Tikhonov regularization

Solve the Tikhonov minimization problem

$$\min_x \{\|Ax - b\|^2 + \mu\|Lx\|^2\} \quad \Longrightarrow \quad x_\mu,$$

where

- $A \in \mathbf{R}^{m\times n}$;
- $L \in \mathbf{R}^{p\times n}$, $p \leq n$, is the regularization operator. Common choices: $L = I$ or a finite difference operator;
- $\mu > 0$ is the regularization parameter. It is important to determine a suitable value; see Engl, Hanke, Neubauer; Hansen; Kilmer; O'Leary; ...
- $\mathcal{N}(L) \cap \mathcal{N}(A) = \{0\} \quad \Longrightarrow \quad x_\mu$ unique for any $\mu > 0$.

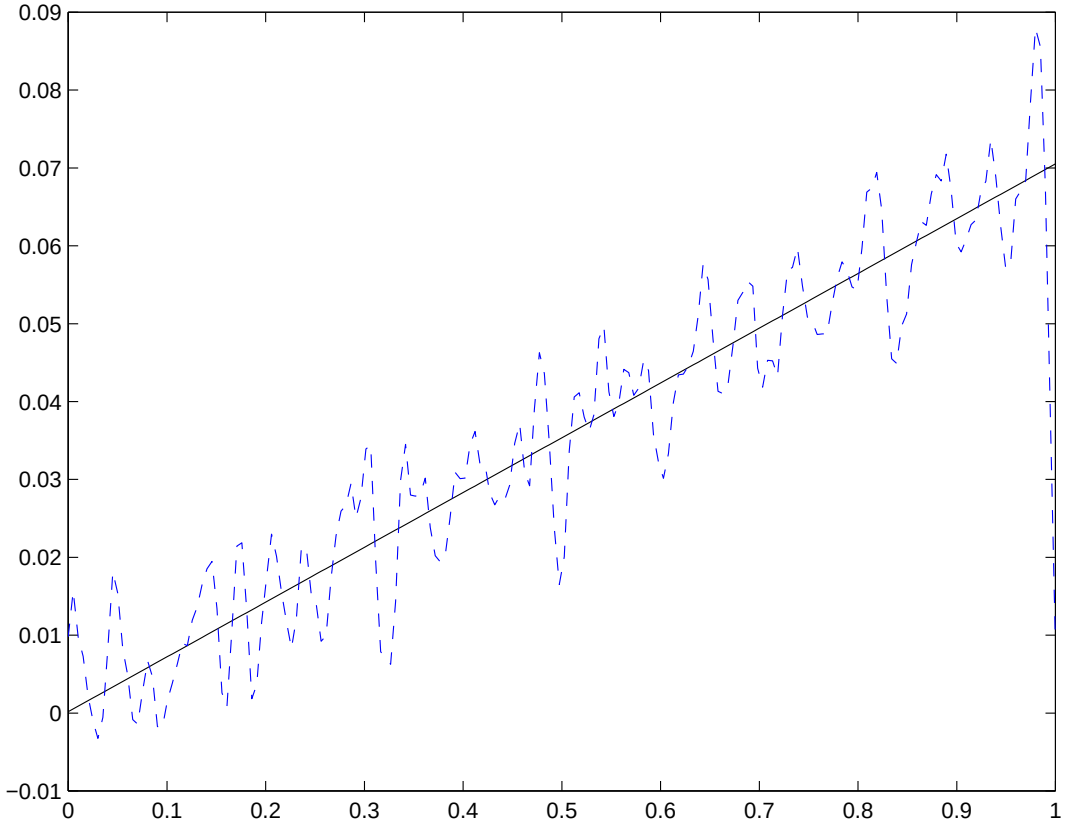
Example 2: Fredholm integral equation of the 1st kind

$$\int_0^1 k(s,t)x(t)dt = \frac{1}{6}(s^2 - s), \qquad 0 \le s \le 1,$$

$$k(s,t) = \begin{cases} s(t-1), & s < t, \\ t(s-1), & s \ge t. \end{cases}$$

Solution $x(t) = t$. Code deriv2 from Regularization Tools by Hansen discretizes by Galerkin method with 200 piecewise constant test and trial functions. Relative error (noise) in rhs 0.1%.

Solution for problem with $L = I$, μ determined by L-curve



Solution obtained with square invertible regularization operator $L = \text{tridiag}[-1, 2, -1]$, μ determined by L-curve