SPECTRAL THEORY OF REDUCIBLE NONNEGATIVE MATRICES IN MAX ALGEBRA

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Chemnitz
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$$\text{max plus } (\mathbb{R}, \text{max}, +)$$
$$\text{max times } (\mathbb{R}_0^+, \text{max}, \times)$$
$$\text{min plus } (\mathbb{R}, \text{min}, +) = \text{tropical}$$
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min plus $(\mathbb{R}, \min, +)$ = tropical
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We do max times

MAX ALGEBRA

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$$a \oplus b = \max(a, b)$$

$$a \otimes b = ab$$

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Just like $(\mathbb{R}_0^+, +, \times)$?

same vs different

$$a \oplus b = 0 \Longrightarrow a = b = 0$$

same vs different

$$a \oplus b = 0 \Longrightarrow a = b = 0$$

$$a \oplus b = a \not\Longrightarrow b = 0$$

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$$a \oplus b = a \Longrightarrow a \geq b$$

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$$a \oplus a = a$$

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Our aim is to compare and contrast the two theories

where is the difference?

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graph recap

$$A \in \mathbb{R}_+^{n \times n}, \; A \geq 0$$

$\mathcal{G}(A)$: Graph of A

Vertex set $\{1, \ldots, n\}$

arcs $i \to j \quad : \quad a_{ij} > 0$

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$$i_0 \xrightarrow{*} i_k \ : \quad \exists (i_1, \ldots, i_{k-1}) \quad i_0 \to i_1 \cdots \to i_{k-1} \to i_k$$
$$\text{or} \quad i = j$$

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cycle γ: $i_0 \xrightarrow{*} i_k,\ i_0 = i_k$

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$$\text{cycle } \gamma\text{: } i_0 \xrightarrow{*} i_k,\ i_0 = i_k$$

$$\text{cycle mean } \bar{\gamma}(A) = (a_{i_0,i_1} \cdots a_{i_{k-1},i_k})^{1/k}$$

$$\rho(A) = \max \bar{\gamma}(A), \gamma(A) \in cG(A)$$

Critical graph

Critical graph $\mathcal{C}(A)$: Graph induced in $\mathcal{G}(A)$ by the vertices of arcs lying on max cycles.

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$$A = \begin{pmatrix} 3/4 & 1 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3/4 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1 & 3/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 3/4 & 0 & 0 & 0 & 0 & 3/4 & 0 \end{pmatrix}$$

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$\rho(A) = 1$

Components of $\mathcal{C}(A)$: $\{1,2,3,4\},\{5\}$

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Very special?

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Not really!

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Not really!

Observation: If $B = X^{-1}AX$, where X is a pos diag matrix then

$$b_{ij} = \frac{x_i a_{ij}}{x_j}$$

$$\mathcal{G}(B) = cG(A)$$

$$\bar{\gamma}(A) = \bar{\gamma}(B), \ \forall \text{ cycles } \gamma$$

$$\rho(B) = \rho(A)$$

$$\mathcal{C}(B) = cC(A)$$

Fiedler-Ptak(1967, 1969), M.Schneider -S (1990)

Theorem

Let $A \in \mathbb{R}_+^{n \times n}$. There exists a pos diag X such that for $B = X^{-1}AX$,

$$\begin{array}{ll} b_{ij} = \rho(B) & \text{if } (i,j) \in \mathcal{C}(B) \\ b_{ij} < \rho(B) & \text{otherwise} \end{array}$$

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We may assume matrix is **strictly visualized**

Frobenius (1912)

Theorem

Let $A \geq 0$ be irreducible. Then its spec rad $\rho(A)$ is its the unique eigenvalue with an assoc nonneg eigenvector

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$\rho(A)$ is the **Perron root** of A

Cunninghame-Greene (1960s)

Theorem

Let $A \geq 0$ be irreducible. Then its max cyc mean $\rho(A)$ is its unique (dist) eigenvalue. There is an (ess) unique associated positive eigenvector for each component of the crit graph, Cunninghame-Greene (1960s)

Theorem

Let $A \geq 0$ be irreducible. Then its max cyc mean $\rho(A)$ is its unique (dist) eigenvalue. There is an (ess) unique associated positive eigenvector for each component of the crit graph, which are the extremals of the max cone of eigenvectors.

$\rho(A)$ will be called the **(max) Perron root** of A

$$A = \begin{pmatrix} 3/4 & 1 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3/4 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1 & 3/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 3/4 & 0 & 0 & 0 & 0 & 3/4 & 0 \end{pmatrix} \quad \begin{matrix} 1 & 3/4 \\ 1 & 3/4 \\ 1 & 3/4 \\ 1 & 3/4 \\ 1/2 & 1 \\ 3/8 & 9/32 \\ 3/4 & 9/16 \end{matrix}$$

Two evectors of $\rho(A) = 1$

collect strong conn cpts [classes] of $\mathcal{G}(A)$
and linearly order them

After permutation similarity

$$A = \begin{bmatrix} A_{11} & 0 & \dots & \dots & 0 \\ A_{21} & A_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ A_{k1} & A_{k2} & \dots & \dots & A_{kk} \end{bmatrix}$$

each diagonal block irreducible

Reduced graph $\mathcal{R}(A)$

$$V = \{1,, \dots, k\}$$

$$i \rightarrow j \in E: \; A_{ij} \neq 0$$

Path from i to j

$$i_0 \rightarrow i_1 \rightarrow \cdots \rightarrow i_{p-1} \rightarrow i_m$$

Transitive closure $\mathcal{R}^*(A)$

$$i \xrightarrow{*} j: \; \text{exists path from } i \text{ to } j$$

Skeleton $S = \mathcal{R}_*(A)$

$$(i,j) \in S : i \xrightarrow{*} k \xrightarrow{*} j \text{ implies } k = i \text{ or } k = j$$

Example

$$\begin{array}{ccccc} (1) & \longleftarrow & (2) & \longleftarrow & (3) \\ & ?\nwarrow & \uparrow & & \\ & & (4) & & \end{array}$$

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$$\begin{pmatrix} \spadesuit & 0 & 0 & 0 \\ \heartsuit & \spadesuit & 0 & 0 \\ \diamondsuit & \heartsuit & \spadesuit & 0 \\ \diamondsuit & \heartsuit & 0 & \spadesuit \end{pmatrix}$$

$\spadesuit$ irred block
$\heartsuit$ nonzero block
$\diamondsuit$ in trans closure of skeleton

Vertex set $\{1, \ldots, k\}$ (classes)

$$i \to j \iff A_{ij} \gneq 0$$

j has access to i in $\mathcal{R}(A)$:

$$i \stackrel{*}{\leftarrow} j$$

Each vertex marked with its (max) Perron root

Distinguished, semi-distinguished vertices

i distinguished

$$i \stackrel{*}{\leftarrow} j \Longrightarrow \rho_i > \rho_j$$

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$$\begin{array}{ccccc} (\rho_1) & \longleftarrow & (\rho_2) & \longleftarrow & (\rho_3) \\ & ?\nwarrow & \uparrow & & \\ & & (\rho_4) & & \end{array}$$

$$\rho_1 = \rho_2 > \rho_3 > \rho_4$$

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$2, 3, 4$ distinguished, 1 semi-distinguished

Gaubert (1990s),
Butkovic&Cuninghame-Green&Gaubert(2009)

Theorem

Let A be a nonnegative matrix in FNF. Then λ is an evalue of (A) if snd only there is a semi-distinguished vertex i with $\rho_i = \lambda$

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Theorem

Let A be a nonnegative matrix in FNF. Then λ *is an evalue of* (A) *if snd only there is a semi-distinguished vertex i with* $\rho_i = \lambda$
The eigenvectors of A correspond to the semi-distinguished vertices of A: for for each semi-distinguished vertex i of $\mathcal{R}(A)$ *there are (nonnegative) eigenvectors* x^i *with* $Ax^i = \rho_i x^i$ *such that*

$$x^i_j > 0 \quad \text{if} \quad i \leftarrow\leftarrow j$$

$$x^i_j = 0 \quad \text{otherwise}$$

MX: eigenvalues, eigenvectors

Gaubert (1990s),
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Properly chosen, these form the exremals oif the cones of eignevectors

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The nonnegative eigenvectors of A correspond to the distinguish vertices of A: for for each distinguished vertex i of $\mathcal{R}(A)$ there is nonnegative eigenvector x^i with $Ax^i = \rho_i x^i$ such that

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$$x_j^i > 0 \quad \text{if} \quad i \leftarrow\leftarrow j$$

$$x_j^i = 0 \quad \text{otherwise}$$

Properly chosen, these are linearly independent, and for any part evalue, form the extremals of the cone of nonneg evectors

NN example

$$A = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \end{pmatrix}$$

$$[4] \quad \leftarrow \quad [5]^{**} \quad \leftarrow [2]^{**}$$

$$\begin{matrix} 0 & . & 0 \\ 1 & . & 0 \\ 1 & . & 0 \\ 1 & . & 1 \end{matrix}$$

MX example

$$A = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \end{pmatrix}$$

$$\left([4]^* \quad \leftarrow \quad [4]^{**} \quad \leftarrow [2]^{**}\right)$$

$$\begin{matrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 3/16 & 3/4 & 0 \\ 9/64 & 3/8 & 1 \end{matrix}$$

P. Butkovic
Max-Linear Systems: Theory and Algorithms
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That's it for today

next time:
Commuting matrices in three incarnations

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THANKS!