Rank structured matrices: theory, algorithms and applications

In numerical linear algebra much attention has been paid to matrices that are sparse, i.e., containing a lot of zeros. For example, to compute the eigenvalues of a general dense symmetric matrix, this matrix is first reduced to a similar tridiagonal one using an orthogonal similarity transformation. The subsequent QR-algorithm performed on this $n \times n$ tridiagonal matrix, takes the sparse structure of this matrix into account resulting in $O(n)$ flops per iteration compared to $O(n^3)$ flops for a general dense matrix. Much less attention was given to so-called rank structured matrices although they share similar theoretical and computational properties. As an example one can look at the inverse of a tridiagonal matrix with nonzero subdiagonal elements. This inverse is generically a dense matrix. However, all submatrices taken from the lower (upper) triangular part of this matrix all have rank one. Such a matrix is called a semiseparable matrix.

The aim of this series of lectures is to gain insight into this class of matrices and some of its generalizations. These matrices are called rank structured matrices. Rank structured matrices can be loosely defined as matrices that have one or more submatrices of low rank. Similar to sparse matrices where the sparsity pattern can be structured or unstructured, rank structured matrices can be considered having a structured or unstructured pattern of the low rank matrices. We will focus in these lectures on the rank structured matrices with a specific pattern in the low rank submatrices.

As time permits, one or more of the following topics will be covered:

- In the literature different names and slightly different definitions are given for rank structured matrices. We will study these different definitions. Besides the different definitions also several different representations can be used. We will discuss the advantages and disadvantages of these representations.

- One of the nice properties of rank structured matrices is the fact that a rank structure can also be found in the inverse, the factors of LR and QR factorizations, ... See, e.g., [3]. Once this structure is detected, this leads to fast algorithms to compute the corresponding factors.

- As described above a tridiagonal matrix can be used as an intermediate form to approximate the eigenvalues (and/or eigenvectors) of a general dense symmetric matrix. In a similar way, semiseparable matrices or diagonal-plus-semiseparable matrices can be used as intermediate form. We will investigate the similarity reduction of a general symmetric matrix using orthogonal similarity transformations into such a form and the convergence properties of this reduction method as well as of the corresponding QR algorithm applied to one of these forms. See, e.g., [7, 4].

- Orthogonal polynomials on the real line satisfy a three term recurrence relation. This relation can be written in matrix notation by using a tridiagonal matrix. Similarly, orthogonal polynomials on the unit circle satisfy a Szegő recurrence relation that corresponds to an (almost) unitary Hessenberg matrix. It turns out that orthogonal rational functions with prescribed poles satisfy a recurrence relation that corresponds to diagonal plus semiseparable matrices. This leads to efficient algorithms for computing the recurrence parameters for these orthogonal rational functions by solving a corresponding inverse eigenvalue problem. See, e.g., [1].

- The insight that we gained on performing operations with rank structured matrices can be used to design efficient $O(n^2)$ eigenvalue solvers for companion and more general matrices corresponding to the problem of finding the zeros of a polynomial. See, e.g., [2].
- We will also look at variants and generalizations of the rank structured matrices as studied before. We will investigate (block) quasiseparable matrices, hierarchical or $\mathcal{H}$, $\mathcal{H}^2$, hierarchically semiseparable matrices, ...

The slides of the lectures will be made available to the participants. For more details on rank structured matrices, one can have a look at the two volumes [5, 6].

References

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[7] R. Vandebril, E. Van Camp, M. Van Barel, and N. Mastronardi. On the convergence properties of the orthogonal similarity transformations to tridiagonal and semiseparable (plus diagonal) form. *Numerische Mathematik*, 104:205–239, 2006.