Structured Rank Matrices
Lecture 2:
Structure Transport

Marc Van Barel and Raf Vandebril
Dept. of Computer Science, K.U.Leuven, Belgium
Chemnitz, Germany, 26-30 September 2011

Contents

Outline

The nullity theorem

Definition (Right null space)

Given a matrix $A \in \mathbb{R}^{m \times n}$. The right null space $N(A)$ equals

$$N(A) = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = 0\}.$$

Definition (Nullity of a matrix)

Given a matrix $A \in \mathbb{R}^{m \times n}$. The nullity $\mathrm{n}(A)$ is defined as the dimension of the right null space of A.

Corollary

The dimension of the right null space corresponds to the rank deficiency of the columns of the matrix A:

$$\mathrm{n}(A) = n - \operatorname{rank}(A) = (\textit{number of columns}) - \operatorname{rank}(A).$$

The nullity theorem

Theorem (Nullity theorem)

Suppose the following invertible matrix $A \in \mathbb{R}^{n\times n}$ *is partitioned as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with A_{11} *of size* $p \times q$. *The inverse* B *of* A *is partitioned as*

$$A^{-1} = B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

with B_{11} *of size* $q \times p$. *Then the nullities* $\mathrm{n}(A_{11})$ *and* $\mathrm{n}(B_{22})$ *are equal:*

$$\mathrm{n}(A_{11}) = \mathrm{n}(B_{22}).$$

Corollaries of the nullity theorem

Corollary

Suppose $A \in \mathbb{R}^{n\times n}$ *is a nonsingular matrix, and* α, β *are nonempty subsets of* N *with* $|\alpha| < n$ *and* $|\beta| < n$. *Then*

$$\mathrm{rank}\left(A^{-1}(\alpha;\beta)\right) = \mathrm{rank}\left(A(N\backslash\beta; N\backslash\alpha)\right) + |\alpha| + |\beta| - n.$$

- **Proof:**
  Permuting the matrix such that $A(N\backslash\beta; N\backslash\alpha)$ moves to the upper left position A_{11}, will move $A^{-1}(\alpha;\beta)$ to the position B_{22}. Using the equalities:

$$\mathrm{n}(A_{11}) = n - |\alpha| - \mathrm{rank}\left(A_{11}\right),$$
$$\mathrm{n}(B_{22}) = |\beta| - \mathrm{rank}\left(B_{22}\right),$$

  gives us the proof.

Corollaries of the nullity theorem

Corollary

Suppose $A \in \mathbb{R}^{n\times n}$ *is a nonsingular matrix, and* α, β *are nonempty subsets of* N *with* $|\alpha| < n$ *and* $|\beta| < n$. *Then*

$$\mathrm{rank}\left(A^{-1}(\alpha;\beta)\right) = \mathrm{rank}\left(A(N\backslash\beta; N\backslash\alpha)\right) + |\alpha| + |\beta| - n.$$

- Examples for 5×5 matrices:

$\alpha = \{1,2\}$ and $\beta = \{1,2\}$ $N\backslash\beta = \{3,4,5\}$ and $N\backslash\alpha = \{3,4,5\}$

$$\begin{bmatrix} \color{red}\times & \color{red}\times & \times & \times & \times \\ \color{red}\times & \color{red}\times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \leftrightarrow \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \color{red}\times & \color{red}\times & \color{red}\times \\ \times & \times & \color{red}\times & \color{red}\times & \color{red}\times \\ \times & \times & \color{red}\times & \color{red}\times & \color{red}\times \end{bmatrix}$$

Corollaries of the nullity theorem

Corollary

Suppose $A \in \mathbb{R}^{n\times n}$ *is a nonsingular matrix, and* α, β *are nonempty subsets of* N *with* $|\alpha| < n$ *and* $|\beta| < n$. *Then*

$$\mathrm{rank}\left(A^{-1}(\alpha;\beta)\right) = \mathrm{rank}\left(A(N\backslash\beta; N\backslash\alpha)\right) + |\alpha| + |\beta| - n.$$

- Examples for 5×5 matrices:

$\alpha = \{1,2\}$ and $\beta = \{1,2,3\}$ $N\backslash\beta = \{4,5\}$ and $N\backslash\alpha = \{3,4,5\}$

$$\begin{bmatrix} \color{red}\times & \color{red}\times & \color{red}\times & \times & \times \\ \color{red}\times & \color{red}\times & \color{red}\times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \leftrightarrow \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \color{red}\times & \color{red}\times & \color{red}\times \\ \times & \times & \color{red}\times & \color{red}\times & \color{red}\times \end{bmatrix}$$

Corollaries of the nullity theorem

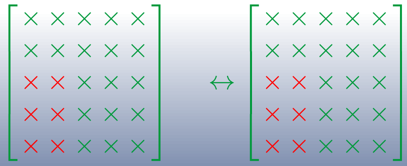
Corollary

Suppose $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix, and α, β are nonempty subsets of N with $|\alpha| < n$ and $|\beta| < n$. Then

$$\operatorname{rank}\left(A^{-1}(\alpha;\beta)\right) = \operatorname{rank}\left(A(N\backslash\beta; N\backslash\alpha)\right) + |\alpha| + |\beta| - n.$$

- Examples for 5×5 matrices:

$\alpha = \{3,4,5\}$ and $\beta = \{1,2\}$ $\quad$ $N\backslash\beta = \{3,4,5\}$ and $N\backslash\alpha = \{1,2\}$



The nullity theorem

Corollaries of the nullity theorem

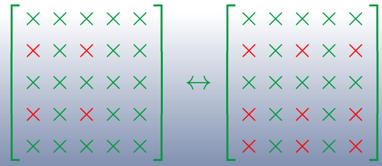
Corollary

Suppose $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix, and α, β are nonempty subsets of N with $|\alpha| < n$ and $|\beta| < n$. Then

$$\operatorname{rank}\left(A^{-1}(\alpha;\beta)\right) = \operatorname{rank}\left(A(N\backslash\beta; N\backslash\alpha)\right) + |\alpha| + |\beta| - n.$$

- Examples for 5×5 matrices:

$\alpha = \{2,4\}$ and $\beta = \{1,3\}$ $\quad$ $N\backslash\beta = \{2,4,5\}$ and $N\backslash\alpha = \{1,3,5\}$



Structured Rank Matrices Lecture 2: Structure Transport 5 / 26

Some corollaries of the nullity theorem

Corollary

For a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ and $\alpha \subseteq N$, we have:

$$\operatorname{rank}\left(A^{-1}(\alpha; N\backslash\alpha)\right) = \operatorname{rank}\left(A(\alpha; N\backslash\alpha)\right).$$

- **Proof:**
  Is a direct consequence of the previous equation:

$$\operatorname{rank}\left(A^{-1}(\alpha;\beta)\right) = \operatorname{rank}\left(A(N\backslash\beta; N\backslash\alpha)\right) + |\alpha| + |\beta| - n,$$

when posing $\beta = N\backslash\alpha$:

$$\operatorname{rank}\left(A^{-1}(\alpha;N\backslash\alpha)\right) = \operatorname{rank}\left(A(\alpha; N\backslash\alpha)\right) + |\alpha| + |N\backslash\alpha| - n.$$

The nullity theorem

Some corollaries of the nullity theorem

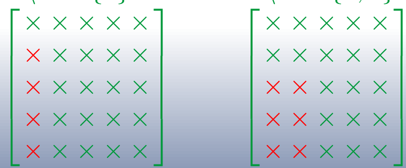
Corollary

For a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ and $\alpha \subseteq N$, we have:

$$\operatorname{rank}\left(A^{-1}(\alpha; N\backslash\alpha)\right) = \operatorname{rank}\left(A(\alpha; N\backslash\alpha)\right).$$

- This means that for a matrix the following blocks always have the same rank in A and in A^{-1}.

$\alpha = \{2,3,4,5\}$ and $N\backslash\alpha = \{1\}$ $\quad$ $\alpha = \{3,4,5\}$ and $N\backslash\alpha = \{1,2\}$



Structured Rank Matrices Lecture 2: Structure Transport 6 / 26

Some corollaries of the nullity theorem

Corollary

For a nonsingular matrix $A \in \mathbb{R}^{n \times n}$ and $\alpha \subseteq N$, we have:

$$\operatorname{rank}\left(A^{-1}(\alpha; N\backslash\alpha)\right) = \operatorname{rank}\left(A(\alpha; N\backslash\alpha)\right).$$

- This means that for a matrix the following blocks always have the same rank in A and in A^{-1}.

$\alpha = \{4,5\}$ and $N\backslash\alpha = \{1,2,3\}$

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ {\color{red}\times} & {\color{red}\times} & {\color{red}\times} & \times & \times \\ {\color{red}\times} & {\color{red}\times} & {\color{red}\times} & \times & \times \end{bmatrix}$$

$\alpha = \{5\}$ and $N\backslash\alpha = \{1,2,3,4\}$

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ {\color{red}\times} & {\color{red}\times} & {\color{red}\times} & {\color{red}\times} & \times \end{bmatrix}$$

The nullity theorem

Some corollaries of the nullity theorem

Corollary

For a nonsingular matrix $A \in \mathbb{R}^{n \times n}$ and $\alpha \subseteq N$, we have:

$$\operatorname{rank}\left(A^{-1}(\alpha; N\backslash\alpha)\right) = \operatorname{rank}\left(A(\alpha; N\backslash\alpha)\right).$$

- This means that for a matrix the following blocks always have the same rank in A and in A^{-1}.

$\alpha = \{3,5\}$ and $N\backslash\alpha = \{1,2,4\}$

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ {\color{red}\times} & {\color{red}\times} & \times & {\color{red}\times} & \times \\ \times & \times & \times & \times & \times \\ {\color{red}\times} & {\color{red}\times} & \times & {\color{red}\times} & \times \end{bmatrix}$$

$\alpha = \{2,3\}$ and $N\backslash\alpha = \{1,4,5\}$

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ {\color{red}\times} & \times & \times & {\color{red}\times} & {\color{red}\times} \\ {\color{red}\times} & \times & \times & {\color{red}\times} & {\color{red}\times} \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}$$

Structured Rank Matrices Lecture 2: Structure Transport 6 / 26

Outline

Different proofs

There exist different strategies to prove the nullity theorem.

- An important remark, the theorem predicts structures but does not provide inversion formulas.
- Fiedler and Markham proved it, working directly on the ranks and nullities of the blocks, their proof was based on a paper by Gustafson.
- Barrett and Feinsilver were very close to an alternative proof, but they only worked with tridiagonal and semiseparable matrices.
- Recently also Strang and Nguyen proved a weaker formulation of the theorem.

Different proofs

Proof (by Fiedler and Markham)

Suppose $\mathrm{n}(A_{11}) \leq \mathrm{n}(B_{22})$. If this is not true, we can prove the theorem for the matrices

$$\begin{bmatrix} A_{22} & A_{21} \\ A_{12} & A_{11} \end{bmatrix}, \quad \begin{bmatrix} B_{22} & B_{21} \\ B_{12} & B_{11} \end{bmatrix},$$

which are also each others inverse. Suppose $\mathrm{n}(B_{22}) > 0$ otherwise $\mathrm{n}(A_{11}) = 0$ and the theorem is proved. When $\mathrm{n}(B_{22}) = c > 0$, then there exists a matrix F with c linearly independent columns, such that $B_{22}F = 0$.
Remember that

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

The nullity theorem

Different proofs

Proof (by Fiedler and Markham)

Hence, multiplying the following equation to the right by F

$$A_{11}B_{12} + A_{12}B_{22} = 0,$$

we get

$$A_{11}B_{12}F = 0. \qquad (1)$$

Applying the same operation to the relation:

$$A_{21}B_{12} + A_{22}B_{22} = I$$

it follows that $A_{21}B_{12}F = F$, and therefore $\mathrm{rank}\,(B_{12}F) \geq c$. Using this last statement together with equation (1), we derive

$$\mathrm{n}(A_{11}) \geq \mathrm{rank}\,(B_{12}F) \geq c = \mathrm{n}(B_{22}).$$

With our assumption $\mathrm{n}(A_{11}) \leq \mathrm{n}(B_{22})$, this proves the theorem.

The nullity theorem

Outline

The nullity theorem

Some real matrix examples

Example (Upper triangular matrix)

- The inverse of an upper triangular matrix is an upper triangular matrix.

Some real matrix examples

Example (Upper triangular matrix)

- The inverse of an upper triangular matrix is an upper triangular matrix.
- The rank of the red marked blocks is maintained by Corollary 2.

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \color{red}0 & \times & \times & \times & \times \\ \color{red}0 & 0 & \times & \times & \times \\ \color{red}0 & 0 & 0 & \times & \times \\ \color{red}0 & 0 & 0 & 0 & \times \end{bmatrix}$$

The nullity theorem

Some real matrix examples

Example (Upper triangular matrix)

- The inverse of an upper triangular matrix is an upper triangular matrix.
- The rank of the red marked blocks is maintained by Corollary 2.

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ \color{red}0 & \color{red}0 & \times & \times & \times \\ \color{red}0 & \color{red}0 & 0 & \times & \times \\ \color{red}0 & \color{red}0 & 0 & 0 & \times \end{bmatrix}$$

Structured Rank Matrices Lecture 2: Structure Transport 11 / 26

The nullity theorem

Some real matrix examples

Example (Upper triangular matrix)

- The inverse of an upper triangular matrix is an upper triangular matrix.
- The rank of the red marked blocks is maintained by Corollary 2.

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ \color{red}0 & \color{red}0 & \color{red}0 & \times & \times \\ \color{red}0 & \color{red}0 & \color{red}0 & 0 & \times \end{bmatrix}$$

Structured Rank Matrices Lecture 2: Structure Transport 11 / 26

The nullity theorem

Some real matrix examples

Example (Upper triangular matrix)

- The inverse of an upper triangular matrix is an upper triangular matrix.
- The rank of the red marked blocks is maintained by Corollary 2.

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ \color{red}0 & \color{red}0 & \color{red}0 & \color{red}0 & \times \end{bmatrix}$$

Structured Rank Matrices Lecture 2: Structure Transport 11 / 26

Some real matrix examples

Example (Quasiseparable matrix)

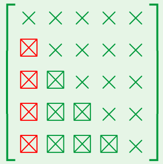
- The inverse of a quasiseparable matrix is a quasiseparable matrix.

Some real matrix examples

Example (Quasiseparable matrix)

- The inverse of a quasiseparable matrix is a quasiseparable matrix.

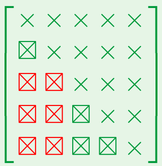
- The rank of the red marked blocks is maintained by Corollary 2.



Some real matrix examples

Example (Quasiseparable matrix)

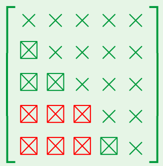
- The inverse of a quasiseparable matrix is a quasiseparable matrix.
- The rank of the red marked blocks is maintained by Corollary 2.



Some real matrix examples

Example (Quasiseparable matrix)

- The inverse of a quasiseparable matrix is a quasiseparable matrix.
- The rank of the red marked blocks is maintained by Corollary 2.



Some real matrix examples

Example (Quasiseparable matrix)

- The inverse of a quasiseparable matrix is a quasiseparable matrix.
- The rank of the red marked blocks is maintained by Corollary 2.

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \boxtimes & \times & \times & \times & \times \\ \boxtimes & \boxtimes & \times & \times & \times \\ \boxtimes & \boxtimes & \boxtimes & \times & \times \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \times \end{bmatrix}$$

Some real matrix examples

Example (Tridiagonal vs. semiseparable)

- The inverse of a tridiagonal matrix is a semiseparable matrix.

Some real matrix examples

Example (Tridiagonal vs. semiseparable)

- The inverse of a tridiagonal matrix is a semiseparable matrix.
- The rank of the left block plus 1 equals the rank of the right block, according to corollary 1

$\alpha = \{3,4,5\}$ and $\beta = \{1\}$ $\qquad$ $N \backslash \beta = \{2,3,4,5\}$ and $N \backslash \alpha = \{1,2\}$

$$\begin{bmatrix} \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 \\ 0 & \times & \times & \times & 0 \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \leftrightarrow \begin{bmatrix} \times & \times & \times & \times & \times \\ \boxtimes & \boxtimes & \times & \times & \times \\ \boxtimes & \boxtimes & \times & \times & \times \\ \boxtimes & \boxtimes & \times & \times & \times \\ \boxtimes & \boxtimes & \times & \times & \times \end{bmatrix}$$

Some real matrix examples

Example (Tridiagonal vs. semiseparable)

- The inverse of a tridiagonal matrix is a semiseparable matrix.
- The rank of the left block plus 1 equals the rank of the right block, according to corollary 1

$\alpha = \{4,5\}$ and $\beta = \{1,2\}$ $\qquad$ $N \backslash \beta = \{3,4,5\}$ and $N \backslash \alpha = \{1,2,3\}$

$$\begin{bmatrix} \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 \\ 0 & \times & \times & \times & 0 \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \leftrightarrow \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \boxtimes & \boxtimes & \boxtimes & \times & \times \\ \boxtimes & \boxtimes & \boxtimes & \times & \times \\ \boxtimes & \boxtimes & \boxtimes & \times & \times \end{bmatrix}$$

Some real matrix examples

Example (Tridiagonal vs. semiseparable)

- The inverse of a tridiagonal matrix is a semiseparable matrix.
- The rank of the left block plus 1 equals the rank of the right block, according to corollary 1

$$\alpha = \{5\} \text{ and } \quad N\backslash\beta = \{4,5\} \text{ and}$$
$$\beta = \{1,2,3\} \quad N\backslash\alpha = \{1,2,3,4\}$$

$$\begin{bmatrix} \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 \\ 0 & \times & \times & \times & 0 \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \leftrightarrow \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \times \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \times \end{bmatrix}$$

Some real matrix examples

Example

- The inverse of a $\{p, q\}$-semiseparable matrix is a $\{p, q\}$-band matrix.
- One can predict the structure of the inverse of a generalized Hessenberg matrix.
- One can predict the structure when inverting hierarchically semiseparable and/or $\mathcal{H}$ matrices.
- Structure related: The off-diagonal structure is maintained. For example the inverse of a rank one matrix plus a diagonal is again a rank 1 matrix plus a diagonal.
- Applicable to all structured rank matrices.

Outline

References for the nullity theorem

- W. H. Gustafson, **A note on matrix inversion**, Linear Algebra and Its Applications **57** (1984), 71–73.
- M. Fiedler, **Basic matrices**, Linear Algebra and Its Applications **373** (2003), 143–151.
- W. W. Barrett, **A theorem on inverse of tridiagonal matrices**, Linear Algebra and Its Applications **27** (1979), 211–217.
- W. W. Barrett and P. J. Feinsilver, **Gaussian families and a theorem on patterned matrices**, Journal of Applied Probability **15** (1978), 514–522.
- W. W. Barrett and P. J. Feinsilver, **Inverses of banded matrices**, Linear Algebra and Its Applications **41** (1981), 111–130.
- G. Strang and T. Nguyen, **The interplay of ranks of submatrices**, SIAM Review 46 (2004), 637–646.

General remarks

LU and QR-decompositions

- Given a matrix $A \in \mathbb{R}^{m \times n}$. $A = LU$ is called an LU-decomposition if L is lower triangular and U is upper triangular.
  - Frequently used for solving systems of equations (Gaussian elimination).
  - Computing eigenvalues of specialized matrices (quotient-difference algorithms).
- Under some mild conditions both factorizations are unique.

Generalizations of the nullity theorem

General remarks

LU and QR-decompositions

- Given a matrix $A \in \mathbb{R}^{m \times n}$. $A = LU$ is called an LU-decomposition if L is lower triangular and U is upper triangular.
  - Frequently used for solving systems of equations (Gaussian elimination).
  - Computing eigenvalues of specialized matrices (quotient-difference algorithms).
- Given a matrix $A \in \mathbb{R}^{m \times n}$. $A = QR$ is called a QR-decomposition if Q is unitary ($QQ^H = Q^H Q = I$) and R is upper triangular.
  - Solving systems of equations (more stable than Gaussian eliminiation).
  - In the top 10 algorithms of the 20th century for computing eigenvalues of arbitrary matrices.
- Under some mild conditions both factorizations are unique.

Outline

The LU-decomposition

Theorem (LU-factorization)

Given an invertible matrix A, with a LU factorization $A = LU$. Let A be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with A_{11} of dimension $p \times q$. Let U be partitioned as

$$U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

with U_{11} of dimension $p \times q$. Then the nullities $\mathrm{n}(A_{12})$ and $\mathrm{n}(U_{12})$ are equal (as well as their ranks).

The LU-decomposition

Example (Structured rank matrices)

- The L and U factor inherit the structure.
- For a semiseparable matrix: U is upper semiseparable, and L is lower semiseparable.
- For a tridiagonal matrix: U is upper bidiagonal, and L is lower bidiagonal.
- For a $\{p, q\}$-semiseparable matrix: U is $\{q\}$-upper semiseparable, and L is $\{p\}$-lower semiseparable.
- For a $\{p, q\}$-band matrix: U is $\{q\}$-upper band, and L is $\{p\}$-lower band.
- Holds for combinations, and even more general structures.

Outline

The QR-decomposition

Theorem (QR-factorization)

Given an invertible matrix A, with a QR-factorization $A = QR$. Let A be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with A_{11} of dimension $p \times q$. Let Q be partitioned as

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

with Q_{11} of dimension $p \times q$. Then the nullities $\mathrm{n}(A_{21})$ and $\mathrm{n}(Q_{21})$ are equal.

The QR-decomposition

Example (Structured rank matrices)

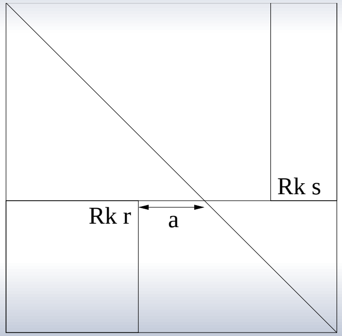
- The Q factor inherits the structure of the lower triangular part.
- The structure of R is more complicated (see next slides).
- For a semiseparable matrix: Q has the lower triangular part of lower semiseparable form, and R has the upper triangular structure of rank 2.
- For a tridiagonal matrix: Q has the lower triangular part of bidiagonal form.
- For a $\{p, q\}$-semiseparable matrix: Q has the lower triangular part of $\{p\}$-semiseparable form.
- For a $\{p, q\}$-band matrix: Q has the lower triangular part of $\{p\}$-band form.
- Holds for combinations, and even more general structures.

Rank structure of the R-factor

We derive this structure by investigating how the original rank structure is transformed when computing the QR-factorization.

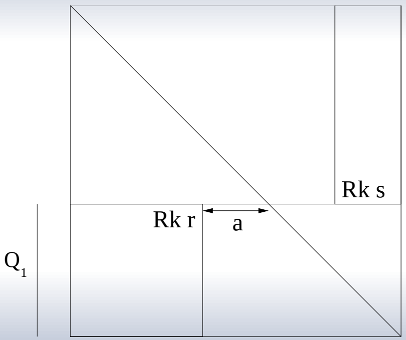
Rank structure of the R-factor

We derive this structure by investigating how the original rank structure is transformed when computing the QR-factorization.
Starting situation:



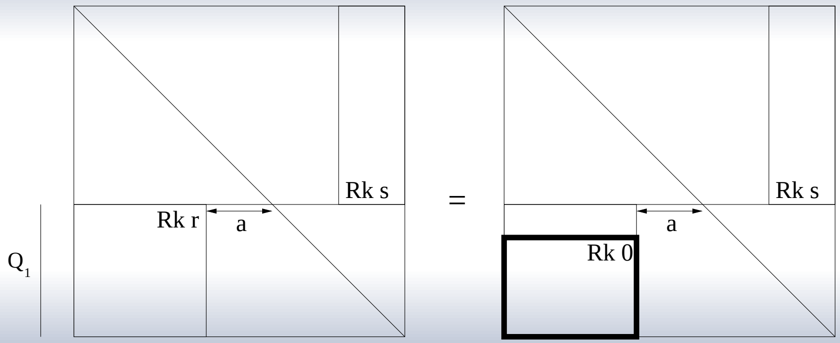
Rank structure of the R-factor

We derive this structure by investigating how the original rank structure is transformed when computing the QR-factorization.
First series of Givens transformations:



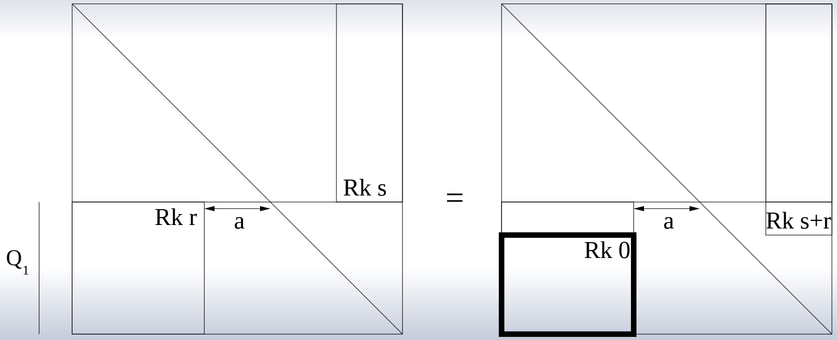
Rank structure of the R-factor

We derive this structure by investigating how the original rank structure is transformed when computing the QR-factorization.
First series of Givens transformations:



Rank structure of the R-factor

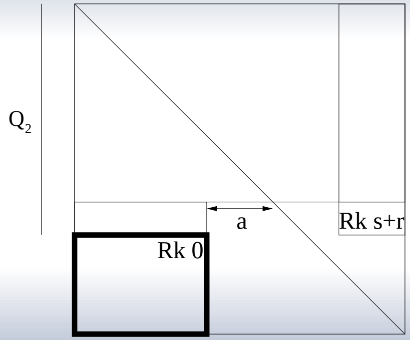
We derive this structure by investigating how the original rank structure is transformed when computing the QR-factorization.
First series of Givens transformations:



Generalizations of the nullity theorem

Rank structure of the R-factor

We derive this structure by investigating how the original rank structure is transformed when computing the QR-factorization.
Second series of Givens transformations:

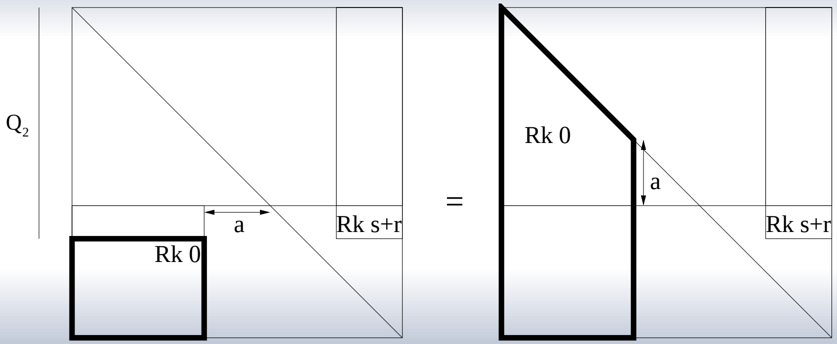


Structured Rank Matrices Lecture 2: Structure Transport 24 / 26

Generalizations of the nullity theorem

Rank structure of the R-factor

We derive this structure by investigating how the original rank structure is transformed when computing the QR-factorization.
Second series of Givens transformations:

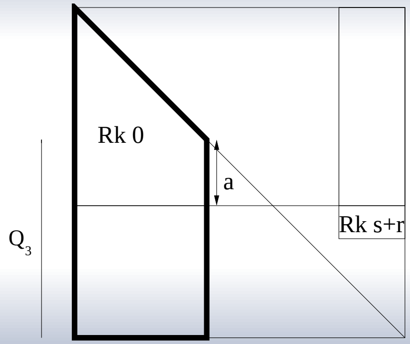


Structured Rank Matrices Lecture 2: Structure Transport 24 / 26

Generalizations of the nullity theorem

Rank structure of the R-factor

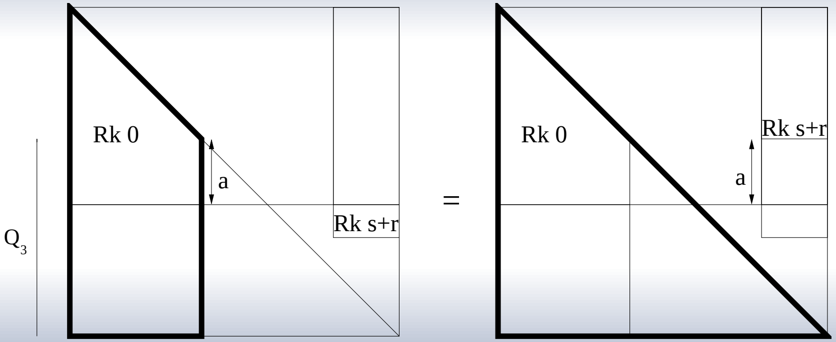
We derive this structure by investigating how the original rank structure is transformed when computing the QR-factorization.
Third series of Givens transformations:



Structured Rank Matrices Lecture 2: Structure Transport 24 / 26

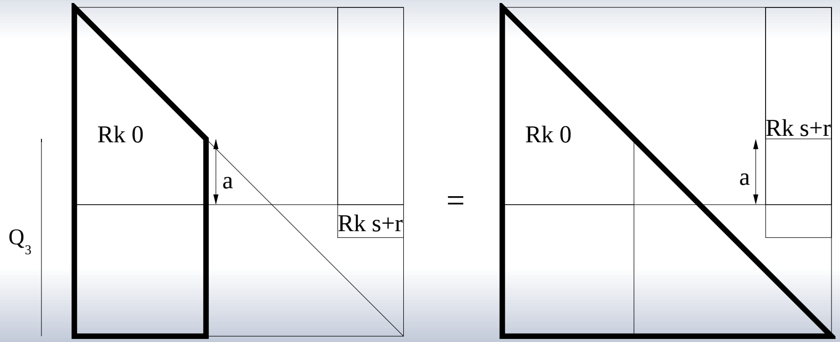
Rank structure of the R-factor

We derive this structure by investigating how the original rank structure is transformed when computing the QR-factorization.
Third series of Givens transformations:



Rank structure of the R-factor

We derive this structure by investigating how the original rank structure is transformed when computing the QR-factorization.
Third series of Givens transformations:



Q.E.D.

Outline

References for these generalizations

- R. Vandebril and M. Van Barel, **A short note on the nullity theorem**, Journal of Computational and Applied Mathematics 189:179–190, 2006.