda \in \mathbb{C} : \| (A - \lambda I)^{-1} \| > 1/\varepsilon \}$$

$$= \bigcup_{\| T \| < \varepsilon} \text{spec } (A + T)$$

$$\supseteq \text{spec } A + \varepsilon \mathbb{D} \quad (\text{“=” if } A \text{ is normal}).$$

The sets $\text{spec}_{\varepsilon} A$, $\varepsilon > 0$, are the so-called $\varepsilon$–pseudospectra of $A$. It holds that

$$\text{spec } A \ = : \ \text{spec}_0 A \subset \text{spec}_{\varepsilon_1} A \subset \text{spec}_{\varepsilon_2} A, \quad 0 < \varepsilon_1 < \varepsilon_2.$$
Reminder: Gershgorin’s circles

Here is our tridiagonal bi-infinite matrix:

\[
\begin{bmatrix}
\alpha & \beta & \gamma \\
\beta & \gamma & 0 \\
\gamma & 0 & \beta \\
\end{bmatrix}
\]
Reminder: Gershgorin’s circles

Here is our tridiagonal bi-infinite matrix:

For every row $k$, consider the disk with center at $a_{k,k}$ and radius $|a_{k,k-1}| + |a_{k,k+1}|$.
Reminder: Gershgorin’s circles

Here is our tridiagonal bi-infinite matrix:

For every row \( k \), consider the disk with

center at \( a_{k,k} \) and radius \( |a_{k,k-1}| + |a_{k,k+1}| \)

Now take the union over all \( k \in \mathbb{Z} \). \( \Rightarrow \) upper bound on spec \( A \).
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Now take the union over all \( k \in \mathbb{Z} \). \( \Rightarrow \text{upper bound} \) on spec \( A \).
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Now take the union over all $k \in \mathbb{Z}$. $\Rightarrow$ upper bound on spec $A$. 
Our new strategy

Look at (pseudo)spectra of the finite principal submatrices of $A$:
Our new strategy

Look at (pseudo)spectra of the finite principal submatrices of $A$:
Look at (pseudo)spectra of the **finite principal submatrices** of $A$:
Our new strategy

Look at (pseudo)spectra of the finite principal submatrices of $A$:
Let $\lambda \in \text{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.

\[ \| (A - \lambda I) x \| < \varepsilon \| x \| \]
Method 1: Finite principal submatrices

Let $\lambda \in \text{spec}_\mathcal{E}(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.

\[ \| (A - \lambda I) x \| < \varepsilon \| x \| \]
Let $\lambda \in \text{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.

Claim: $\exists k \in \mathbb{Z}$:

$$\|(A - \lambda I)x\| < \varepsilon \|x\|$$

$$\|(A_{n,k} - \lambda I_n)x_{n,k}\| < (\varepsilon + \varepsilon_n) \|x_{n,k}\|$$
Let $\lambda \in \text{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.

\[ \| (A - \lambda I)x \| < \varepsilon \| x \| \]

**Claim:** $\exists k \in \mathbb{Z}:$

\[
\| (A_{n,k} - \lambda I_n)x_{n,k} \| < (\varepsilon + \varepsilon_n) \| x_{n,k} \|
\]

\[
\sum_k \| (A_{n,k} - \lambda I_n)x_{n,k} \|^2 < (\varepsilon + \varepsilon_n)^2 \sum_k \| x_{n,k} \|^2
\]
Let $\lambda \in \text{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.

\[
\| (A - \lambda I) x \| < \varepsilon \| x \|
\]

**Claim:** $\exists k \in \mathbb{Z}$:

\[
\| (A_{n,k} - \lambda I_n) x_{n,k} \| < (\varepsilon + \varepsilon_n) \| x_{n,k} \|
\]

\[
\| (A_{n,k} - \lambda I_n) x_{n,k} \|^2 < (\varepsilon + \varepsilon_n)^2 \| x_{n,k} \|^2
\]
Let $\lambda \in \text{spec}_e(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.

\[ \| (A - \lambda I) x \| < \varepsilon \| x \| \]

\[ \| (A_{n,k} - \lambda I_n) x_{n,k} \| < (\varepsilon + \varepsilon_n) \| x_{n,k} \| \]
Let $\lambda \in \text{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.

\[
\| (A - \lambda I)x \| < \varepsilon \| x \|
\]

\[
\| (A_{n,k} - \lambda I_n)x_{n,k} \| < (\varepsilon + \varepsilon_n) \| x_{n,k} \|
\]

\[
\varepsilon_n < \frac{1}{\sqrt{n}} (\| \alpha \|_\infty + \| \gamma \|_\infty)
\]
Method 1: Finite principal submatrices

Let \( \lambda \in \text{spec}_{\varepsilon}(A) \) and let \( x \in \ell^2 \) be a corresponding pseudomode.

\[
\| (A - \lambda I)x \| < \varepsilon \| x \|
\]

\[
\| (A_{n,k} - \lambda I_n)x_{n,k} \| < (\varepsilon + \varepsilon_n) \| x_{n,k} \|
\]

\[
\varepsilon_n < \frac{\pi}{n}(\| \alpha \|_{\infty} + \| \gamma \|_{\infty})
\]
Method 1: Finite principal submatrices

Let $\lambda \in \text{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.

\[
\| (A - \lambda I)x \| \ < \ \varepsilon \| x \|
\]

\[
\| (A_{n,k} - \lambda I_n) x_{n,k} \| 
\ < (\varepsilon + \varepsilon_n) \| x_{n,k} \|
\]

\[
\varepsilon_n \ < \ \frac{\pi}{n} (\| \alpha \|_\infty + \| \gamma \|_\infty)
\]

\[
\Rightarrow \lambda \in \text{spec}_{\varepsilon + \varepsilon_n}(A_{n,k})
\]
Let $\lambda \in \text{spec}_\varepsilon (A)$ and let $x \in \ell^2$ be a corresponding pseudomode.

\[
\| (A - \lambda I) x \| < \varepsilon \| x \|
\]

\[
\| (A_{n,k} - \lambda I_n) x_{n,k} \| < (\varepsilon + \varepsilon_n) \| x_{n,k} \|
\]

\[
\varepsilon_n < \frac{\pi}{n} (\| \alpha \|_\infty + \| \gamma \|_\infty)
\]

\[
\Rightarrow \lambda \in \text{spec}_{\varepsilon + \varepsilon_n} (A_{n,k})
\]
Method 1: Finite principal submatrices

Let $\lambda \in \text{spec}_{\varepsilon}(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.

$$\| (A - \lambda I)x \| < \varepsilon \| x \|$$

$$\| (A_{n,k} - \lambda I_n)x_{n,k} \| < (\varepsilon + \varepsilon_n) \| x_{n,k} \|$$

$$\varepsilon_n < \frac{\pi}{n}(\| \alpha \|_\infty + \| \gamma \|_\infty)$$

$$\Rightarrow \lambda \in \text{spec}_{\varepsilon + \varepsilon_n}(A_{n,k})$$
Let $\lambda \in \text{spec}_\varepsilon(A)$ and let $x \in \ell^2$ be a corresponding pseudomode.

\[
\| (A - \lambda I) x \| < \varepsilon \| x \|
\]

\[
\| (A_{n,k} - \lambda I_n) x_{n,k} \| < (\varepsilon + \varepsilon_n) \| x_{n,k} \|
\]

\[
\varepsilon_n < \frac{\pi}{n}(\| \alpha \|_{\infty} + \| \gamma \|_{\infty})
\]

$\Rightarrow \lambda \in \text{spec}_{\varepsilon + \varepsilon_n}(A_{n,k})$
So one gets

\[ \text{Upper Bound} \]

\[ \text{spec}_\varepsilon(A) \subset \bigcup_{k \in \mathbb{Z}} \text{spec}_{\varepsilon + \varepsilon_n}(A_n,k), \]

where

\[ \varepsilon_n < \frac{\pi}{n} (\|\alpha\|_\infty + \|\gamma\|_\infty). \]

In particular, \( \varepsilon_n \to 0 \) as \( n \to \infty \).
Method 2: **Periodised** finite principal submatrices

If the finite submatrices $A_{n,k}$ are “periodised”,

\[ \text{spec}(A) \subset [k \in \mathbb{Z} \mid \text{spec}(A_{\text{per}, n, k})] \]

but this upper bound on $\text{spec}(A)$ generally seems sharper than in method 1.
Method 2: **Periodised** finite principal submatrices

If the finite submatrices $A_{n,k}$ are “periodised”,

![Diagram of periodised submatrices](image-url)
Method 2: *Periodised* finite principal submatrices

If the finite submatrices $A_{n,k}$ are “periodised”,

very similar computations show that, again,

$$\text{spec}_\varepsilon(A) \subset \bigcup_{k \in \mathbb{Z}} \text{spec}_{\varepsilon + \varepsilon_n}(A_{n,k}^{\text{per}})$$

with

$$\varepsilon_n < \frac{\pi}{n}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

but this upper bound on $\text{spec}_\varepsilon(A)$ generally seems sharper than in method 1.
Look at the shift operator

\[(Ax)(i) = x(i + 1), \text{ i.e. } A = \begin{pmatrix} \cdots & 0 & 1 & 1 & \cdots \\ \cdots & 0 & 1 & \cdots \\
\end{pmatrix}.\]
Look at the shift operator

$$(Ax)(i) = x(i+1), \text{ i.e. } A = \begin{pmatrix} \cdot & 0 & 1 & \cdot \\ 0 & 1 & 1 & 0 \\ \cdot & 0 & 1 & \cdot \end{pmatrix}. \quad (1)$$

$$A_{n,k} = \begin{pmatrix} 0 & 1 & 1 & \cdot \\ 0 & 1 & 1 & 0 \\ \cdot & 0 & 1 & 1 \end{pmatrix}$$
Look at the shift operator

\[(Ax)(i) = x(i + 1), \quad \text{i.e.} \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}.\]

\[A_{n,k} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{spec} \ A_{n,k} = \begin{pmatrix} \text{points} \end{pmatrix} \]
Look at the shift operator

\[(Ax)(i) = x(i + 1), \quad \text{i.e.} \quad A = \begin{pmatrix}
0 & 1 \\
0 & 1 \\
\end{pmatrix}.\]

\[A_{n,k} = \begin{pmatrix}
0 & 1 & 1 \\
0 & 1 & 1 \\
\end{pmatrix} \quad \text{spec } A_{n,k} = \]
Method 1 vs. Method 2: An Example

Look at the shift operator

\[(Ax)(i) = x(i + 1), \quad \text{i.e.} \quad A = \begin{pmatrix} \cdot & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.\]

\[A_{n,k} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{spec } A_{n,k} = \]

\[A_{n,k}^{\text{per}} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \]
Method 1 vs. Method 2: An Example

Look at the shift operator

\[(Ax)(i) = x(i + 1), \text{i.e. } A = \begin{pmatrix} \cdot & 0 & 1 & 1 \\ \cdot & 0 & 1 & 0 \\ \cdot & 0 & 1 & 0 \\ \cdot & 0 & 1 & 0 \end{pmatrix}.\]

\[A_{n,k} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}\quad \text{spec } A_{n,k} = \]

\[A_{n,k}^{\text{per}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}\quad \text{spec } A_{n,k}^{\text{per}} = \]
Method 1 vs. Method 2: An Example

Look at the shift operator

\[(Ax)(i) = x(i + 1), \text{ i.e. } A = \begin{pmatrix} \cdot & 1 & 0 & 0 \\ 0 & \cdot & 1 & 0 \\ 0 & 0 & \cdot & 1 \\ 0 & 0 & 0 & \cdot \end{pmatrix}.\]

\[A_{n,k} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{spec } A_{n,k} = \]

\[A_{n,k}^{\text{per}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{spec } A_{n,k}^{\text{per}} = \]
Method 1 vs. Method 2: An Example

Look at the shift operator

\[(Ax)(i) = x(i + 1), \text{ i.e. } A = \begin{pmatrix} \cdots & 0 & 1 & 1 \\ \cdots & 0 & 0 & 1 \\ \cdots & 0 & 0 & 0 \\ \end{pmatrix}.\]

\[A_{n,k} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ \end{pmatrix} \quad \text{spec}_{\varepsilon_n} A_{n,k} = \]

\[A_{n,k}^{\text{per}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ \end{pmatrix} \quad \text{spec } A_{n,k}^{\text{per}} = \]
Method 1 vs. Method 2: An Example

Look at the shift operator

\[(Ax)(i) = x(i + 1), \text{ i.e. } A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.\]

\[A_{n,k} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{spec}_{\varepsilon_n} A_{n,k} = \]

\[A_{n,k}^{\text{per}} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{spec}_{\varepsilon_n} A_{n,k}^{\text{per}} = \]
Method 1 vs. Method 2: An Example

Look at the shift operator

\[(Ax)(i) = x(i + 1), \quad \text{i.e.} \quad A = \begin{pmatrix}
0 & 1 & 1 \\
\vdots & 0 & \vdots \\
\end{pmatrix}.\]
Look at the shift operator

$$(Ax)(i) = x(i + 1), \text{ i.e. } A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$
Method 1 vs. Method 2: An Example

Look at the shift operator

\[(Ax)(i) = x(i + 1), \quad \text{i.e.} \quad A = \begin{pmatrix} 0 & 1 & \cdots \\ 0 & 0 & 1 \end{pmatrix}.\]
Look at the shift operator

\[(Ax)(i) = x(i + 1), \text{ i.e. } A = \begin{pmatrix} 0 & 1 & 1 \\ \vdots & \ddots & \end{pmatrix} \].
Method 1 vs. Method 2: An Example

Look at the shift operator

$$(Ax)(i) = x(i+1), \text{ i.e. } A = \begin{pmatrix} \cdot & 0 & 1 & 1 \\ 0 & \cdot & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$
Method 1 vs. Method 2: An Example

Look at the shift operator

\[(Ax)(i) = x(i + 1), \text{ i.e. } A = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & \cdots \\ \end{pmatrix}.\]
Look at the shift operator

\[(Ax)(i) = x(i + 1), \quad \text{i.e.} \quad A = \begin{pmatrix} 0 & 1 & 1 \\ \cdot & 0 & 1 \\ \cdot & \cdot & 0 \end{pmatrix}.\]
Method 1 vs. Method 2: An Example

Look at the shift operator

\[(Ax)(i) = x(i + 1), \quad \text{i.e.} \quad A = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & & \\ & & & \end{pmatrix}.\]
Method 1 vs. Method 2

Summary on Methods 1 & 2

- Both methods give upper bounds on $\text{spec } A$ and $\text{spec}_\varepsilon A$. 
Method 1 vs. Method 2

Summary on Methods 1 & 2

- Both methods give **upper bounds** on $\text{spec } A$ and $\text{spec}_\varepsilon A$.
- The bound from Method 2 always appears to be **sharper**.
Method 1 vs. Method 2

Summary on Methods 1 & 2

- Both methods give upper bounds on $\text{spec } A$ and $\text{spec}_\varepsilon A$.
- The bound from Method 2 always appears to be sharper.
- Conjecture: Method 2 converges to $\text{spec}_\varepsilon A$ as $n \to \infty$. 
Both methods give **upper bounds** on \( \text{spec} A \) and \( \text{spec}_\varepsilon A \).

The bound from Method 2 always appears to be **sharper**.

Conjecture: Method 2 **converges** to \( \text{spec}_\varepsilon A \) as \( n \to \infty \).

Method 1 also works for **semi-infinite** and **finite** matrices \( A \)!
Here is another idea: Method 3

Instead of

\[ \alpha, \beta, \gamma \]

\[ k+1 \]

\[ k+n \]
We do a “one-sided” truncation.
Here is another idea: Method 3

We do a **one-sided** truncation.

In a sense, we work with **rectangular** finite submatrices.

This is motivated by work of Davies 1998 and Hansen 2008.
(Also see Heinemeyer/ML/Potthast [SIAM Num. Anal. 2007].)
For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let $P_{n,k} : \ell^2 \to \ell^2$ denote the projection

$$(P_{n,k}x)(i) := \begin{cases} x(i), & i \in \{k + 1, \ldots, k + n\}, \\ 0 & \text{otherwise}. \end{cases}$$

Further, we put $X_{n,k} := \text{im} \ P_{n,k}$ and identify it with $\mathbb{C}^n$ in the obvious way.
Method 3: Truncations

Method 1:

\[ P_{n,k}(A - \lambda I)P_{n,k}|_{x_{n,k}} \]

Method 3:

\[ (A - \lambda I)P_{n,k}|_{x_{n,k}} \]
Method 3: Truncations

Method 1:

\[ P_{n,k}(A - \lambda I)P_{n,k}|x_{n,k} \]

Method 3:

\[ (A - \lambda I)P_{n,k}|x_{n,k} \]
Method 1:

\( \lambda \in \text{spec}_\epsilon(A) \implies \text{For some } k \in \mathbb{Z} : \)

\( \lambda \in \text{spec}_{\epsilon + \epsilon_n}(P_n,kAP_n,k|X_{n,k}) \)
Method 3: Method 1 revisited

Method 1:

\[ \lambda \in \text{spec}_\varepsilon(A) \implies \text{For some } k \in \mathbb{Z} : \]

\[ \lambda \in \text{spec}_{\varepsilon + \varepsilon_n}(P_n,kAP_n,k|X_{n,k}) \]

i.e. \[ s_{\text{min}}(P_n,k(A - \lambda I)P_n,k) < \varepsilon + \varepsilon_n \]
Method 3: Method 1 revisited

Method 1:

\[ \lambda \in \text{spec}_\varepsilon(A) \implies \text{For some } k \in \mathbb{Z} : \]

\[ \lambda \in \text{spec}_{\varepsilon + \varepsilon_n}(P_{n,k}AP_{n,k}|x_{n,k}) \]

i.e. \[ s_{\text{min}}(P_{n,k}(A - \lambda I)P_{n,k}) < \varepsilon + \varepsilon_n \]

\[ \min \text{spec} \left( (P_{n,k}(A - \lambda I)P_{n,k})^* (P_{n,k}(A - \lambda I)P_{n,k}) \right) < (\varepsilon + \varepsilon_n)^2 \]
Method 3: Method 1 revisited

Method 1:

\[ \lambda \in \text{spec}_\varepsilon(A) \implies \text{For some } k \in \mathbb{Z} : \]

\[ \lambda \in \text{spec}_{\varepsilon + \varepsilon_n}(P_{n,k}AP_{n,k}|_{x_{n,k}}) \]

i.e. \[ s_{\min}(P_{n,k}(A - \lambda I)P_{n,k}) < \varepsilon + \varepsilon_n \]

\[ \min \text{spec} \left( (P_{n,k}(A - \lambda I)^*P_{n,k})(P_{n,k}(A - \lambda I)P_{n,k}) \right) < (\varepsilon + \varepsilon_n)^2 \]
Method 3: Method 1 revisited

Method 1:

\[ \lambda \in \text{spec}_\varepsilon(A) \implies \text{For some } k \in \mathbb{Z} : \]

\[ \lambda \in \text{spec}_{\varepsilon+\varepsilon_n}(P_{n,k}AP_{n,k}|x_{n,k}) \]

i.e. \[ s_{\min}(P_{n,k}(A - \lambda I)P_{n,k}) < \varepsilon + \varepsilon_n \]

\[ \min \text{spec}\left(P_{n,k}(A - \lambda I)^*P_{n,k}P_{n,k}(A - \lambda I)P_{n,k}\right) < (\varepsilon + \varepsilon_n)^2 \]
Method 3: Method 1 revisited

Method 1:

\[ \lambda \in \text{spec}_\varepsilon(A) \implies \text{For some } k \in \mathbb{Z} : \]

\[ \lambda \in \text{spec}_{\varepsilon + \varepsilon_n}(P_{n,k}AP_{n,k}|x_{n,k}) \]

i.e. \[ s_{\min}(P_{n,k}(A - \lambda I)P_{n,k}) < \varepsilon + \varepsilon_n \]

\[ \min \text{spec} \left( P_{n,k}(A - \lambda I)^*P_{n,k}(A - \lambda I)P_{n,k} \right) < (\varepsilon + \varepsilon_n)^2 \]
Method 1:

\[ \lambda \in \text{spec}_\varepsilon(A) \implies \text{For some } k \in \mathbb{Z} : \]

\[ \lambda \in \text{spec}_{\varepsilon + \varepsilon_n} (P_n,k A P_{n,k} | x_{n,k}) \]

i.e. \[ s_{\text{min}} (P_{n,k} (A - \lambda I) P_{n,k}) < \varepsilon + \varepsilon_n \]

Idea: \[ \min \text{spec} \left( P_{n,k} (A - \lambda I) \ast P_{n,k} (A - \lambda I) P_{n,k} \right) < (\varepsilon + \varepsilon_n)^2 \]
Method 3: Method 1 revisited

Method 1:

\[ \lambda \in \text{spec}_{\varepsilon}(A) \implies \text{For some } k \in \mathbb{Z} : \]

\[ \lambda \in \text{spec}_{\varepsilon + \varepsilon_n}(P_{n,k}AP_{n,k}|x_{n,k}) \]

i.e. \[ \min_{\lambda} \left( P_{n,k}(A - \lambda I)P_{n,k} \right) < \varepsilon + \varepsilon_n \]

Idea: \[ \min \text{spec} \left( P_{n,k}(A - \lambda I)^*P_{n,k}(A - \lambda I)P_{n,k} \right) < \left( \varepsilon + \varepsilon_n \right)^2 \]
Method 1:

\[ \lambda \in \text{spec}_\varepsilon(A) \implies \text{For some } k \in \mathbb{Z} : \]

\[ \lambda \in \text{spec}_{\varepsilon + \varepsilon_n}(P_{n,k}AP_{n,k} | x_{n,k}) \]

\[ \text{i.e. } s_{\min}(P_{n,k}(A - \lambda I)P_{n,k}) < \varepsilon + \varepsilon_n \]

\[ \text{Idea: } \min \text{ spec } (P_{n,k}(A - \lambda I)^*P_{n,k}(A - \lambda I)P_{n,k}) < (\varepsilon + \varepsilon_n)^2 \]
Method 1:

Idea: \[
\min \text{spec} \left( P_{n,k}(A - \lambda I)^* P_{n,k}(A - \lambda I) P_{n,k} \right) < (\varepsilon + \varepsilon_n)^2
\]
Method 1:

Idea: \[ \min \text{spec} \left( P_{n,k}(A - \lambda I)^*P_{n,k}(A - \lambda I)P_{n,k} \right) < (\varepsilon + \varepsilon_n)^2 \]

Method 3

Let \( \gamma_{\varepsilon}^{n,k}(A) \) be the set of all \( \lambda \in \mathbb{C} \), for which

\[
\min \text{spec} \left( P_{n,k}(A - \lambda I)^*(A - \lambda I)P_{n,k} \right) < (\varepsilon + \varepsilon_n)^2
\]
Method 3: Method 1 revisited

Method 1:

Idea: \( \min \text{spec} \left( P_{n,k}(A - \lambda I)^* P_{n,k}(A - \lambda I) P_{n,k} \right) < (\varepsilon + \varepsilon_n)^2 \)

Method 3

Let \( \gamma_{\varepsilon}^{n,k}(A) \) be the set of all \( \lambda \in \mathbb{C} \), for which

\[
\min \text{spec} \left( P_{n,k}(A - \lambda I)^* (A - \lambda I) P_{n,k} \right) < (\varepsilon + \varepsilon_n)^2
\]

and

\[
\min \text{spec} \left( P_{n,k}(A - \lambda I)(A - \lambda I)^* P_{n,k} \right) < (\varepsilon + \varepsilon_n)^2.
\]
Method 1:

Idea: \( \min \text{spec} \left( P_{n,k}(A - \lambda I)^* P_{n,k}(A - \lambda I) P_{n,k} \right) < (\varepsilon + \varepsilon_n)^2 \)

Method 3

Let \( \gamma_{\varepsilon}^{n,k}(A) \) be the set of all \( \lambda \in \mathbb{C} \), for which

\[
\min \text{spec} \left( P_{n,k}(A - \lambda I)^* (A - \lambda I) P_{n,k} \right) < (\varepsilon + \varepsilon_n)^2 \quad \text{and} \quad \min \text{spec} \left( P_{n,k}(A - \lambda I)(A - \lambda I)^* P_{n,k} \right) < (\varepsilon + \varepsilon_n)^2.
\]

Then put

\[
\Gamma_{\varepsilon}^{n}(A) := \bigcup_{k \in \mathbb{Z}} \gamma_{\varepsilon}^{n,k}(A).
\]
Method 3: Spectral bounds

Again we get (as in Methods 1 & 2)

**Upper Bound**

\[
\text{spec}_\varepsilon(A) \subset \bigcup_{k \in \mathbb{Z}} \gamma^{n,k}_{\varepsilon+\varepsilon_n}(A) = \Gamma^{n}_{\varepsilon+\varepsilon_n}(A)
\]

with \( \varepsilon_n < \frac{\pi}{n}(\|\alpha\|_\infty + \|\gamma\|_\infty) \)

and this time the upper bound looks even sharper than before.
Again we get (as in Methods 1 & 2)

**Upper Bound**

\[
\text{spec}_{\varepsilon}(A) \subset \bigcup_{k \in \mathbb{Z}} \gamma_{\varepsilon + \varepsilon n}^{n,k}(A) = \Gamma_{\varepsilon + \varepsilon n}^n(A)
\]

with \( \varepsilon_n < \frac{\pi}{n}(\|\alpha\|_{\infty} + \|\gamma\|_{\infty}) \)

and this time the upper bound looks even sharper than before. But now we also have

**Lower Bound**

\[
\Gamma_{\varepsilon}^n(A) \subset \text{spec}_{\varepsilon}(A).
\]
Method 3: Spectral bounds

From the lower and upper bound

\[ \Gamma^n_\epsilon(A) \subset \text{spec}_\epsilon(A) \quad \text{and} \quad \text{spec}_\epsilon(A) \subset \Gamma^n_{\epsilon+\epsilon_n}(A) \]

we get

**Sandwich 1**

\[ \Gamma^n_\epsilon(A) \subset \text{spec}_\epsilon(A) \subset \Gamma^n_{\epsilon+\epsilon_n}(A) \]
Method 3: Spectral bounds

From the lower and upper bound

\[ \Gamma^n_\varepsilon(A) \subset \text{spec}_\varepsilon(A) \quad \text{and} \quad \text{spec}_\varepsilon(A) \subset \Gamma^n_{\varepsilon+\varepsilon_n}(A) \]

we get

**Sandwich 1**

\[ \Gamma^n_\varepsilon(A) \subset \text{spec}_\varepsilon(A) \subset \Gamma^n_{\varepsilon+\varepsilon_n}(A) \]

**Sandwich 2**

\[ \text{spec}_\varepsilon(A) \subset \Gamma^n_{\varepsilon+\varepsilon_n}(A) \subset \text{spec}_{\varepsilon+\varepsilon_n}(A). \]
Method 3: Spectral bounds

From the lower and upper bound

\[ \Gamma_n^\varepsilon(A) \subset \text{spec}_\varepsilon(A) \quad \text{and} \quad \text{spec}_\varepsilon(A) \subset \Gamma_n^{\varepsilon+\varepsilon}(A) \]

we get

**Sandwich 1**

\[ \Gamma_n^\varepsilon(A) \subset \text{spec}_\varepsilon(A) \subset \Gamma_n^{\varepsilon+\varepsilon}(A) \]

**Sandwich 2**

\[ \text{spec}_\varepsilon(A) \subset \Gamma_n^{\varepsilon+\varepsilon}(A) \subset \text{spec}_{\varepsilon+\varepsilon}(A). \]

In particular, it follows that

\[ \Gamma_n^{\varepsilon+\varepsilon}(A) \to \text{spec}_\varepsilon(A), \quad n \to \infty \]

in the Hausdorff metric.
Methods 1, 2 & 3: The Shift Operator

Marko Lindner

Spectra and Finite Sections of Band Operators
Methods 2 & 3: The Shift Operator

Marko Lindner
Spectra and Finite Sections of Band Operators
We now look at a matrix $A$ with 3-periodic diagonals:

main diagonal: $\cdots, -\frac{3}{2}, 1, 1, \cdots$

super-diagonal: $\cdots, 1, 2, 1, \cdots$
Methods 1, 2 & 3: Second Example

Marko Lindner
Spectra and Finite Sections of Band Operators
We now look at a matrix $A$ with 3-periodic diagonals:

- main diagonal: $\cdots, -\frac{1}{2}, 1, 1, \cdots$
- super-diagonal: $\cdots, 1, 1, 1, \cdots$
Methods 1, 2 & 3: Third Example

Spectral inclusion set: $n = 4$

Spectral inclusion set: $n = 8$

Spectral inclusion set: $n = 16$

Spectral inclusion set: $n = 32$

Marko Lindner  Spectra and Finite Sections of Band Operators
Method 3: Schrödinger operator with Cantor spectrum

Marko Lindner
Spectra and Finite Sections of Band Operators
1. Classes of Infinite Matrices
2. The Finite Section Method, Part I
3. Limit Operators
4. The Spectrum: Formulas and Bounds
5. Spectral Bounds: An Example
6. The Finite Section Method, Part II
Exploring Non-Hermitian Random Matrices

largely terra incognita!

Consider

\[ H = \begin{pmatrix}
1 & 0 & \pm 1 \\
0 & 1 & \pm 1 \\
\pm 1 & 0 & 1
\end{pmatrix} \]

+1 or -1 with probability \(\frac{1}{2}\)

If \(H\) regarded as Hamiltonian of quantum particle

\[ \sum_j H_{ij} \psi_j = E \psi_i \]

\[ \Rightarrow \psi_{i-1} + t \psi_{i+1} = E \psi_i, \quad t = \pm 1 \]

Feinberg & Zee.
Nuc. Phys.

What does the spectrum look like?
Look at the bi-infinite matrix

\[ A^b = \begin{pmatrix} \ddots & & & & \vdots \\ \vdots & 0 & 1 & & \vdots \\ b_{-1} & 0 & 1 & & \vdots \\ b_0 & 0 & 1 & & \vdots \\ b_1 & 0 & \ddots & & \end{pmatrix}, \]

where \( b = (\cdots, b_{-1}, b_0, b_1, \cdots) \in \{\pm 1\}^\mathbb{Z} \) is a pseudoergodic sequence.
Look at the bi-infinite matrix

\[ A^b = \begin{pmatrix}
\ldots & \ldots & \ldots & \ldots & 0 & 1 \\
\ldots & \ldots & \ldots & \ldots & b_{-1} & 0 & 1 \\
\ldots & \ldots & \ldots & \ldots & b_0 & 0 & 1 \\
\ldots & \ldots & \ldots & \ldots & b_1 & 0 & \ddots \\
\ldots & \ldots & \ldots & \ldots & \ddots & \ldots & \ddots
\end{pmatrix}, \]

where \( b = (\ldots, b_{-1}, b_0, b_1, \ldots) \in \{-1,1\}^\mathbb{Z} \) is a pseudoergodic sequence; that means:

every finite pattern of \( \pm 1 \)'s can be found somewhere in the infinite sequence \( b \).
Example [Feinberg/Zee 1999]

Spectral Formula

If $b$ is pseudoergodic then

$$\text{spec } A^b = \text{spec}_{\text{ess }} A^b = \bigcup_{c \in \{\pm 1\}^\mathbb{Z}} \text{spec}_{\text{point }} A^c.$$
If $b$ is pseudoergodic then

$$\text{spec } A^b = \text{spec}_{\text{ess}} A^b = \bigcup_{c \in \{\pm 1\}^\mathbb{Z}} \text{spec}_{\text{point}} A^c.$$  

**Idea:** Try to “exhaust” the RHS by running through all periodic $\pm 1$ sequences $c$.  

Example [Feinberg/Zee 1999]  

[Marko Lindner: Spectra and Finite Sections of Band Operators]
Example [Feinberg/Zee 1999]

Period 1
Period 2

Periods 1, 2
Example [Feinberg/Zee 1999]

Period 3

Periods 1, ..., 3
Example [Feinberg/Zee 1999]

Period 4

Periods 1, ..., 4
Example [Feinberg/Zee 1999]

Period 5

Periods 1, ..., 5
Example [Feinberg/Zee 1999]

Period 6

Periods 1, ..., 6
Example [Feinberg/Zee 1999]

Period 7

Periods 1, ..., 7
Example [Feinberg/Zee 1999]

Period 8

Periods 1, ..., 8
Example [Feinberg/Zee 1999]

Period 9

Periods 1, ..., 9
Example [Feinberg/Zee 1999]

Period 10

Periods 1, ..., 10
Example [Feinberg/Zee 1999]

Period 11

Periods 1, ..., 11
Example [Feinberg/Zee 1999]

Period 12

Periods 1, ..., 12
Example [Feinberg/Zee 1999]

Period 13

Periods 1, ..., 13
Example [Feinberg/Zee 1999]

Period 14

Periods 1, ..., 14
Example [Feinberg/Zee 1999]

Period 15

Periods 1, ..., 15
Example  [Feinberg/Zee 1999]

Period 16

Periods 1, ..., 16
Example [Feinberg/Zee 1999]

Period 17

Periods 1, ..., 17
Example [Feinberg/Zee 1999]

Period 18

Periods 1, ..., 18
Example [Feinberg/Zee 1999]

Period 19

Periods 1, ..., 19
Example [Feinberg/Zee 1999]

Period 20

Periods 1, ..., 20
Example [Feinberg/Zee 1999]

Period 21

Periods 1, ..., 21
Example [Feinberg/Zee 1999]

Period 22

Periods 1, ..., 22
Example [Feinberg/Zee 1999]

Period 23  

Periods 1, ..., 23
Example [Feinberg/Zee 1999]

Period 24

Periods 1, ..., 24
Example [Feinberg/Zee 1999]

Period 25

Periods 1, ..., 25
Example [Feinberg/Zee 1999]

Period 26

Periods 1, ..., 26
Period 27

Periods 1, ..., 27
Example [Feinberg/Zee 1999]

Period 28

Periods 1, ..., 28
Example [Feinberg/Zee 1999]

Period 29

Periods 1, ..., 29
Example [Feinberg/Zee 1999]

Period 30

Periods 1, ..., 30
Recall our “Sandwich 1”: In this example, one has

$$\bigcup_{k \in \mathbb{Z}} \text{spec}_\varepsilon(P_{n,k}A^b P_{n,k}) \subset \text{spec}_\varepsilon(A^b) \subset \bigcup_{k \in \mathbb{Z}} \text{spec}_{\varepsilon+\varepsilon_n}(P_{n,k}A^b P_{n,k})$$

$$=: \sigma^\varepsilon_n$$

$$\Sigma^\varepsilon := \sigma^{\varepsilon+\varepsilon_n}$$

for all $n \in \mathbb{N}$, so let’s look at $\sigma^\varepsilon_n$ for $\varepsilon = 0$.

Here are the $n \times n$ matrix eigenvalues

$$\sigma^0_n = \bigcup_{k \in \mathbb{Z}} \text{spec}(P_{n,k}A^b P_{n,k})$$

for $n = 1, \ldots, 30$: 
Example [Feinberg/Zee 1999]

Size 1
Size 2
Example [Feinberg/Zee 1999]

Size 3
Example [Feinberg/Zee 1999]

Size 4
Size 5
Example [Feinberg/Zee 1999]

Size 6
Example [Feinberg/Zee 1999]

Size 7
Size 8

Example [Feinberg/Zee 1999]
Size 9

Example [Feinberg/Zee 1999]
Example [Feinberg/Zee 1999]

Size 10
Size 11
Example [Feinberg/Zee 1999]

Size 12
Example [Feinberg/Zee 1999]

Size 13
Size 14
Size 15
Size 17
Size 19
Example [Feinberg/Zee 1999]

Size 20
Example [Feinberg/Zee 1999]

Size 21
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Example [Feinberg/Zee 1999]

Size 24
Example [Feinberg/Zee 1999]

Size 25
Example [Feinberg/Zee 1999]

Size 26
Example [Feinberg/Zee 1999]

Size 27
Example [Feinberg/Zee 1999]

Size 28
Example [Feinberg/Zee 1999]

Size 29
Example [Feinberg/Zee 1999]

Size 30
Zoom into Region $1 + i$ of $\sigma_{25}^0$
The finite matrix spectra $\sigma_n^0$ are even contained in the periodic (infinite) matrix spectra shown before.

More precisely, the spectra of all $n \times n$ principal submatrices are contained in the set of all $(2n+2)$-periodic matrices:

\[
\begin{array}{cccccc}
\vdots & & \vdots & & \vdots & \\
\vdots & \cdots & \cdots & \cdots & \vdots & \\
\vdots & \cdots & \cdots & \cdots & \vdots & \\
\end{array}
\quad \subset
\]

Size $n = 5$ \quad Period $2n + 2 = 12$

Here we demonstrate this inclusion for some values of $n$. 
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 2$
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 3$
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 4$

Marko Lindner

Spectra and Finite Sections of Band Operators
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 5$
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 8$
Finite Matrix Spectra in Periodic Matrix Spectra: $n = 9$

Marko Lindner

Spectra and Finite Sections of Band Operators
Conjecture: \( \text{spec } A^b \) if \( b \) is pseudoergodic
Upper bound on spec $A^b$ by the closed numerical range
...in comparison with plots of $\sum_0^0 = \sigma_0^{0+\varepsilon_n}$

$n = 2$
...in comparison with plots of $\sum_0^n = \sigma_n^{0+\epsilon_n}$
...in comparison with plots of $\sum_{n=0}^{\infty} = \sigma_n^{0+\epsilon_n}$

$n = 4$
...in comparison with plots of $\sum_{n}^{0} = \sigma_{n}^{0+\varepsilon_{n}}$

$n = 5$
in comparison with plots of $\sum_{n}^{0} = \sigma_{n}^{0+\epsilon}$

$n = 6$
...in comparison with plots of $\sum_{n}^{0} = \sigma_{n}^{0+\varepsilon_{n}}$

$n = 7$
in comparison with plots of $\sum_{n}^{0} = \sigma_{n}^{0+\varepsilon_{n}}$

$n = 8$
...in comparison with plots of $\sum_{n}^{0} = \sigma_{n}^{0+\varepsilon_{n}}$

$n = 9$
...in comparison with plots of $\sum_{n}^{0} = \sigma_{n}^{0+\epsilon_{n}}$
...in comparison with plots of $\sum_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 11$
...in comparison with plots of $\sum_{n}^{0} = \sigma_{n}^{0+\varepsilon_{n}}$

$n = 12$
...in comparison with plots of $\sum_{n}^{0} = \sigma_{n}^{0+\varepsilon_{n}}$
...in comparison with plots of $\sum_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 14$
...in comparison with plots of $\sum_{n=0}^0 = \sigma_{n+\varepsilon_n}$

$n = 15$
...in comparison with plots of $\sum_{n}^{0} = \sigma_{n}^{0+\epsilon_n}$

$n = 16$
...in comparison with plots of $\sum_n^0 = \sigma_n^{0+\varepsilon_n}$

$n = 17$
...in comparison with plots of $\sum_{n=0}^{0} = \sigma_{n}^{0+\varepsilon_{n}}$

$n = 18$
Where does $\sum_{n}^{0}$ go as $n \rightarrow \infty$?

Computational cost for these pics: $n \cdot 2^{n-1} \times$ number of pixels.
Where does $\sum_{n}^{0}$ go as $n \to \infty$?

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So let us focus on just one point (pixel) $\lambda$:
Where does $\sum_{n}^{0}$ go as $n \rightarrow \infty$?

Computational cost for these pics: $n \cdot 2^{n-1} \times$ number of pixels.
So let us focus on just one point (pixel) $\lambda$:

\[ \lambda = 1.5 + 0.5i \notin \sum_{36}^{0} \supset \text{spec } A^b \]

We found a hole!

- Center: $1.5 + 0.5i$
- Radius: 0.01
Now we come back to the FSM and bring in our knowledge on Fredholm indices.
Recall the following two facts in the bi-infinite case:
Recall the following two facts in the bi-infinite case:

So \( \text{ind } A_+ = 0 = \text{ind } A_- \) is necessary for applicability of the FSM!
Abbreviate \( \text{ind } A_+ =: \kappa_+ \) and \( \text{ind } A_- =: \kappa_- \).

\[ \kappa_+ = 0 = \kappa_- \text{ is necessary for applicability of the FSM!} \]

Otherwise: **Shift** the system up or down accordingly, i.e. place the corners of your finite sections \( A_n \) on another (the \( \kappa_- \text{th} \)) diagonal!
Abbreviate $\text{ind } A_+ =: \kappa_+$ and $\text{ind } A_- =: \kappa_-.$

$\kappa_+ = 0 = \kappa_-$ is **necessary** for applicability of the FSM!

Otherwise: **Shift** the system up or down accordingly, i.e. place the corners of your finite sections $A_n$ on another (the $\kappa_-^{th}$) diagonal!

This means: Replace $Ax = b$ by $V_{\kappa_+}Ax = V_{\kappa_+}b.$

Reason: The new system has plus-index zero:

$$\text{ind } (V_{\kappa_+}A)_+ = \text{ind } (V_{\kappa_+})_+ + \text{ind } A_+ = -\kappa_+ + \kappa_+ = 0$$

This process is called **index cancellation**.

$\Rightarrow$ We have found the (from the FSM perspective) 
“proper” main diagonal of $A$!
Example: FSM for slowly oscillating operators

Suppose $A \in BDO(E)$ has slowly oscillating diagonals. We want to solve $Ax = b$ by the FSM.

**Assumption (minimal):** $A$ be invertible.

**Step 1:** Compute the plus-index $\kappa_+ := \text{ind } A_+$. 
Example: FSM for slowly oscillating operators

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**Assumption (minimal):** \( A \) be invertible.

**Step 1:** Compute the plus-index \( \kappa_+ := \text{ind } A_+ \).

Therefore, take an arbitrary limop \( B \) of \( A \) at \( +\infty \) and recall that \( \text{ind } B_+ = \text{ind } A_+ \).

\( A \) is slowly oscillating \( \Rightarrow \) \( B_+ \) is Toeplitz \( \Rightarrow \kappa_+ = -\text{wind}(a(\mathbb{T}), 0) \)
Example: FSM for slowly oscillating operators

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**Step 1:** Compute the plus-index $\kappa_+ := \text{ind } A_+.$

**Step 2:** Perform index cancellation (i.e. shift down by $\kappa_+$ rows).

Remarkable fact: We can truncate at arbitrary points $l$ and $r$! Reason: All limops $B$ and $C$ (w.r.t. subsequences of $r$ and $l$, resp) are Laurent operators. So all $B -$ and all $C +$ are Toeplitz operators that are Fredholm of index zero (after index cancellation).

Coburn’s lemma $\Rightarrow$ all $B -$ and all $C +$ are invertible, as well as $A$.

FSM theorem $\Rightarrow$ The FSM applies.
Suppose $A \in \text{BDO}(E)$ has slowly oscillating diagonals. We want to solve $Ax = b$ by the FSM.

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**Step 1:** Compute the plus-index $\kappa_+ := \text{ind } A_+$. 

**Step 2:** Perform index cancellation (i.e. shift down by $\kappa_+$ rows).

**Step 3:** Truncate.

Remarkable fact: We can truncate at arbitrary points $l_n$ and $r_n$! 
Reason: All limops $B$ and $C$ (w.r.t. subsequences of $r$ and $l$, resp) are Laurent operators. So all $B_-$ and all $C_+$ are Toeplitz operators that are Fredholm of index zero (after index cancellation).
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FSM theorem $\Rightarrow$ The FSM applies.
Final example: Back to our random Jacobi operator

\[
A = \begin{pmatrix}
\ddots & \ddots \\
\ddots & v_{-2} & w_{-1} \\
& u_{-2} & v_{-1} & w_0 \\
& u_{-1} & v_0 & w_1 \\
& u_0 & v_1 & w_2 \\
& & u_1 & v_2 & \ddots \\
& & & & \ddots & \ddots
\end{pmatrix}
\]

with iid entries \(u_i \in U, v_i \in V\) and \(w_i \in W\).

Assumption (minimal): \(A\) is invertible.

How do we truncate \(A\) to get an applicable FSM?
Assumption (minimal): $A \in \Psi E(U, V, W)$ is invertible. How do we truncate $A$ to get an applicable FSM?

**Step 1.** Compute $\kappa_+ := \text{ind } A_+$
Assumption (minimal): $A \in \Psi E(U, V, W)$ is invertible. How do we truncate $A$ to get an applicable FSM?

**Step 1.** Compute $\kappa_+ := \text{ind } A_+$

We know that $\text{ind } A_+ = \text{ind } B_+$ for all limops $B$ of $A$ at $+\infty$. Let’s take a Laurent operator $B$. So pick arbitrary $u \in U$, $v \in V$ and $w \in W$. Then $\kappa_+ = \text{ind } A_+ = \text{ind } B_+ = -\text{wind}(E_{u,w}, v)$.
Assumption (minimal): \( A \in \Psi E(U, V, W) \) is invertible.
How do we truncate \( A \) to get an applicable FSM?

**Step 1.** Compute \( \kappa_+ := \text{ind } A_+ \)

We know that \( \text{ind } A_+ = \text{ind } B_+ \) for all limops \( B \) of \( A \) at \( +\infty \).
Let’s take a Laurent operator \( B \). So pick arbitrary \( u \in U, \ v \in V \) and \( w \in W \). Then \( \kappa_+ = \text{ind } A_+ = \text{ind } B_+ = -\text{wind}(E_{u,w}, \nu) \).

It’s very simple:

- if \( \nu \) is outside \( E_{u,w} \): \( \kappa_+ = 0 \)
- if \( \nu \) is inside \( E_{u,w} \): \( \kappa_+ = \pm 1 \)
  - if \( |u| > |w| \): \( \kappa_+ = -1 \)
  - if \( |u| < |w| \): \( \kappa_+ = +1 \)

**Note:** The result \( \kappa_+ \) is independent of \( u \in U, \ v \in V, \ w \in W \)!
Step 1. Compute $\kappa_+ := \text{ind } A_+$

Step 2. Perform index cancellation.
Step 1. Compute $\kappa_+ := \text{ind } A_+$

Step 2. Perform index cancellation.

- $\kappa_+ = -1$: shift up

- $\kappa_+ = 0$: leave as it is

- $\kappa_+ = +1$: shift down

In either case, the new system $\tilde{A}x = \tilde{b}$ has $\text{ind } \tilde{A}_+ = 0 = \text{ind } \tilde{A}_-$. 
Step 3. Do the truncations.

Choose the truncation points $\cdots < l_2 < l_1 < r_1 < r_2 < \cdots$ so that

\[
\begin{pmatrix}
v_{l_n} & w_{l_n+1} \\
u_{l_n} & & \\
& & \\
& & \\
& & \\
& & \\
u_{r_n} & w_{r_n} & \\
u_{r_n-1} & v_{r_n} \\
\end{pmatrix} \rightarrow \begin{pmatrix}v & w \\
u & & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{pmatrix} =: C_+
\]

and

\[
\begin{pmatrix}
\cdots & \cdots & w_{r_n} \\
\cdots & \cdots & \\
u_{r_n-1} & v_{r_n} \\
\end{pmatrix} \rightarrow \begin{pmatrix}
\cdots & \cdots & w \\
\cdots & \cdots & \\
u & v \\
\end{pmatrix} =: B_-
\]

as $n \to \infty$, for some fixed $u \in U$, $v \in V$ and $w \in W$. 
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\[
\begin{pmatrix}
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u_{l_n} & \ddots & \ddots \\
\ddots & \ddots & \ddots
\end{pmatrix}
\rightarrow
\begin{pmatrix}
v & w \\
u & \ddots & \ddots \\
\ddots & \ddots & \ddots
\end{pmatrix} =: C_+
\]

and

\[
\begin{pmatrix}
\ddots & \ddots & \ddots \\
\ddots & \ddots & w_{r_n} \\
u_{r_n-1} & v_{r_n} & \ddots
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\ddots & \ddots & \ddots \\
\ddots & \ddots & w \\
\ddots & u & v
\end{pmatrix} =: B_-
\]

as \( n \to \infty \), for some fixed \( u \in U, v \in V \) and \( w \in W \).

Both \textbf{Toeplitz} operators \( C_+ \) and \( B_- \) are Fredholm of index 0 (because \( \text{ind } A_+ = 0 = \text{ind } A_- \)) so they are \textbf{invertible} (Coburn).
But how about the ‘full’ FSM?

The previous truncation pattern was specially adapted to the operator \( A \in \Psi E(U, V, W) \) at hand.

The **full (or usual) FSM** uses the cut-off sequences \( l = (-1, -2, ...) \) and \( r = (1, 2, ...) \).

---

**Theorem**  
ML, Roch 2011

For \( A \in \Psi E(U, V, W) \) we have the following results:

1. The following are equivalent: the full FSM is applicable to \( A \), the full FSM is applicable to all \( B \in J(U, V, W) \), all operators \( B + \) with \( B \in J(U, V, W) \) are invertible.
2. If \( 0 \in U, W \) and \( A \) is invertible then the full FSM is applicable.
3. If \( A + := \text{ind} \) \( A + = \pm 1 \) and \( A \) is invertible then, after index cancellation, the full FSM is applicable.
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3. If \( \kappa_+ := \text{ind} A_+ = \pm 1 \) and \( A \) is invertible then, after index cancellation, the full FSM is applicable.
Thank you!


M. Lindner and S. Roch: Finite sections of random Jacobi operators, to appear in SIAM J Numerical Analysis