

# SPECTRAL THEORY OF REDUCIBLE NONNEGATIVE MATRICES IN MAX ALGEBRA

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Chemnitz  
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maxspectchmn 21 Sept 2010, 14:30

max plus  $(\mathbb{R}, \max, +)$   
max times  $(\mathbb{R}_0^+, \max, \times)$   
min plus  $(\mathbb{R}, \min, +)$  = tropical  
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We do max times

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$$a \oplus b = \max(a, b)$$

$$a \otimes b = ab$$

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Just like  $(\mathbb{R}_0^+, +, \times)$ ?

$$a \oplus b = 0 \implies a = b = 0$$

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$$a \oplus b = a \not\Rightarrow b = 0$$



$$a \oplus b = 0 \implies a = b = 0$$

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$$a \oplus a = a$$

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The spectral theory in **MX** *"is extremely similar to the well-known Perron-Frobenius theory"* in **NN**

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Our aim is to compare and contrast the two theories

$$A > 0, x \not\geq y \geq 0 \implies Ax > Ay$$

$$A > 0, x \succeq y \geq 0 \implies Ax > Ay$$

$$A > 0, x \succeq y \geq 0 \not\implies A \otimes x > A \otimes y$$

$$\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$



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$$A \in \mathbb{R}_+^{n \times n}, A \geq 0$$

$\mathcal{G}(A)$ : Graph of  $A$   
Vertex set  $\{1, \dots, n\}$   
arcs  $i \rightarrow j$  :  $a_{ij} > 0$

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$$i_0 \xrightarrow{*} i_k : \quad \exists (i_1, \dots, i_{k-1}) \quad i_0 \rightarrow i_1 \cdots \rightarrow i_{k-1} \rightarrow i_k$$

or  $i = j$

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cycle  $\gamma$ :  $i_0 \xrightarrow{*} i_k, i_0 = i_k$

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cycle  $\gamma: i_0 \xrightarrow{*} i_k, i_0 = i_k$

$$\text{cycle mean } \bar{\gamma}(A) = (a_{i_0, i_1} \cdots a_{i_{k-1}, i_k})^{1/k}$$

$$\rho(A) = \max \bar{\gamma}(A), \gamma(A) \in c\mathcal{G}(A)$$

Critical graph  $\mathcal{C}(A)$ : Graph induced in  $\mathcal{G}(A)$  by the vertices of arcs lying on max cycles.

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$$A = \begin{pmatrix} 3/4 & 1 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3/4 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1 & 3/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 3/4 & 0 & 0 & 0 & 0 & 3/4 & 0 \end{pmatrix}$$

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$$\rho(A) = 1$$

Components of  $\mathcal{C}(A)$ :  $\{1, 2, 3, 4\}, \{5\}$



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Very special?

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Not really!

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Observation: If  $B = X^{-1}AX$ , where  $X$  is a pos diag matrix then

$$b_{ij} = \frac{x_i a_{ij}}{x_j}$$

$$\mathcal{G}(B) = c\mathcal{G}(A)$$

$$\bar{\gamma}(A) = \bar{\gamma}(B), \quad \forall \text{ cycles } \gamma$$

$$\rho(B) = \rho(A)$$

$$C(B) = cC(A)$$

Fiedler-Ptak(1967, 1969), M.Schneider -S (1990)

### Theorem

Let  $A \in \mathbb{R}_+^{n \times n}$ . There exists a pos diag  $X$  such that for  $B = X^{-1}AX$ ,

$$\begin{aligned} b_{ij} &= \rho(B) && \text{if } (i,j) \in C(B) \\ b_{ij} &< \rho(B) && \text{otherwise} \end{aligned}$$

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We may assume matrix is **strictly visualized**

Frobenius (1912)

## Theorem

*Let  $A \geq 0$  be irreducible. Then its spec rad  $\rho(A)$  is its the unique eigenvalue with an assoc nonneg eigenvector*

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$\rho(A)$  is the **Perron root** of  $A$

Cunninghame-Greene (1960s)

## Theorem

*Let  $A \geq 0$  be irreducible. Then its max cyc mean  $\rho(A)$  is its unique (dist) eigenvalue. There is an (ess) unique associated positive eigenvector for each component of the crit graph,*

Cunninghame-Greene (1960s)

## Theorem

*Let  $A \geq 0$  be irreducible. Then its max cyc mean  $\rho(A)$  is its unique (dist) eigenvalue. There is an (ess) unique associated positive eigenvector for each component of the crit graph, which are the extremals of the max cone of eigenvectors.*

$\rho(A)$  will be called the **(max) Perron root** of  $A$

$$A = \begin{pmatrix} 3/4 & 1 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3/4 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1 & 3/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 3/4 & 0 & 0 & 0 & 0 & 3/4 & 0 \end{pmatrix} \begin{matrix} 1 & 3/4 \\ 1 & 3/4 \\ 1 & 3/4 \\ 1 & 3/4 \\ 1/2 & 1 \\ 3/8 & 9/32 \\ 3/4 & 9/16 \end{matrix}$$

Two e vectors of  $\rho(A) = 1$

collect strong conn cpts [classes] of  $\mathcal{G}(A)$   
and linearly order them

After permutation similarity

$$A = \begin{bmatrix} A_{11} & 0 & \dots & \dots & 0 \\ A_{21} & A_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ A_{k1} & A_{k2} & \dots & \dots & A_{kk} \end{bmatrix}$$

each diagonal block irreducible

Reduced graph  $\mathcal{R}(A)$

$$V = \{1, \dots, k\}$$

$$i \rightarrow j \in E : A_{ij} \neq 0$$

Path from  $i$  to  $j$

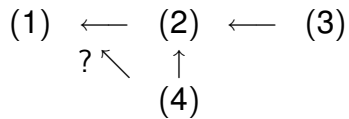
$$i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_{p-1} \rightarrow i_m$$

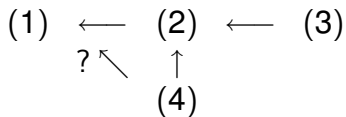
Transitive closure  $\mathcal{R}^*(A)$

$$i \xrightarrow{*} j : \text{exists path from } i \text{ to } j$$

Skeleton  $\mathcal{S} = \mathcal{R}_*(A)$

$$(i, j) \in \mathcal{S} : i \xrightarrow{*} k \xrightarrow{*} j \text{ implies } k = i \text{ or } k = j$$





$$\begin{pmatrix}
 \spadesuit & 0 & 0 & 0 \\
 \heartsuit & \spadesuit & 0 & 0 \\
 \diamondsuit & \heartsuit & \spadesuit & 0 \\
 \diamondsuit & \heartsuit & 0 & \spadesuit
 \end{pmatrix}$$

- $\spadesuit$  irred block
- $\heartsuit$  nonzero block
- $\diamondsuit$  in trans closure of skeleton



Vertex set  $\{1, \dots, k\}$  (classes)

$$i \rightarrow j \iff A_{ij} \underset{\neq}{\geq} 0$$

$j$  has access to  $i$  in  $\mathcal{R}(A)$ :

$$i \overset{*}{\leftarrow} j$$

Each vertex marked with its (max) Perron root

$i$  distinguished

$$i \overset{*}{\leftarrow} j \implies \rho_i > \rho_j$$

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$$i \leftarrow^* j \implies \rho_i > \rho_j$$

$i$  semi-distinguished

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$$\begin{array}{ccccc} (\rho_1) & \longleftarrow & (\rho_2) & \longleftarrow & (\rho_3) \\ & & ? \swarrow & & \uparrow \\ & & & & (\rho_4) \end{array}$$

$$\rho_1 = \rho_2 > \rho_3 > \rho_4$$

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2, 3, 4 distinguished, 1 semi-distinguished

Gaubert (1990s),  
Butkovic&Cuninghame-Green&Gaubert(2009)

## Theorem

*Let  $A$  be a nonnegative matrix in FNF. Then  $\lambda$  is an eigenvalue of  $(A)$  if and only if there is a semi-distinguished vertex  $i$  with  $\rho_i = \lambda$*

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*The eigenvectors of  $A$  correspond to the semi-distinguished vertices of  $A$ : for each semi-distinguished vertex  $i$  of  $\mathcal{R}(A)$  there are (nonnegative) eigenvectors  $x^i$  with  $Ax^i = \rho_i x^i$  such that*

$$x_j^i > 0 \quad \text{if} \quad i \leftarrow \leftarrow j$$

$$x_j^i = 0 \quad \text{otherwise}$$

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*Properly chosen, these form the extremals of the cones of eigenvectors.*



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*Properly chosen, these are linearly independent, and for any part evaluate, form the extremals of the cone of nonneg evectors*

$$A = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \end{pmatrix}$$

$$[4] \leftarrow [5]** \leftarrow [2]**$$

$$\begin{array}{ccc} 0 & . & 0 \\ 1 & . & 0 \\ 1 & . & 0 \\ 1 & . & 1 \end{array}$$

$$A = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \end{pmatrix}$$

$$([4]^* \leftarrow [4]** \leftarrow [2]**)$$

$$\begin{array}{ccc} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 3/16 & 3/4 & 0 \\ 9/64 & 3/8 & 1 \end{array}$$

P. Butkovic  
Max-Linear Systems: Theory and Algorithms  
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That's it for today

next time:  
Commuting matrices in three incarnations

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THANKS!