Geometric Perron-Frobenius theory in finite dimensions

Hans Schneider

Chemnitz
October 2010

geompfchmn 2010.09.08, 11:05 September 8, 2010

Definition

K a cone in $\mathbb{R}^n$

1. $K + K \subseteq K$
   $(x, y \in K \Longrightarrow x + y \in K)$
2. $\mathbb{R}_+ K \subseteq K$
   $(\alpha \geq 0, x \in K \Longrightarrow \alpha x \in K$
3. K pointed
   $K \cap K = \{0\}$
   $(x, -x \in$K$\Longrightarrow x = 0)$

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We assume K closed (in Euclidean topology of $\mathbb{R}^n$)

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cone K proper:
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$$y \geq x : y - x \in K$$

$$x \geq 0 \iff x \in K$$

$$x > 0 \iff x \in \text{int } K$$

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F a face of cone $K - -F \trianglelefteq K$:
F is a subcone of K and
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F a proper face of K:
$F \neq 0,\ F \neq K$

three examples

$$\mathbb{R}^n_+$$

Faces: $(+, 0, +0, \ldots, 0, +)$

$\mathbb{R}^n_+$

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$(x_1^2 + x_2^2 + \cdots + x_{n-1}^2)^(1/2) \leq x_n$

Proper faces:

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Extremals

$\mathbb{R}^n_+$
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Extremals

(Real) Space: Hermitian $n \times n$ matrices
Cone: Positive semi-definite matrices
Faces: Matrices simult similar to $0_k \oplus X$

nonnegative operators

K a proper cone in $\mathbb{R}^n$

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$A \geq 0$ irreducible:

A leaves no proper face of K invariant:

$\iff (I + A)^{n-1} > 0$

Aim of talk

The aim is to relate the structure of invariant faces of the cone to the spectral properties of an operator nonnegative w.r.t. the cone.

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The aim is to relate the structure of invariant faces of the cone to the spectral properties of an operator nonnegative w.r.t. the cone.
As we related the graph theoretic and spectral properties of a nonnegative matrix both in classical nonneg alg and max alg.

Theorem

$A \geq 0$ *irreducible*

- *spec rad* $\rho(A) \in \mathrm{spec}(A)$
- $\rho(A)$ *simple eigenvalue*
- $\exists! x > 0, \; Ax = \rho x$
- $y \gneq 0, Ay = \lambda y \Longrightarrow \lambda = \rho, \; y = x$

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And for reducible matrices?

$\rho_i := \rho(A_{ii})$

i is a *distinguished vertex*: $i \stackrel{*}{\leftarrow} j \quad \Longrightarrow \quad \rho_i > \rho_j$

Theorem

Let A be a nonnegative matrix in FNF. Then the nonnegative eigenvectors of A correspond to the distinguish vertices of A: for for each distinguished vertex i of $\Delta(A)$ there is nonnegative eigenvector x^i with $Ax^i = \rho_i x^i$ such that

$$x_j^i > 0 \quad \text{if} \quad i \stackrel{*}{\leftarrow} j$$

$$x_j^i = 0 \quad \text{otherwise}$$

These are linearly independent, and all others are nonneg lin combs.

$F \trianglelefteq K$: F an invariant face of K

$\rho_F = \rho(A|F)$

$F \trianglelefteq K$ distinguished:
F, G invariant, $G \triangleleft F \implies \rho_G < \rho_F$

Theorem

K a proper cone in $\mathbb{R}^n$. Each distinguished face contains an eigenvector of ρ_F in its relative interior.
For any particular eigenvalue, the eigenvectors so obtained are the extremals of the cone of nonnegative eigenvectors.

chains of semi-dist inv faces

$F \trianglelefteq K$ is *semi-distinguished* :

$$G \trianglelefteq F, \; G \text{ invariant} \Longrightarrow \rho_G \leq \rho_F$$

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$A \in \mathbb{R}_+^{n\times n}$ in FNF
$\text{ind}_\lambda(A)$: = max size of J–block for λ
= $\min\{k\ :\ \mathcal{N} = \mathcal{N}(\lambda I - A)^{k+1} = \mathcal{N}(\lambda I - A)^k\}$

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Invariant $F\lambda$-face: $\rho_F = \lambda$

$$K \trianglerighteq F_1 \triangleright F_2 \triangleright \cdots \triangleright F$$

: proper chain of inv semi-dist λ faces
ord_λ = max length of such a chain of faces

Tam-S (2001)

Theorem

- $ind_\lambda \leq ord_\lambda$
- *If K is polyhedral (fin gen),* $ind_\rho = ord_\rho$

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- *$ind_\lambda \leq ord_\lambda$*
- *If K is polyhedral (fin gen), $ind_\rho = ord_\rho$*

If $K = \mathbb{R}^n_+$ this is a generalization by Hershkowitz-S of Rothblum's theorem (case $\lambda = \rho$)

Tam-S, Matrices leaving a cone invariant , Art. 26, Handbook of Linear Algebra (ed. L. Hogben) 2007

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THANK YOU

And please allow me one more slide!

17.10.50

LINEAR OPERATIONS IN PROBABILITY.

Prof. A.C. AITKEN

Fundamental Thms. in prob.:

(I) "Addition thm" for mutually exclusive events:

If $E_1, E_2 \ldots E_n$ are n events, mutually exclusive, respect. prob., $p_1 p_2 \ldots p_n$, [illegible] $\Sigma p_i = 1$.

then the prob. in a single trial of $E_\alpha, E_\beta \ldots E_\mu$ happen. is

$p_\alpha + p_\beta \quad + p_\mu$.

(II) Multiplication thm., for indep. events

If $E_1 E_2, E_k$ are indep. events, or events belonging to indep. systems, [the p_i are not nec. sum s.t. $\Sigma p_i = 1$] the prob. that all events occur together is $p_1 p_2 \ldots p_k$.

(III) Multiplication for dep. events in a prescribed