Numerical Methods for Ill-Posed Problems III

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Outline of Lecture 3:

- Tikhonov regularization of large-scale problems
  - The discrepancy principle
  - Solution norm constraint
  - Nonnegativity constraint
- Truncated iteration
- Multilevel methods
- Alternating iterative methods

Tikhonov regularization and the discrepancy principle

Write the Tikhonov minimization problem in the form

$$\min\{\|Ax - b\|^2 + \frac{1}{\mu}\|x\|^2\}, \qquad \mu > 0.$$

Define discrepancy associated with x_μ,

$$d_\mu = b - Ax_\mu$$

Assume that an upper bound ϵ for norm of error e in b known.

The discrepancy principle prescribes that μ should be chosen so that

$$\|d_\mu\| = \epsilon.$$

To avoid underregularization, choose $\hat{\mu}$ so that

$$\epsilon \leq \|d_{\hat{\mu}}\| \leq \epsilon\eta$$

for some $\eta > 1$ and compute approximation of $x_{\hat{\mu}}$.

Define

$$\phi(\mu) := \|b - Ax_\mu\|^2.$$

After ℓ Lanczos bidiagonalization steps, we can evaluate the ℓ-point Gauss rule

$$\phi_\ell(\mu) := \|b\|^2 e_1^T (\mu C_\ell C_\ell^T + I_\ell)^{-2} e_1$$

and the $(\ell + 1)$-point Gauss-Radau rule

$$\bar{\phi}_\ell(\mu) := \|b\|^2 e_1^T (\mu \bar{C}_\ell \bar{C}_\ell^T + I_{\ell+1})^{-2} e_1.$$

Recall that

$$\phi_\ell(\mu) < \phi(\mu) < \bar{\phi}_\ell(\mu).$$

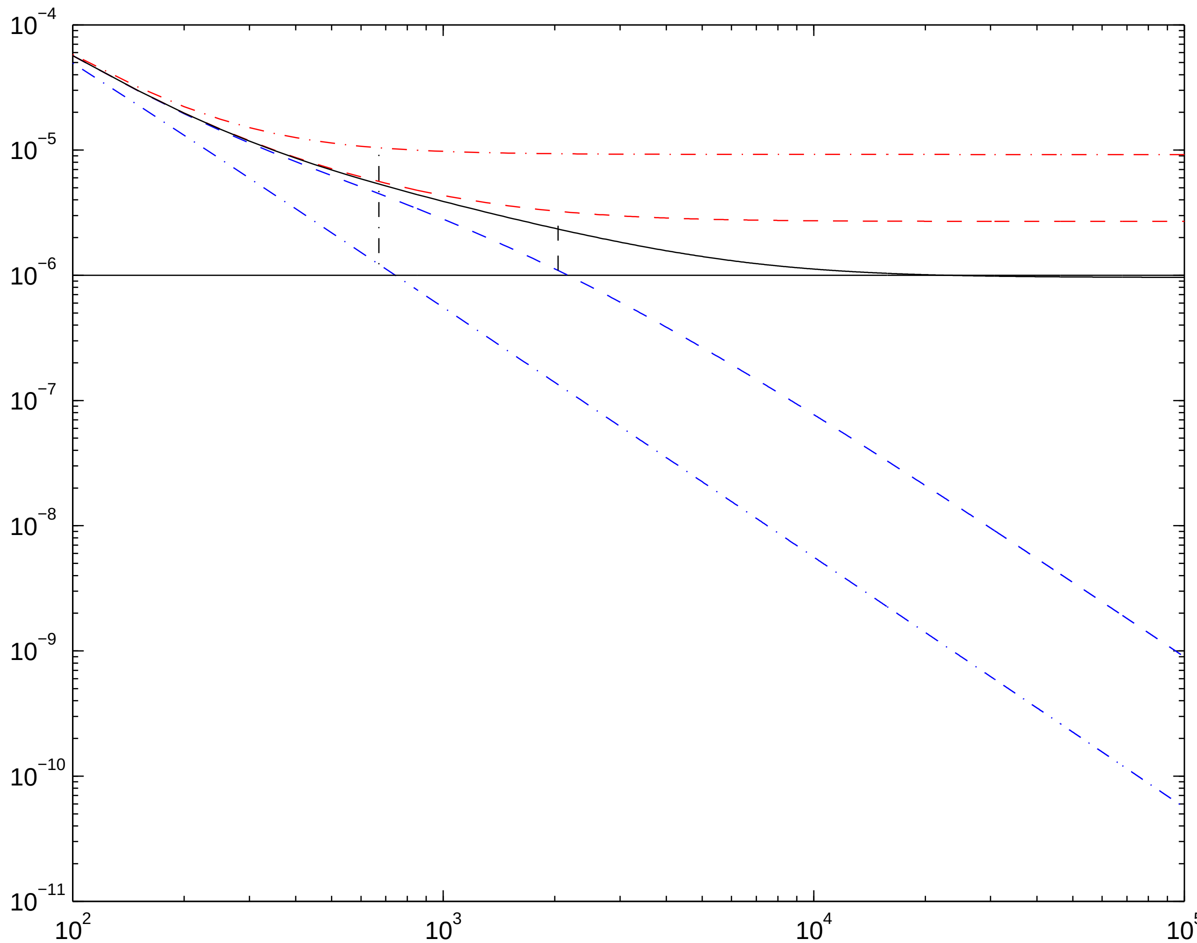
We want to determine ℓ and $\hat{\mu}$ such that

$$\epsilon^2 \leq \phi_\ell(\hat{\mu}) \qquad \bar{\phi}_\ell(\hat{\mu}) \leq \eta^2\epsilon^2,$$

from which it follows that

$$\epsilon^2 \leq \phi(\mu) \leq \eta^2\epsilon^2.$$

Let μ_* solve $\phi(\mu_*) = \epsilon^2$.



10^−4
10^−5
10^−6
10^−7
10^−8
10^−9
10^−10
10^−11
10^2
10^3
10^4
10^5

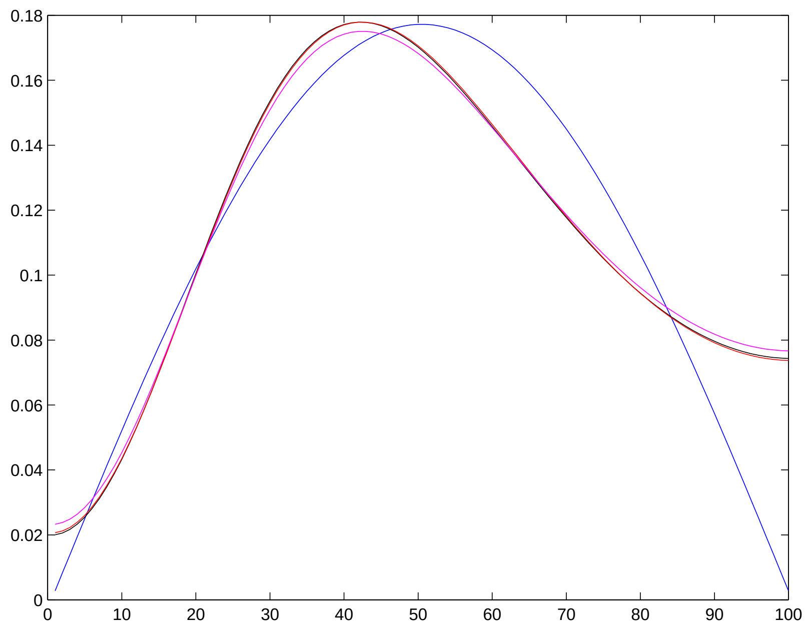
Selection of $\hat{\mu}$ according to Morozov discrepany principle.

1. Choose $\mu < \mu_*$, e.g., $\mu = 0$.
2. Solve $\phi_\ell(\mu) = \epsilon^2$ e.g. by Newton's method.
   $\hat{\mu}$ = solution.
3. If $\bar{\phi}_\ell(\hat{\mu}) \le \epsilon^2\eta^2$ then accept $\hat{\mu}$. Done!
4. $\ell = \ell + 1$
5. Go to 2) using $\hat{\mu}$ as starting value for Newton's method

Monotonic convergence to solution from the left, thus avoiding underregularization

Baart, $n = 100$, $\frac{\|e\|}{\|b\|} = 10^{-3}$

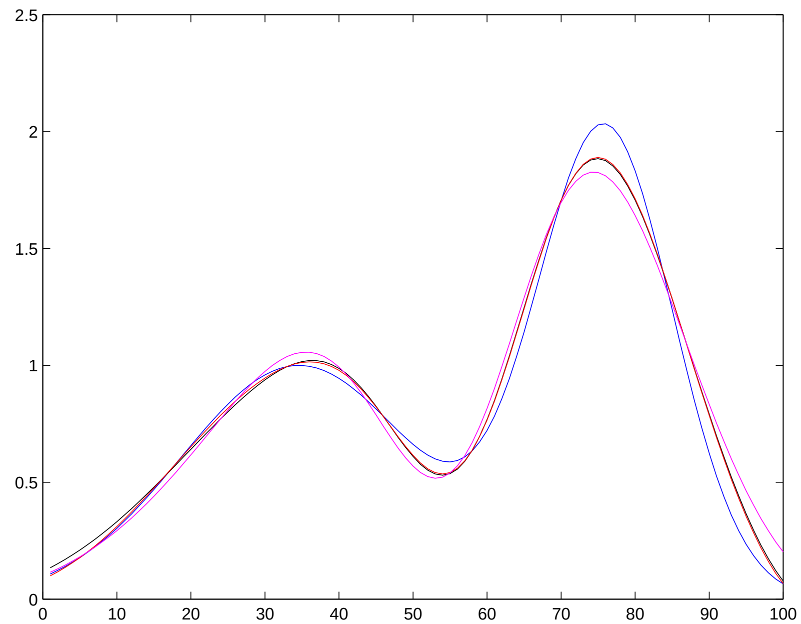
$\eta = 2$, $\ell = 3 \Longrightarrow$ 6 mat-vec; $\|x_{\mu_3,3} - \hat{x}\| = 1.2 \cdot 10^{-1}$



$\hat{x}$ (blue), $x_{\mu_3,3}$ (red), Tikhonov solution, x_{μ_3} (green)

Shaw, $\quad n = 100, \quad \frac{\|e\|}{\|b\|} = 10^{-3}$

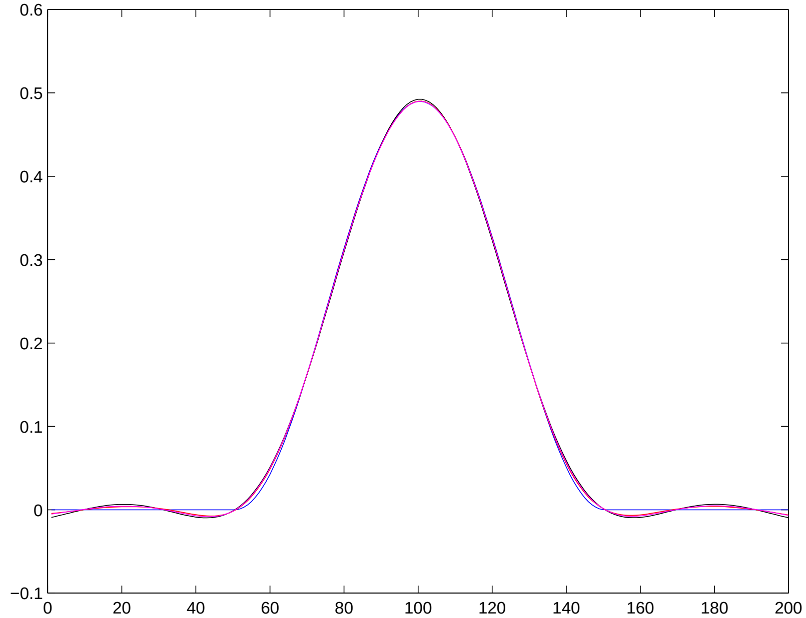
$\eta = 2, \quad \ell = 5 \Longrightarrow$ 10 mat-vec; $\quad \|x_{\mu_5,5} - \hat{x}\| = 7.3 \cdot 10^{-2}$



$\hat{x}$ (blue), $x_{\mu_5,5}$ (red), Tikhonov solution, x_{μ_5} (green)

Phillips: $n = 200$, $\frac{\|e\|}{\|b\|} = 10^{-3}$

$\eta = 2$, $\ell = 4 \Longrightarrow 8$ mat-vec



$\hat{x}$ (blue), $x_{\mu_4,4}$ (red), Tikhonov solution, x_{μ_4} (green)

Tikhonov regularization with a solution norm constraint

Assume that $\|\hat{x}\| = \Delta$ is known. Solve

$$\min_{\|x\|\|=\Delta} \|b - Ax\|$$

(a so-called trust-region subproblem)

Denote the solution by x_*.

Theorem: Under suitable conditions, there exist a unique $\mu_* > 0$, such that

$$x_* = (A^T A + \mu_* I)^{-1} A^T b.$$

Cf. Tikhonov regularization.

Thus

$$\|x_\mu\|^2 = b^T A(A^T A + \mu I)^{-2} A^T b.$$

We use Gauss and Gauss-Radau quadrature rules to obtain lower and upper bounds for $\|x_\mu\|^2$.

Determine ℓ and μ, so that

$$\eta^2 \Delta^2 \leq \hat{G}_\ell(\phi), \quad \hat{R}_\ell(\phi) \leq \Delta^2$$

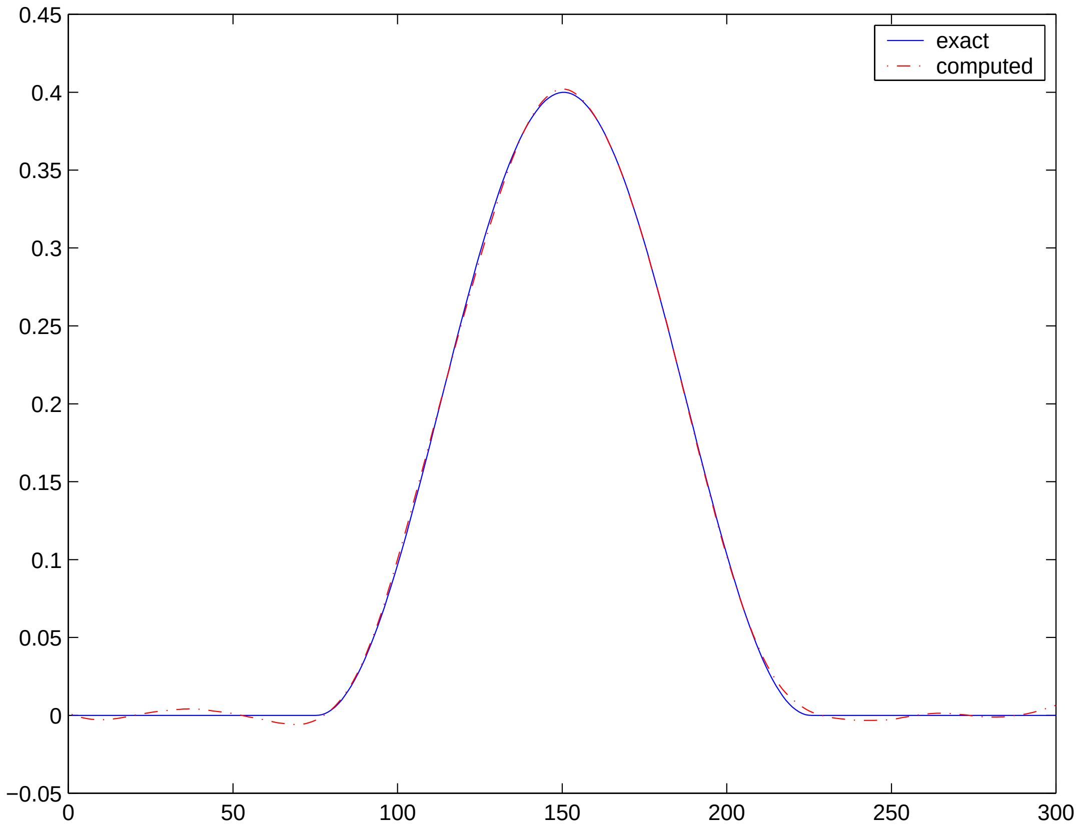
for some $0 < \eta < 1$. Then

$$\eta^2 \Delta^2 \leq \|x_\mu\|^2 \leq \Delta^2.$$

Generally, $\eta \approx 1$.

Phillips; $n = 300$, $\eta = 0.999$, $\frac{\|e\|}{\|b\|} = 6.5 \cdot 10^{-3}$.

$\Delta = 2.9999$, $\ell = 8$, $\Longrightarrow$ 16 mat-vec.



Tikhonov regularization with nonnegativity constraint

Assume $\|\hat{x}\| = \Delta$ known and $\hat{x} \geq 0$.

Consider

$$\min_{\substack{\|x\|=\Delta \\ x \geq 0}} \|b - Ax\|$$

Introduce barrier function

$$f_\gamma(x) = \frac{1}{2}\|b - Ax\|^2 - \gamma \sum_{i=1}^{n} \log \xi_i,$$

$$x = [\xi_1, \xi_2, \ldots, \xi_n]^T.$$

Minimization problem

$$\min_{\|x\|=\Delta} f_\gamma(x)$$

Quadratic model

$$q_\gamma(x+h) = f_\gamma(x) + \nabla f_\gamma(x)^T h + \frac{1}{2} h^T \nabla^2 f_\gamma(x) h.$$

Trust-region subproblem

$$\min_{h, \quad \|x+h\|=\Delta} q_\gamma(x+h)$$

Solution $x_\mu = x + h$ is of the form

$$x_\mu = (A^T A + \gamma X^{-2} + \mu I)^{-1} (A^T b + 2\gamma X^{-1} e),$$

where

$$X = \operatorname{diag}[x_\mu], \qquad e = [1, 1, \ldots, 1]^T,$$

$\mu \geq 0$ chosen so that $\|x_\mu\| = \Delta$.

0. Solve problem without nonnegativity constraint for initial approximate solution x of constrained problem. Let $x := \max\{x, \delta\}$ for some $\delta > 0$. Let $X = \text{diag}[x]$.

Until convergence do

1. Apply ℓ Lanczos steps to the matrix $A^T A + \gamma X^{-2}$ with initial vector $A^T b + 2\gamma X^{-1} e$.

   Use Gauss rules to determine ℓ. Gives $x_{\mu,\ell} \approx x_\mu$ with μ such that $\|x_{\mu,\ell}\| = \Delta$.

2. Update γ, let $X = \text{diag}[x_{\mu,\ell}]$.

3. Go to 1.

Phillips, $n = 300$, $\epsilon = \frac{\|e\|}{\|\hat{b}\|} = 5 \cdot 10^{-3}$

a) Linear problem (in x)

$\ell = 8$, $\frac{\|x_{\mu,\ell} - \hat{x}\|}{\|\hat{x}\|} = 1.9 \cdot 10^{-2}$

$x_{\mu,\ell,0} = \max\{x_{\mu,\ell}, 0\}$ componentwise

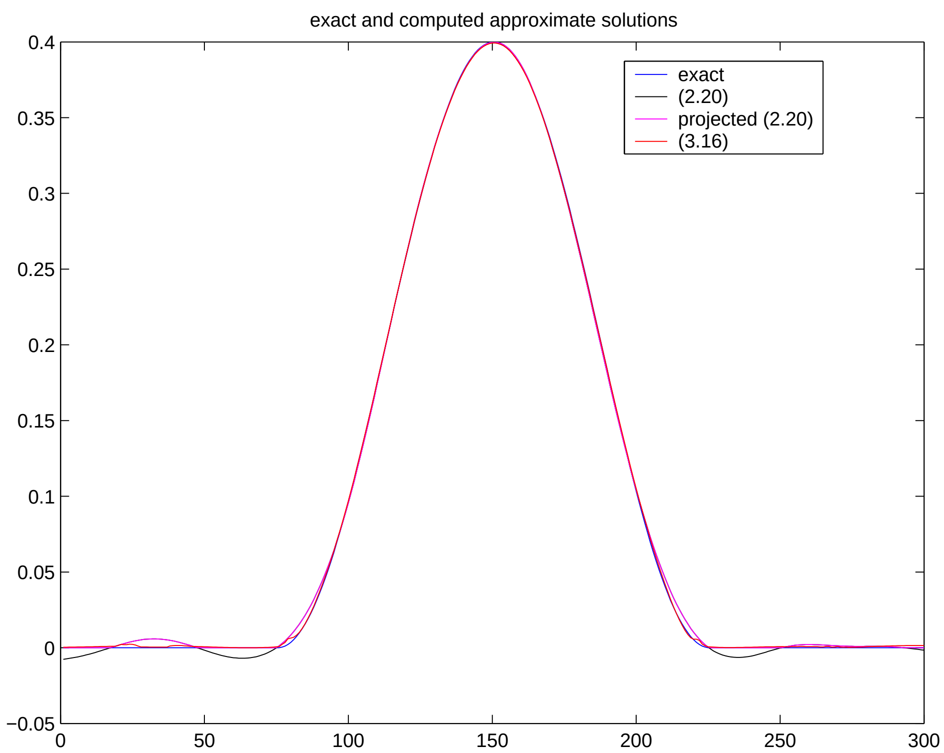
$$\frac{\|x_{\mu,\ell,0} - \hat{x}\|}{\|\hat{x}\|} = 1.4 \cdot 10^{-2}$$

b) Nonlinear problem

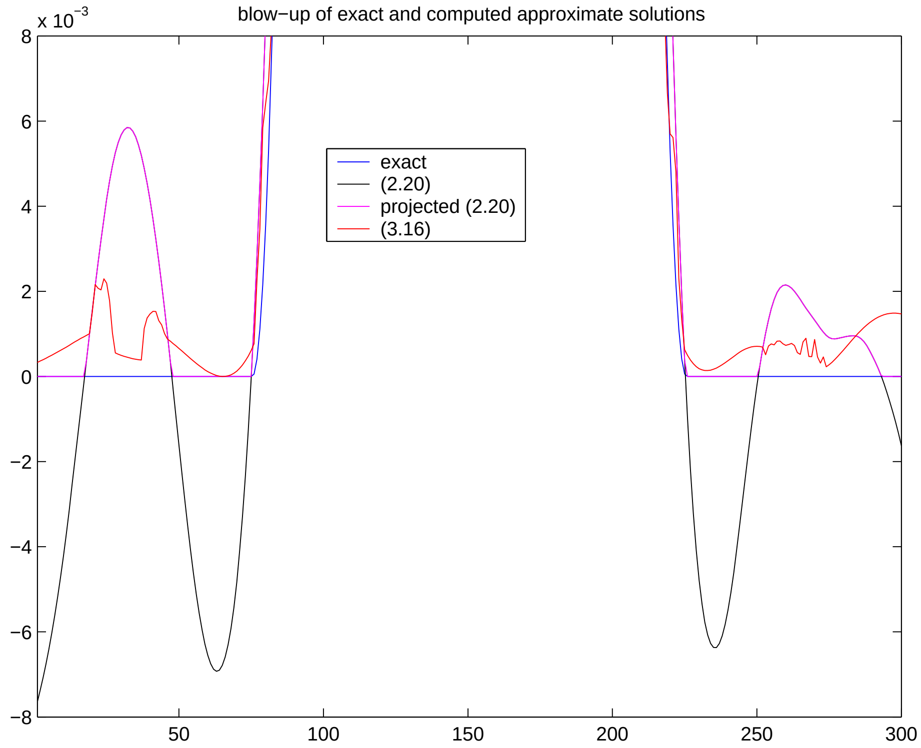
21+8 Lanczos steps give $\tilde{x}_\mu$

$$\frac{\|\tilde{x}_\mu - \hat{x}\|}{\|\hat{x}\|} = 5.4 \cdot 10^{-3}$$

Total number of mat-vec products: 79


exact and computed approximate solutions
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−0.05
0
50
100
150
200
250
300
exact
(2.20)
projected (2.20)
(3.16)

Blow-up



blow−up of exact and computed approximate solutions
x 10^−3
8
6
4
2
0
−2
−4
−6
−8
50
100
150
200
250
300
exact
(2.20)
projected (2.20)
(3.16)

Hemispheres, $n = 256^2$, $\epsilon = \frac{\|e\|}{\|\hat{b}\|} = 10^{-3}$,

Nonsymmetric blurring matrix that models Gaussian blur.

a) Linear problem (in x)

$\ell = 26$, $\frac{\|x_{\mu,\ell} - \hat{x}\|}{\|\hat{x}\|} = 8.3 \cdot 10^{-2}$

$x_{\mu,\ell,0} = \max\{x_{\mu,\ell}, 0\}$ componentwise

$$\frac{\|x_{\mu,\ell,0} - \hat{x}\|}{\|\hat{x}\|} = 8.1 \cdot 10^{-2}$$

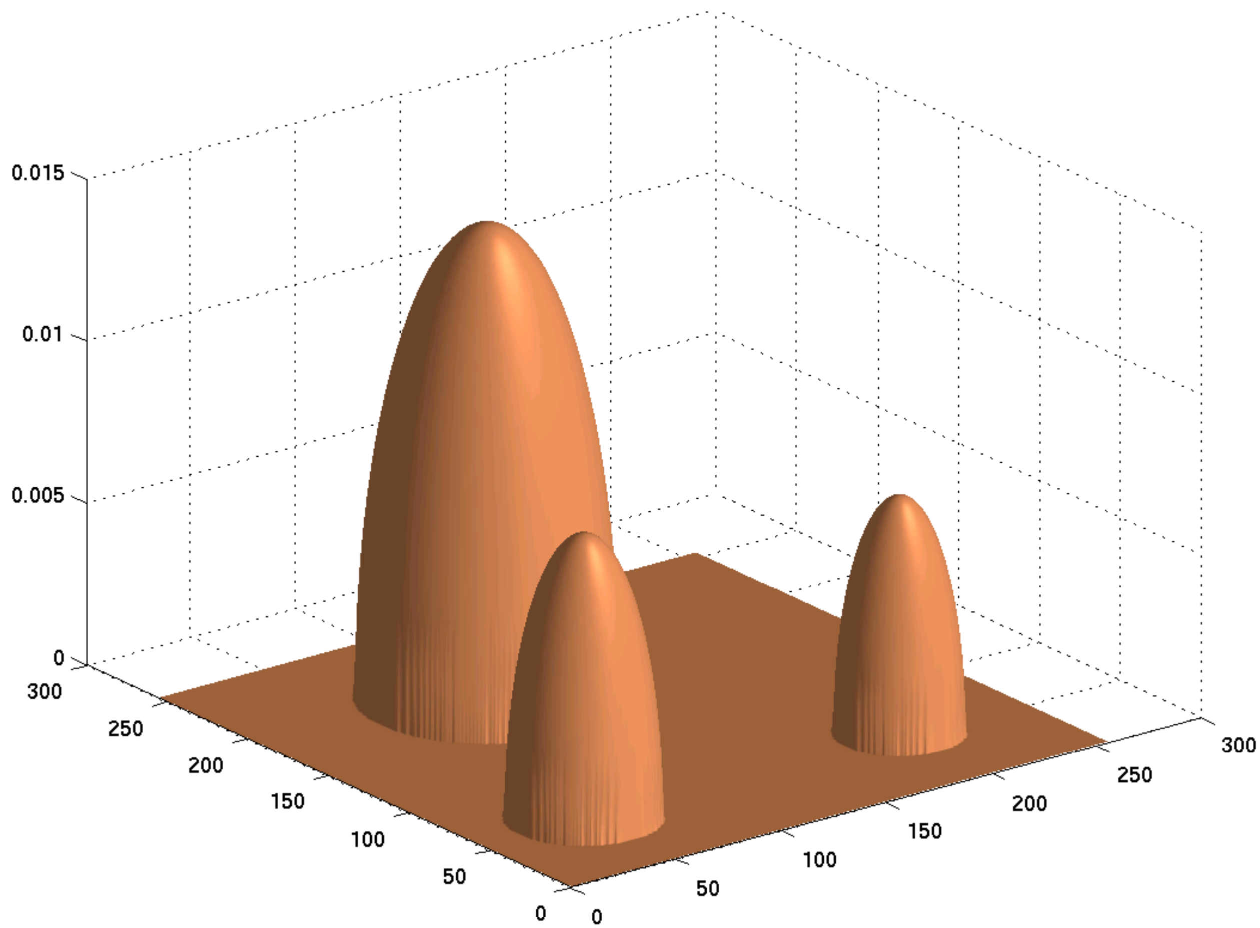
b) Nonlinear problem

27 Lanczos steps give $\tilde{x}_\mu$

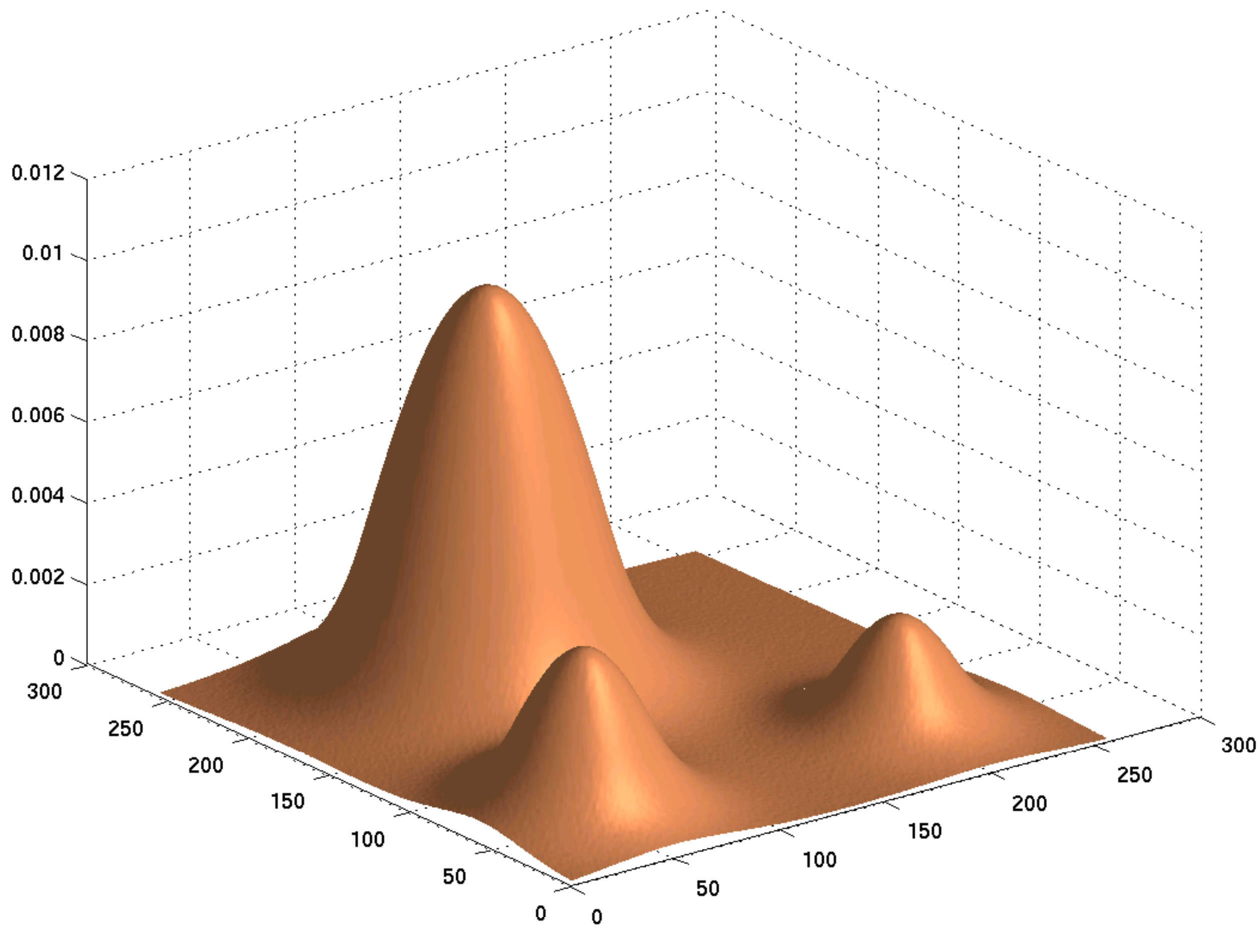
$$\frac{\|\tilde{x}_\mu - \hat{x}\|}{\|\hat{x}\|} = 7.3 \cdot 10^{-2}$$

Total number of mat-vec products: 110

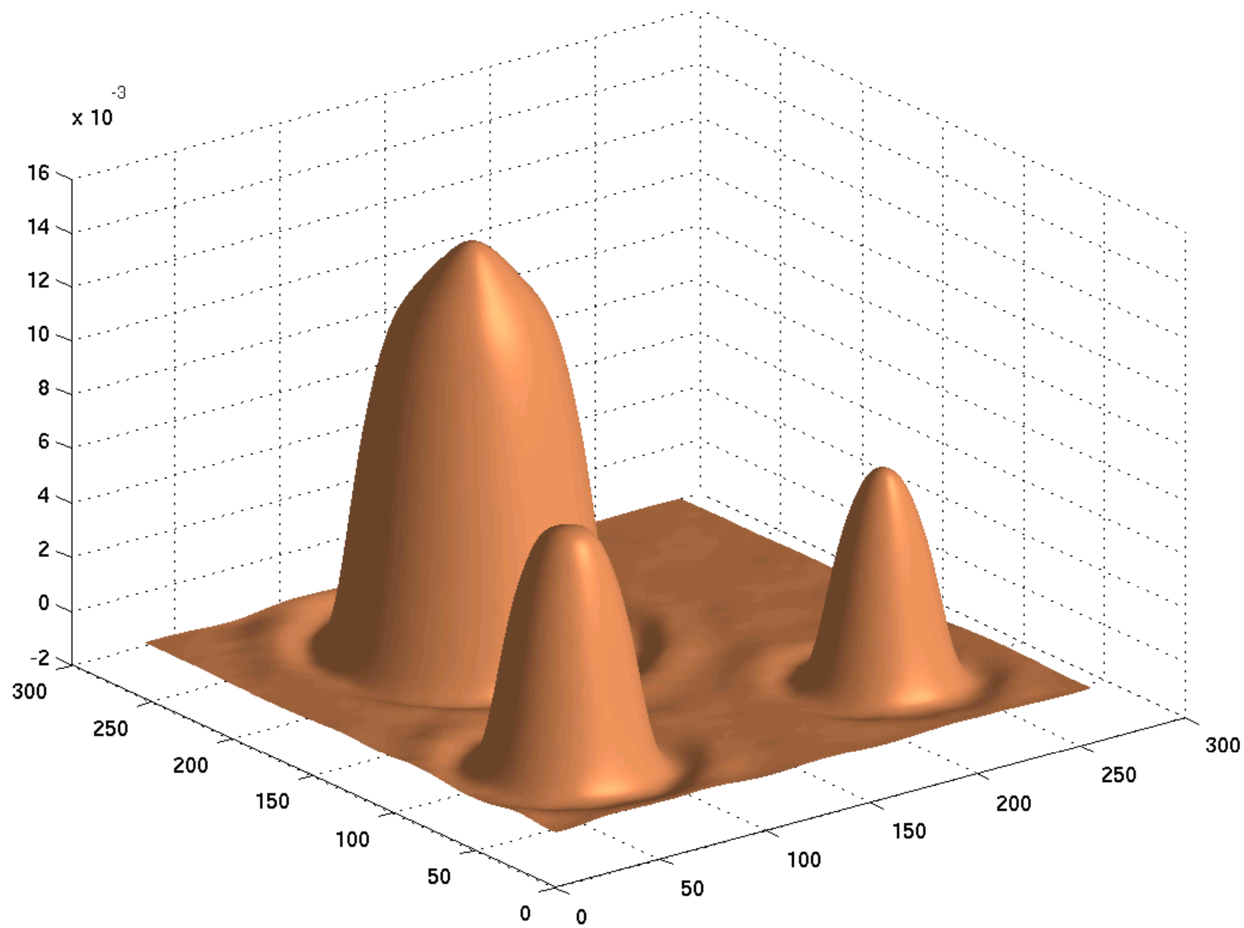
Blur- and noise-free image



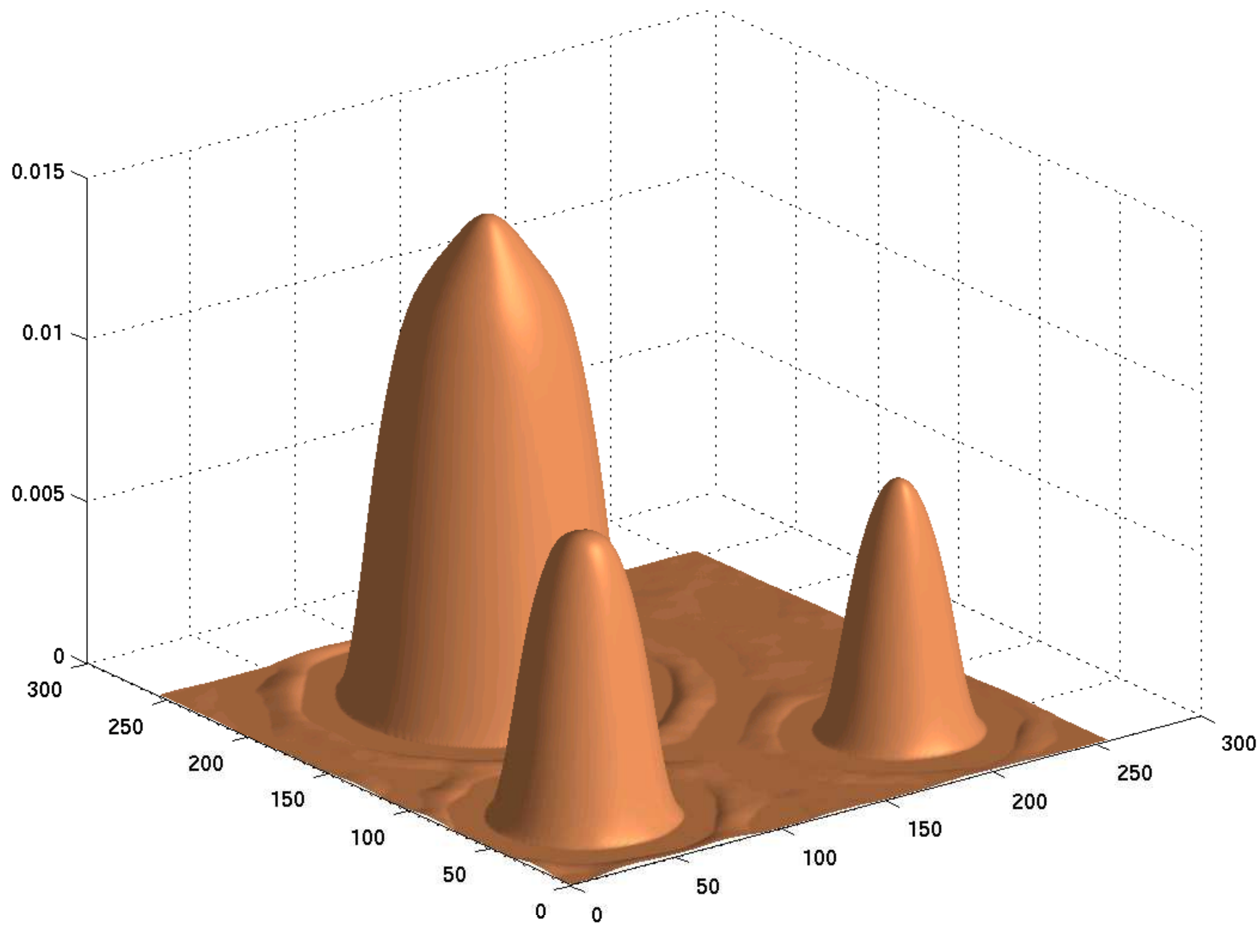
Blurred and noisy image



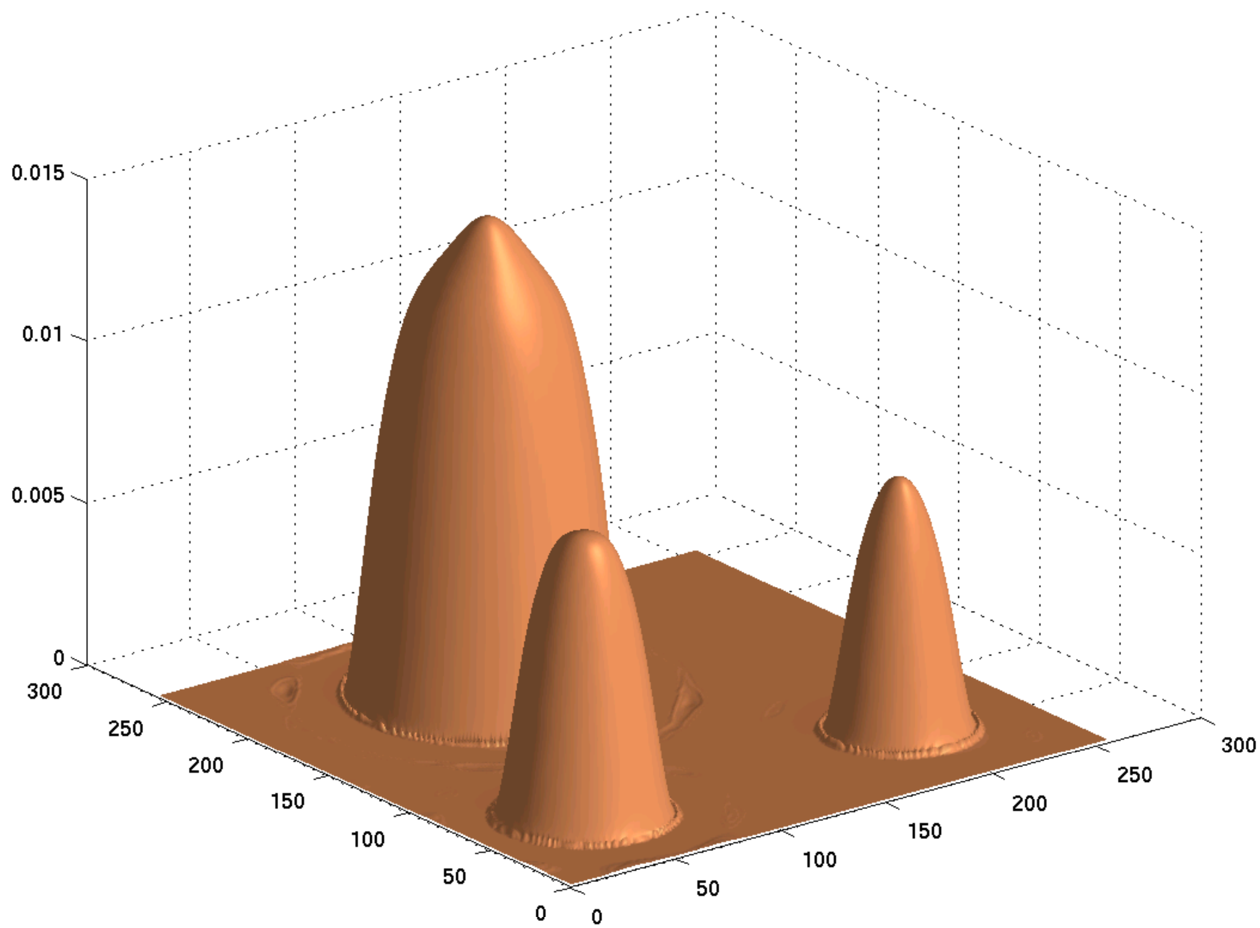
Computed solution without nonnegativity constraint



Solution without nonnegativity constraint after projection



Computed solution with nonnegativity constraint



Regularization by truncated iteration

CGNR: CG applied to $A^*Ax = A^*b$

Define the Krylov subspace

$$\mathcal{K}_m(A^*A, A^*b) = \text{span}\{A^*b, (A^*A)A^*b, \ldots, (A^*A)^{m-2}A^*b, (A^*A)^{m-1}A^*b\}.$$

Then $x_m \in \mathcal{K}_m(A^*A, A^*b)$ and

$$\|Ax_m - b\| = \min_{x \in \mathcal{K}_m(A^*A, A^*b)} \|Ax - b\|$$

Therefore discrepancy $d_j = b - Ax_j$ satisfies

$$\|b\| \geq \|d_1\| \geq \ldots \geq \|d_m\|.$$

Stopping Criterion

Discrepancy principle

Let $\alpha > 1$ be fixed, $\|e\| = \|\hat{b} - b\| = \delta$. The iterate x_m satisfies the discrepancy principle if

$$\|Ax_m - b\| \leq \alpha\delta$$

Stopping rule

Terminate the iterations as soon as iterate x_m satisfies

$$\|Ax_m - b\| \leq \alpha\delta$$

$$\|Ax_{m-1} - b\| > \alpha\delta$$

Denote the termination index by $m(\delta)$.

An iterative method is a regularization method if

$$\lim_{\delta \searrow 0} \sup_{\|e\| \leq \delta} \|x_{m(\delta)} - \hat{x}\| = 0$$

CGNR is a regularization method; see Nemirovskii, Hanke.

Other iterative methods:

Range-restrcited minimal residual methods for symmetric problems

Define the Krylov subspace

$$\mathcal{K}_m(A, Ab) = \text{span}\{Ab, A^2b, \ldots, A^m b\}.$$

Then $x_m \in \mathcal{K}_m(A, Ab)$ and

$$\|Ax_m - b\| = \min_{x \in \mathcal{K}_m(A,Ab)} \|Ax - b\|.$$

Hanke showed they are regularization methods.

GMRES: A minimal residual method for nonsymmetric problems

Define the Krylov subspace

$$\mathcal{K}_m(A, b) = \text{span}\{b, Ab, \ldots, A^{m-1}b\}.$$

Then $x_m \in \mathcal{K}_m(A, b)$ and

$$\|Ax_m - b\| = \min_{x \in \mathcal{K}_m(A,b)} \|Ax - b\|.$$

This is a regularization method under stronger conditions than CGNR.

RRGMRES (Range Restricted GMRES): A minimal residual method for nonsymmetric problems

Define the Krylov subspace

$$\mathcal{K}_m(A, Ab) = \mathrm{span}\{Ab, A^2b, \ldots, A^m b\}.$$

Then $x_m \in \mathcal{K}_m(A, Ab)$ and

$$\|Ax_m - b\| = \min_{x \in \mathcal{K}_m(A,Ab)} \|Ax - b\|.$$

This is a regularization method under the same conditions as GMRES.

PDE methods
for image restoration

Continuous image degradation model

$$f(x) = \int_\Omega h(x-y)\hat{u}(y)dy + \eta(x), \quad x \in \Omega,$$

with h a (for now) symmetric point spread function.

The integral equation can be expressed as

$$f = h * \hat{u} + \eta.$$

Discretization yields

$$b = Au$$

with matrix A symmetric block Toeplitz with Toeplitz blocks.

Nonlinear smoothing operators

Return to Tikhonov regularization:

$$\min_u \left\{ \int_\Omega (h * u - f)^2 + \mu R(u) dx \right\}.$$

Euler-Lagrange equations with gradient descent:

$$\frac{\partial u}{\partial t} = -h * (h * u - f) + \mu\, D(u), \qquad u^0 = f.$$

Examples:

$$R(u) = |\nabla u|^2 \quad \Longrightarrow \quad D(u) = \Delta u$$

$$R(u) = |\nabla u| \quad \Longrightarrow \quad D(u) = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) \quad \text{TV operator}$$

Perona-Malik operator:

$$D(u) = \nabla \cdot (g(|\nabla u|^2)\nabla u), \quad g(s) = 1/(1 + s/\rho), \quad \rho > 0,$$

where g diffusivity.

Semi-discretization of Euler-Lagrange equation:

$$\frac{du}{dt} = (\mu L(u) - A^2)u + Ab,$$

where $L(u)$ a nonlinear operator.

- Explicit integration method (Euler): CFL condition imposes tiny time steps $\Rightarrow$ method expensive
- Semi-implicit integration method:

$$[I - \tau(\mu L(u^{j-1}) - A^2)]u^j = u^{j-1} + \tau Ab,$$

  where $u^j \approx u(j\tau)$. System expensive to solve and small time steps τ required $\Rightarrow$ method expensive

Use Perona-Malik operator D to define a nonlinear prolongation operator for the multilevel scheme.

Discretization of

$$\frac{\partial u}{\partial t} = D(u)$$

gives

$$\frac{du}{dt} = L(u)u, \quad u \in W_i, \quad u(0) = P_i u_{i-1}.$$

May integrate for a few, say ≤ 10, (non-tiny) time steps by explicit method.

Cascadic multilevel methods

Consider

$$\int_\Omega k(s,t)x(s)ds = b(t), \qquad t \in \Omega$$

Let

$$W_1 \subset W_2 \subset \ldots \subset W_\ell \subset L_2(\Omega) \qquad \text{nested subspaces}$$

$$\begin{aligned} R_i &: L_2(\Omega) \rightarrow W_i && \text{restriction operator} \\ Q_i^* &: W_i \rightarrow L_2(\Omega) && \text{prolongation operator, e.g., } Q_i^* = R_i^* \end{aligned}$$

$$\hat{b}_i = R_i \hat{b}, \qquad b_i = R_i b, \qquad A_i = R_i A Q_i^*$$

$$P_i : W_{i-1} \rightarrow W_i \qquad \text{prolongation operator}$$

Algorithm 1 *Multilevel Algorithm*

Input: A, b, δ, $\ell \geq 1$ *(number of levels);*

Output: approximate solution $\tilde{x} \in W_\ell$*;*

Determine A_i *and* b_i *for* $1 \leq i \leq \ell$*;*

$x_0 := 0;$

for $i := 1, 2, \ldots, \ell$ *do*

$x_{i,0} := P_i x_{i-1};$

$\Delta x_{i,m_i} := IM(A_i, b_i - A_i x_{i,0});$

Correction step: $x_i := x_{i,0} + \Delta x_{i,m_i};$

endfor

$\tilde{x} := x_\ell;$

Noise reducing restrictions R_i:

Solve weighted local least-squares problems (inspired by Buades et al.):

1D problems: Define $M_i : W_i \to W_{i-1}$ by solving

$$\min_{a_0,a_1} \sum_{s\in\{0,\pm 1\}} \left(x_i^{(2j+s)} - (a_0 + a_1 s)\right)^2 \omega_i^{(2j)}(s) \qquad \forall j$$

where

$$\omega_i^{(2j)}(s) := \exp\left(-\gamma \left(x_i^{(2j+s)} - x_i^{(2j)}\right)^2\right), \quad \gamma > 0$$

Solution $\{\hat{a}_0, \hat{a}_1\}$.

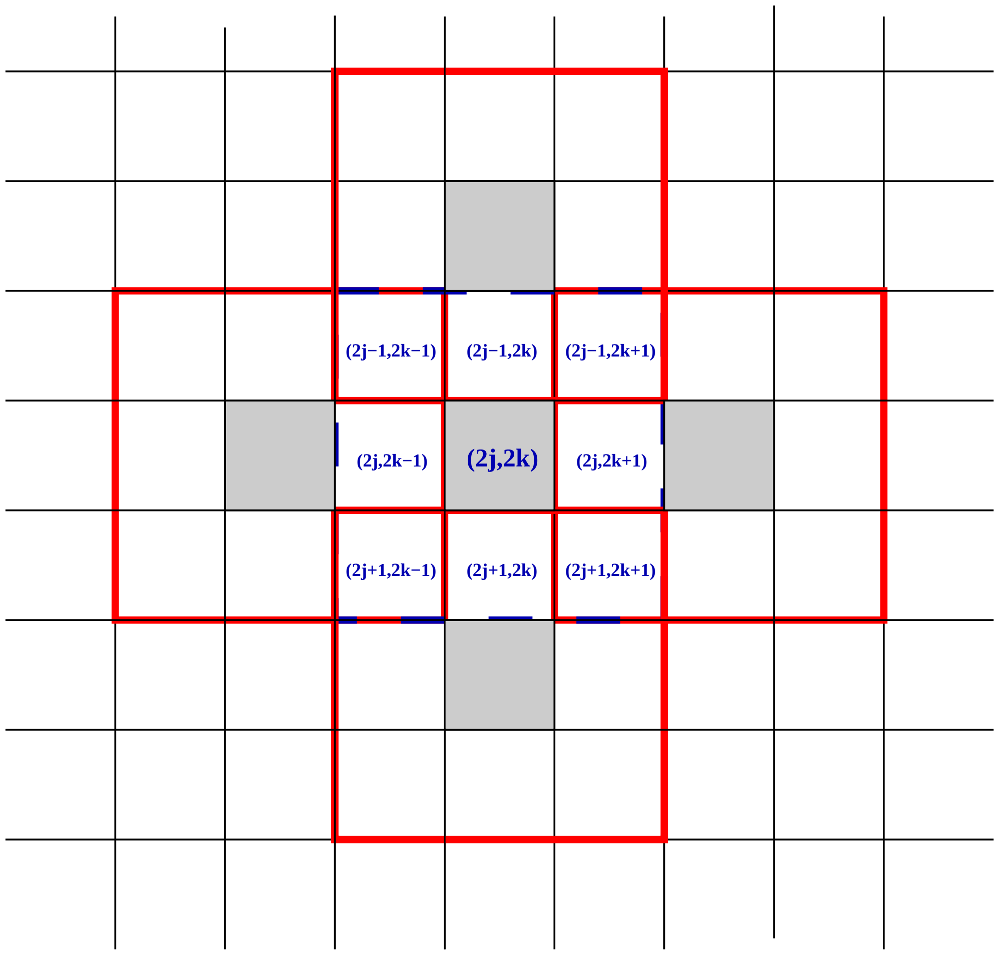
Let $x_{i-1}^{(j)} := \hat{a}_0$ for all j.

Define

$$R_i = M_{i+1} M_{i+2} \ldots M_\ell, \quad 1 \leq i < \ell.$$

2D problems:

Pixel mask for 2D problems



Assume that

$$\|\hat{b} - b\| = \delta.$$

Then, under some simplifying assumptions, such as $\gamma = 0$, one can show that for 1D problems one can expect the projected errors to satisfy the bounds

$$\|\hat{b}_i - b_i\| \leq c_i\delta, \quad c_i = \frac{1}{\sqrt{3}}c_{i+1}, \quad 1 \leq i < \ell, \quad c_\ell = 1.$$

For 2D problems,

$$c_i = \frac{1}{3}c_{i+1}.$$

Edge preserving nonlinear prolongation

The operator P_i consists of two parts:

$x_{i-1} \in W_{i-1} \to L_i x_{i-1} \in W_i$ piecewise linear interpolation

$L_i x_{i-1}$ is smoothed by solving an IBVP for a (discretized) Perona-Malik diffusion equation,

$$\frac{\partial x}{\partial \tau} = \frac{\partial}{\partial s}\left(\psi'\left(\left|\frac{\partial}{\partial s}x\right|^2\right)\frac{\partial}{\partial s}x\right), \quad x = x(\tau, s), \quad a \le s \le b,$$

over a short τ-interval with $\psi'(s) = \rho/(s+\rho)$, $\rho > 0$. This removes noise and preserves edges.

Theorem: Let

$$\|\hat{b}_i - b_i\| \leq c_i \delta_i, \qquad c_i > 1, \qquad 1 \leq i \leq \ell.$$

Assume that the P_i are linear(ized) and that

$$\mathcal{R}(P_i) \subset \mathcal{R}(A_i^*), \qquad 2 \leq i \leq \ell.$$

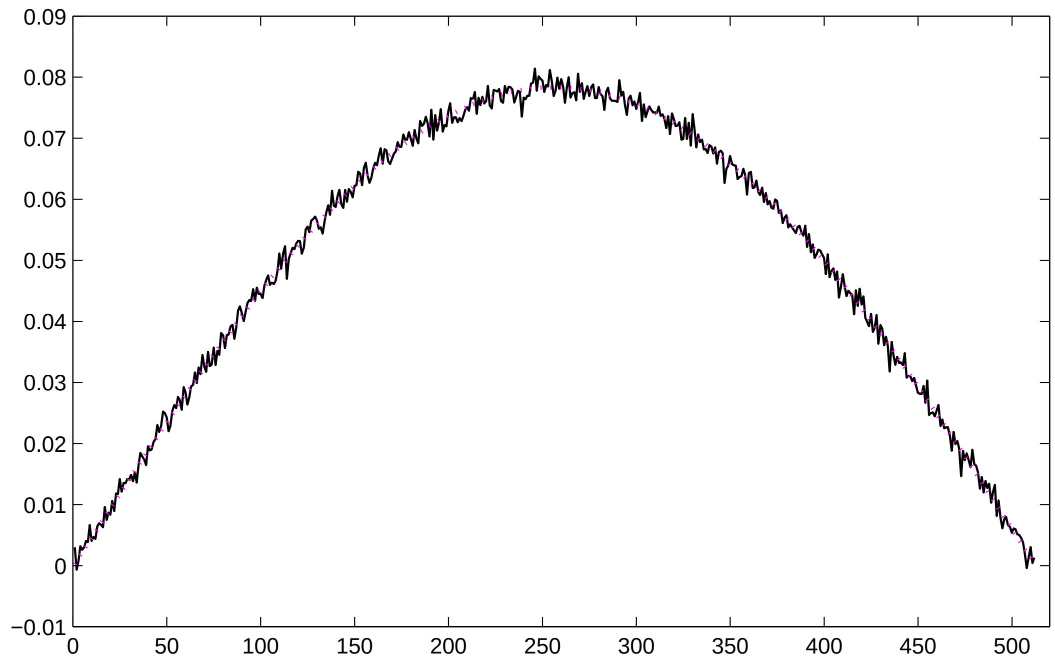
Then

$$\lim_{\delta_i \searrow 0} \sup_{||\hat{b}_i - b_i|| \leq c_i \delta_i} ||\hat{x}_i - x_{i,m_i(\delta_i)}|| = 0, \qquad 1 \leq i \leq \ell,$$

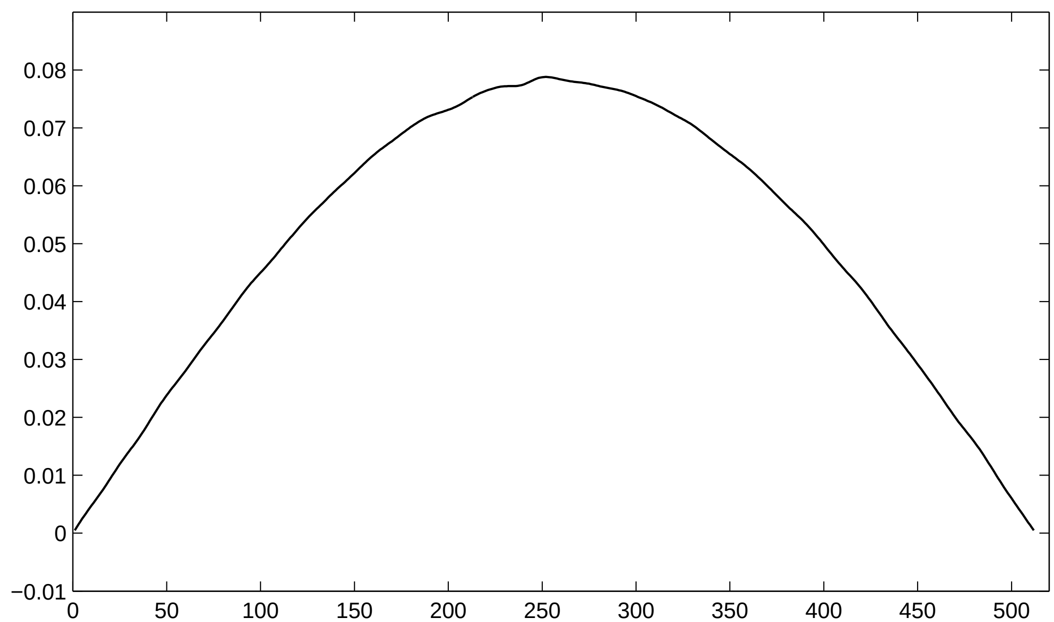
where $\hat{x}_i = A_i^\dagger \hat{b}_i$. Thus, the multilevel is a regularization method.

Examples: Noise reduction by Perona-Malik integration

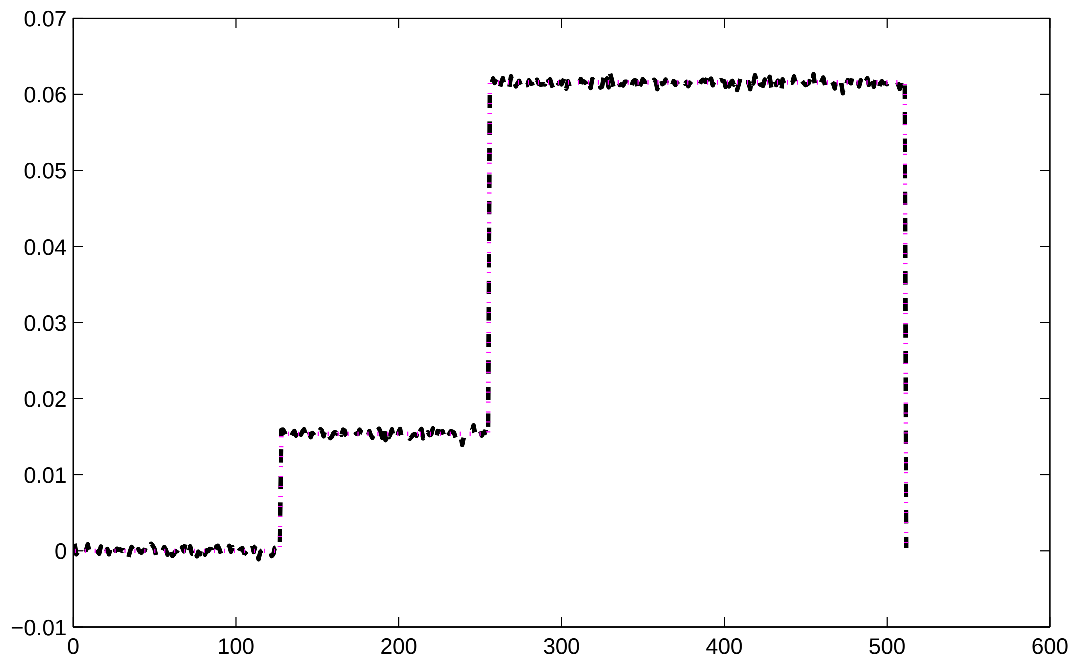
Example 2: Signal contaminated by 1% Gaussian noise. The noise-free signal is smooth.



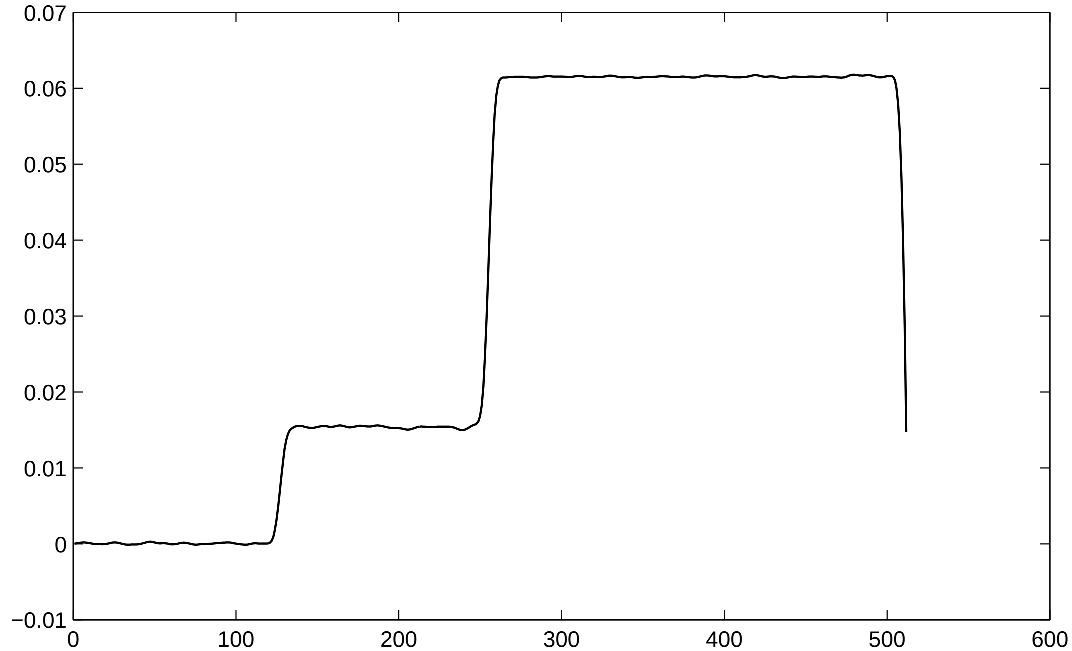
Example 2: Signal denoised by integrating the Perona-Malik equation 10 steps of size $\Delta\tau = 0.3$.



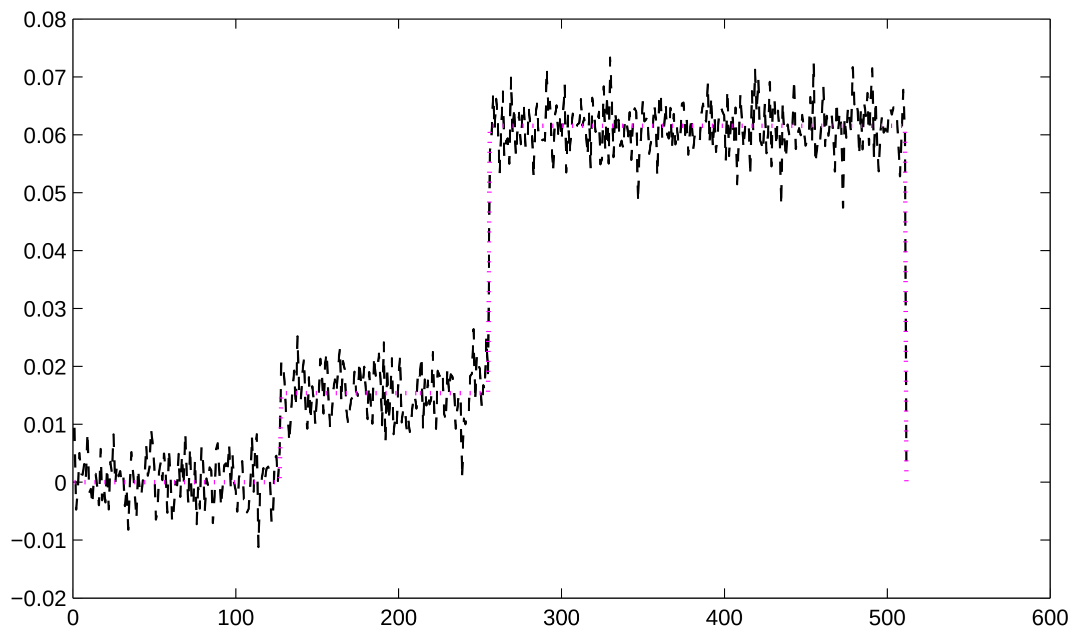
Example 3: Signal contaminated by 1% Gaussian noise. The noise-free signal is piecewise constant.



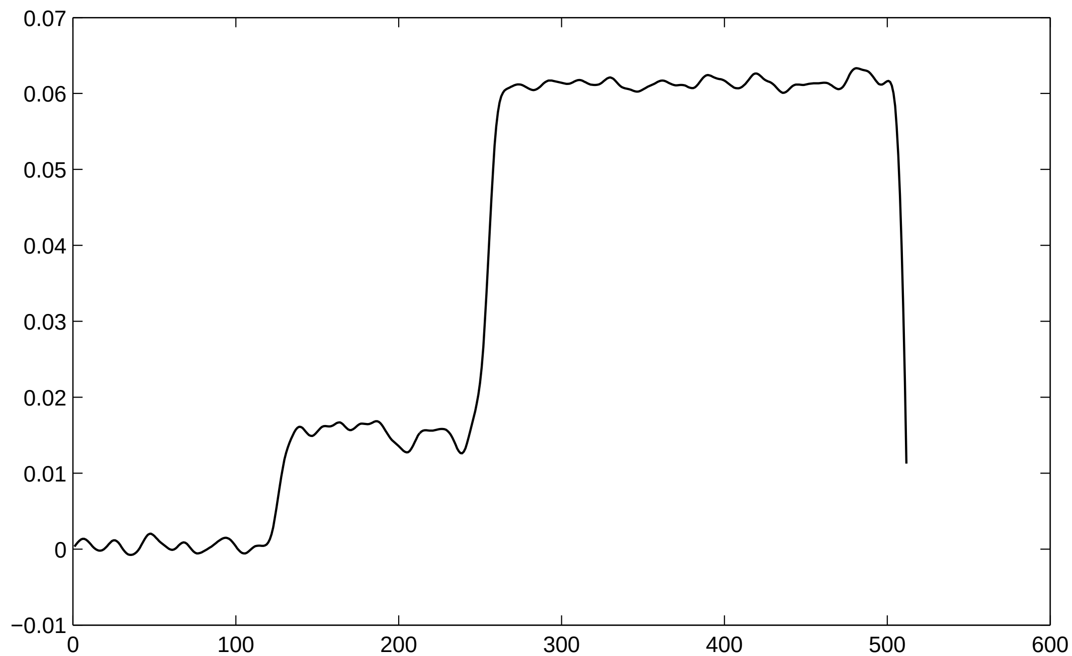
Example 3: Signal denoised by integrating the Perona-Malik equation 10 steps of size $\Delta\tau = 0.3$.



Example 4: Signal contaminated by 10% Gaussian noise. The noise-free signal is piecewise constant.



Example 4: Signal denoised by integrating the Perona-Malik equation.



Example 5: Fredholm integral equation of the 1st kind

$$\int_0^{\pi} \exp(-st)x(t)dt = 2\frac{\sinh(s)}{s}, \quad 0 \leq s \leq \frac{\pi}{2};$$

same as in Example 1. Discretization by Galerkin method using 512 piecewise constant test and trial functions. Relative error ν.

Matrix sizes for 5-level method:

$$\begin{aligned} A_1 &\in \mathrm{R}^{32\times 32}, \quad A_2 \in \mathrm{R}^{64\times 64}, \quad A_3 \in \mathrm{R}^{128\times 128}, \\ A_4 &\in \mathrm{R}^{256\times 256}, \quad A = A_5 \in \mathrm{R}^{512\times 512}. \end{aligned}$$

Minimal residual Iterative method for symmetric problems.

		P_i		L_i	
ℓ	ν	$\|\tilde{x}-\hat{x}\|/\|\hat{x}\|$	# iter	$\|\tilde{x}-\hat{x}\|/\|\hat{x}\|$	# iter
1	$1 \cdot 10^{-2}$	$3.51 \cdot 10^{-2}$	3		
2	$1 \cdot 10^{-2}$	$3.26 \cdot 10^{-2}$	3 1	$3.41 \cdot 10^{-2}$	3 1
3	$1 \cdot 10^{-2}$	$3.10 \cdot 10^{-2}$	3 1 1	$3.41 \cdot 10^{-2}$	3 1 1
5	$1 \cdot 10^{-2}$	$2.51 \cdot 10^{-2}$	3 2 1 1 1	$3.35 \cdot 10^{-2}$	3 2 1 1 1
1	$1 \cdot 10^{-3}$	$3.53 \cdot 10^{-2}$	3		
2	$1 \cdot 10^{-3}$	$3.34 \cdot 10^{-2}$	3 1	$3.57 \cdot 10^{-2}$	3 1
3	$1 \cdot 10^{-3}$	$3.08 \cdot 10^{-2}$	3 2 1	$3.56 \cdot 10^{-2}$	3 2 1
5	$1 \cdot 10^{-3}$	$2.00 \cdot 10^{-2}$	3 2 2 3 1	$3.50 \cdot 10^{-2}$	3 2 2 2 1

Example 6: Signal deblurring by solution of Fredhom integral equation of the 1st kind with a Gaussian convolution kernel. Discretization gives symmetric Toeplitz matrices. Relative error 10%.

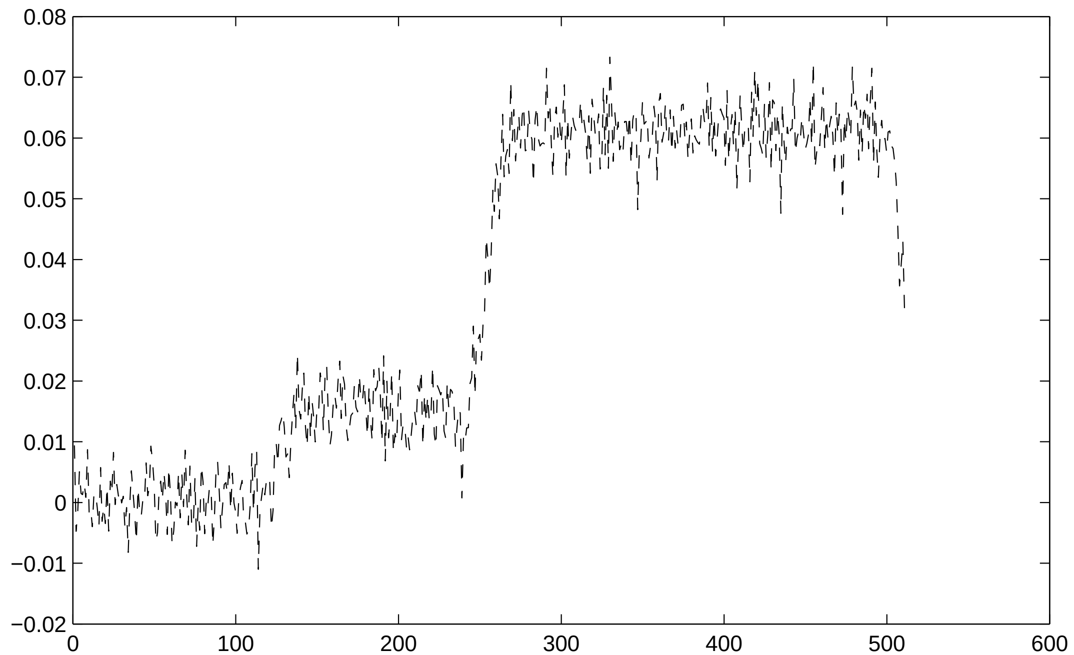
Matrix sizes for 3-level method:

$$A_1 \in \mathrm{R}^{128\times 128}, A_2 \in \mathrm{R}^{256\times 256}, \quad A = A_3 \in \mathrm{R}^{512\times 512}$$

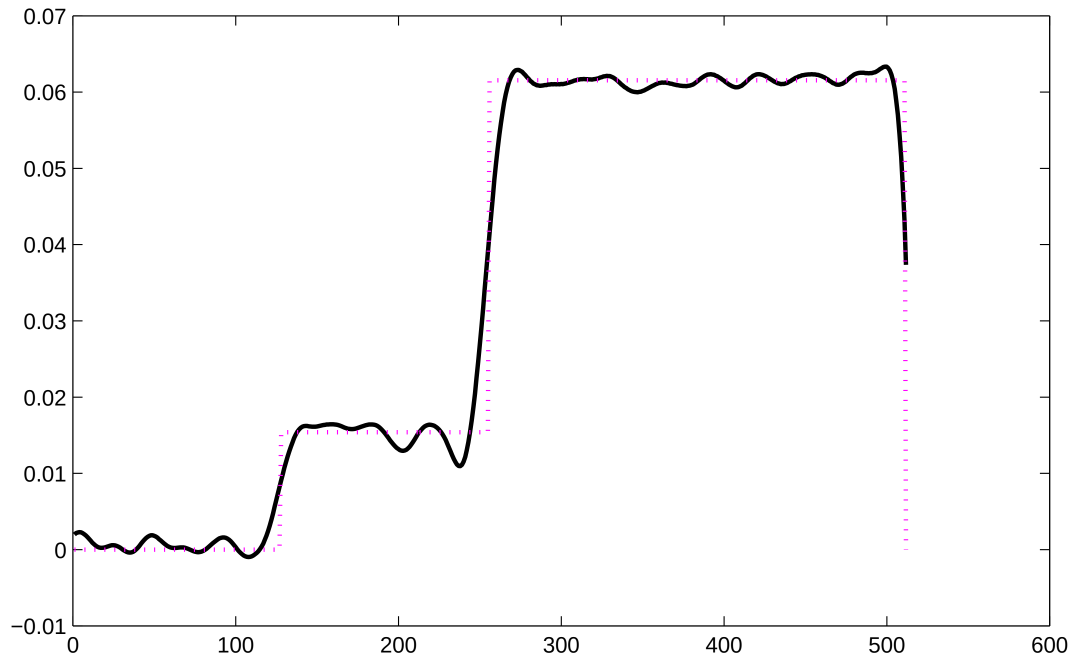
$$\kappa(A) = 1 \cdot 10^{17}.$$

Iterative method: MR-II

Example 6: Signal contaminated by 10% Gaussian noise and blur. The noise-free signal is piecewise constant.



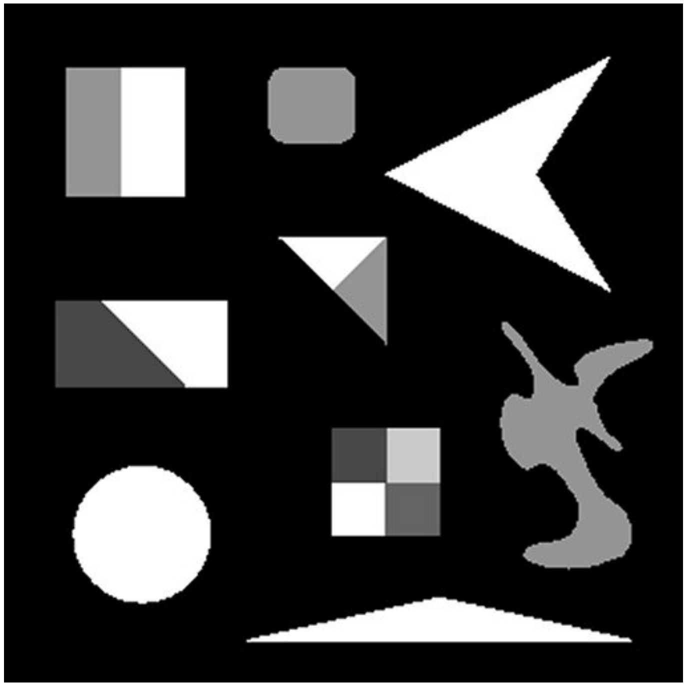
Example 6: Restored signal by multilevel method with 3 levels.



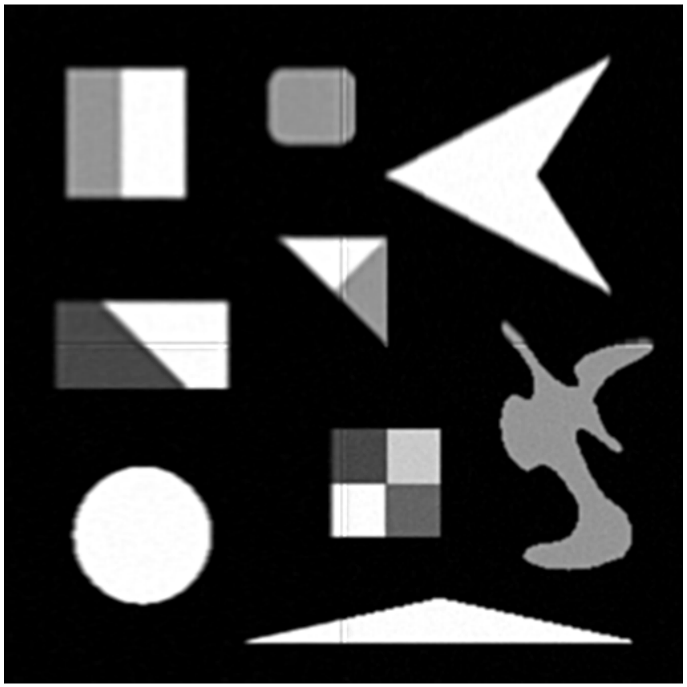
	P_i		L_i	
ℓ	$\|\tilde{x} - \hat{x}\|/\|\hat{x}\|$	# iter	$\|\tilde{x} - \hat{x}\|/\|\hat{x}\|$	# iter
1	$1.02 \cdot 10^{-1}$	1		
2	$8.70 \cdot 10^{-2}$	1 1	$8.78 \cdot 10^{-2}$	1 1
3	$8.08 \cdot 10^{-2}$	4 1 1	$8.88 \cdot 10^{-2}$	4 1 1

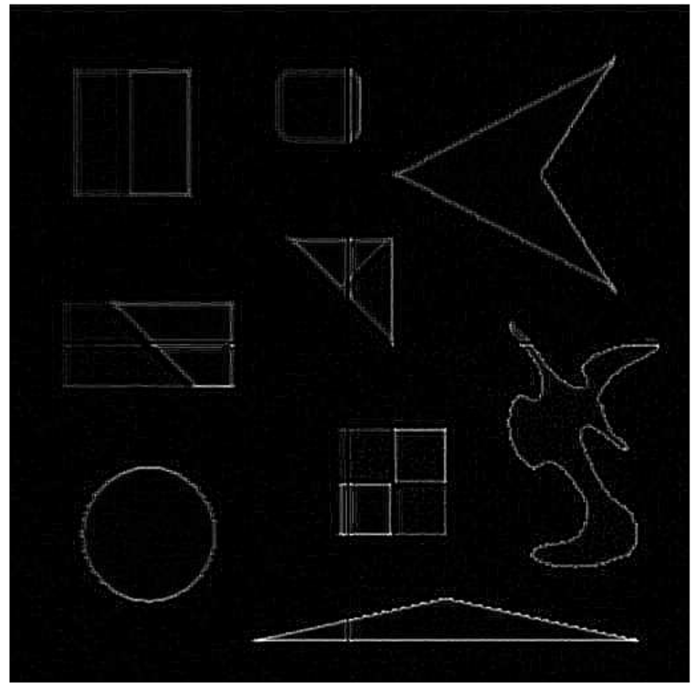
Application to image restoration

Example 7. Blur- and noise-free 512×512-pixel image.



Restoration of image with 5% noise, Gaussian blur, determined by 2 RRGMRES iterations.





Quantitative measure: PSNR$= 20 \log_{10} \frac{255}{||u_\ell - \hat{u}||} = 27.98$ dB.

		RRGMRES		CGNR	
ℓ	ν	PSNR	# iter	PSNR	# iter
1	$1 \cdot 10^{-2}$	29.77	3	32.36	9
2	$1 \cdot 10^{-2}$	31.14	3 2	34.09	5 7
1	$5 \cdot 10^{-2}$	27.98	2	28.74	3
2	$5 \cdot 10^{-2}$	28.80	2 1	29.90	2 2

Example 8. Original lizard image and image contaminated by 10% noise and motion blur.

Lizard restorations:

CGNR

PSNR=25.11

CGNR-based 3-level method

PSNR=26.11

Example 9: Original 256×256 image.

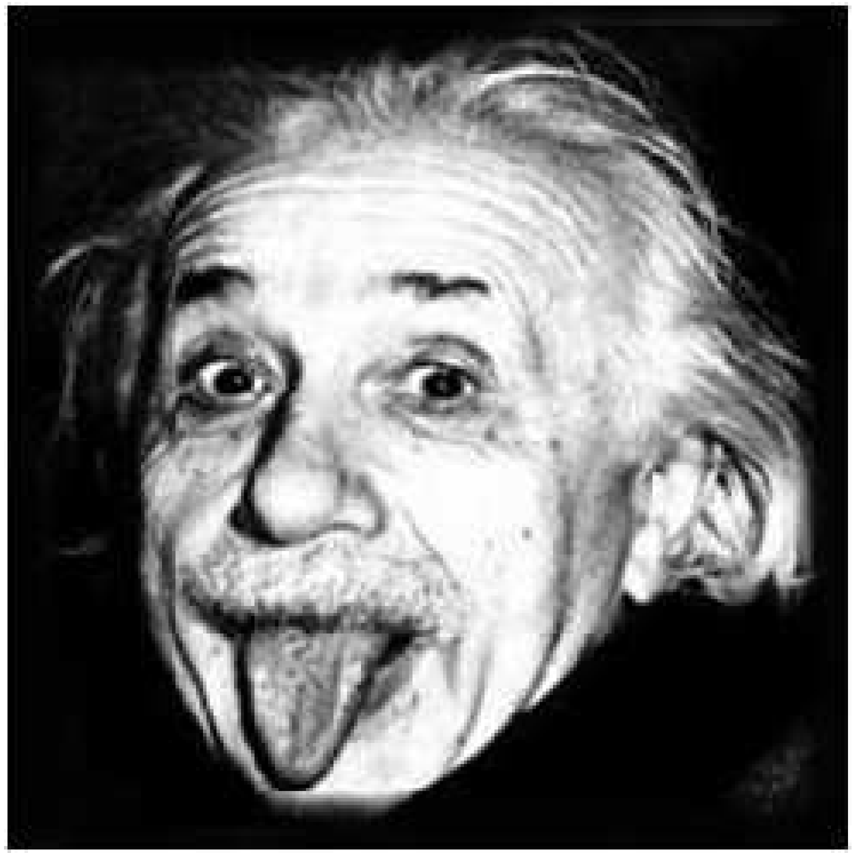
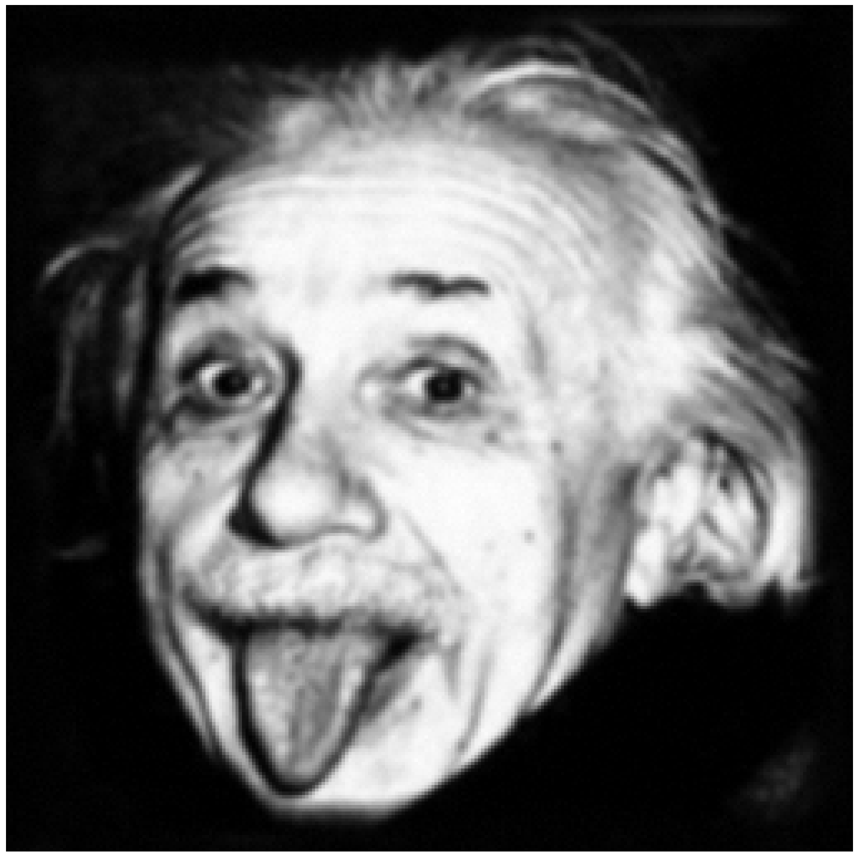
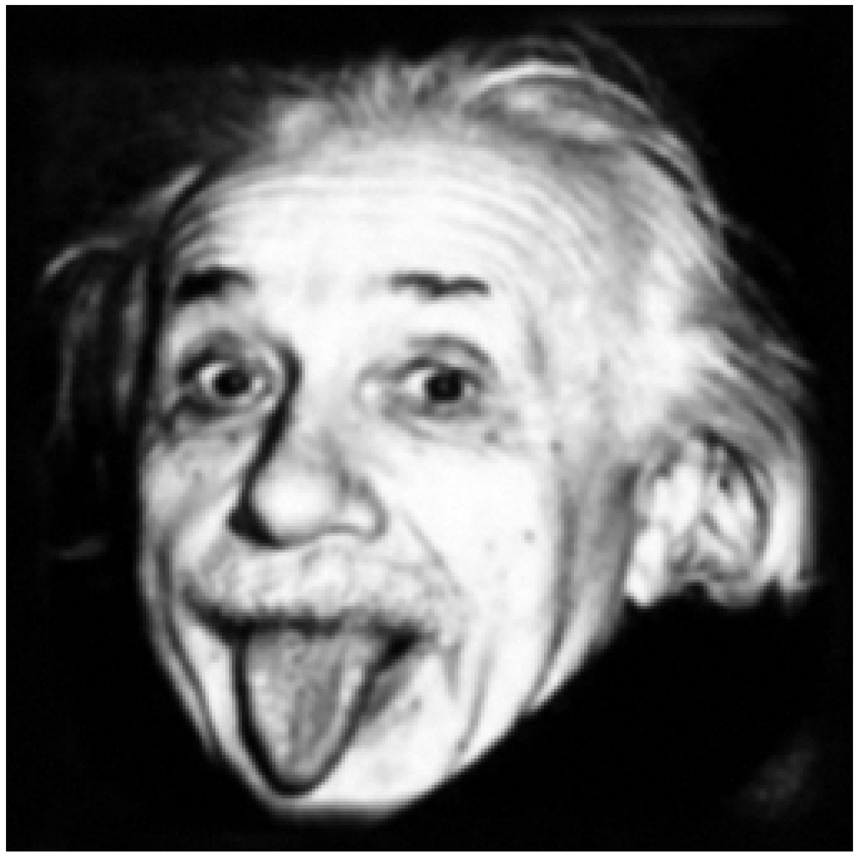


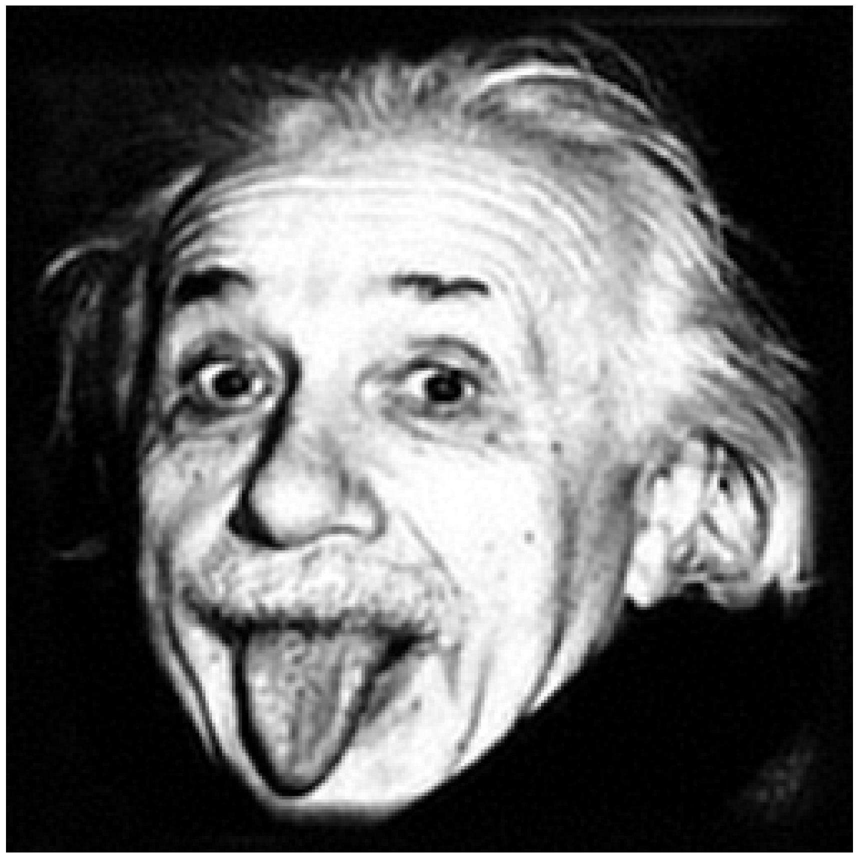
Image contaminated by Gaussian blur and 1% noise.



Restoration by solving Euler-Lagrange equation, Perona-Mailk regularization operator, 500 time-steps, 1000 mvp, PSNR=28.84.



Restoration by 3-level method, Perona-Malik-based prolongation operator, 4 mvp, PSNR=35.85.



Carrying out too many iterations may give rise to instability and produce unexpected results ...

such as ...

or …

Application to image segmentation

Algorithm 2 *Enhanced Multilevel Algorithm*

Input: A, b, δ, $\ell \geq 1$ *(number of levels);*

Output: approximate solution $\tilde{x} := x_\ell \in W_\ell$*;*

Determine A_i *and* b_i *for* $1 \leq i \leq \ell$*;*

$x_0 := 0$; ϕ_0 *:=initial contour;*

for $i := 1, 2, \ldots, \ell$ *do*

$x_{i,0} := P_i x_{i-1}$; $\phi_{i,0} := S_i \phi_{i-1}$;

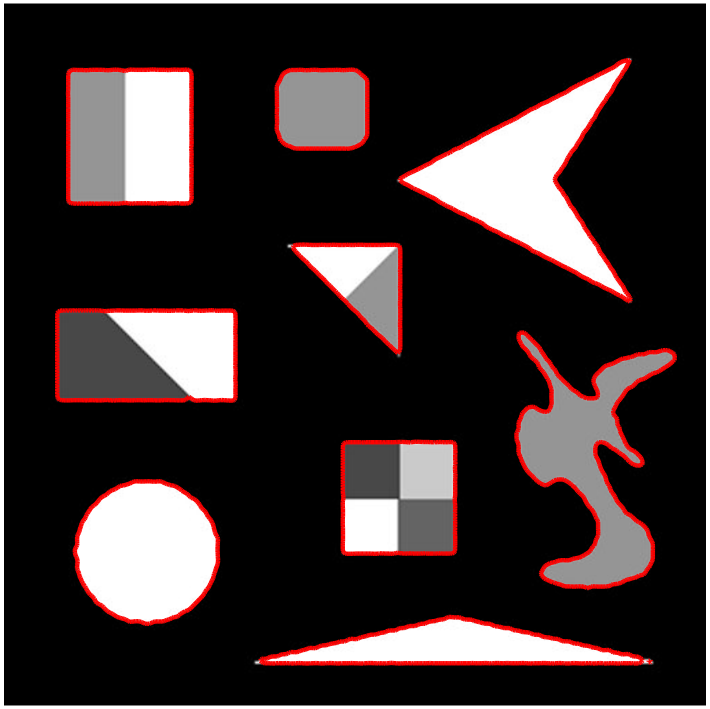
$\Delta x_{i,m_i} := IM(A_i, b_i - A_i x_{i,0})$;

Correction step: $x_i := x_{i,0} + \Delta x_{i,m_i}$;

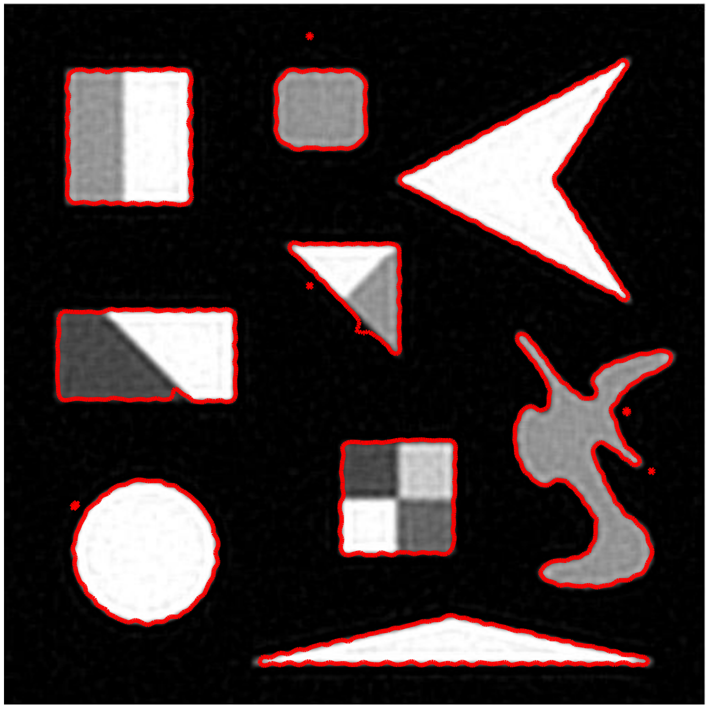
Segmentation step: $\phi_i := GAC(\phi_{i,0}, x_i)$;

endfor

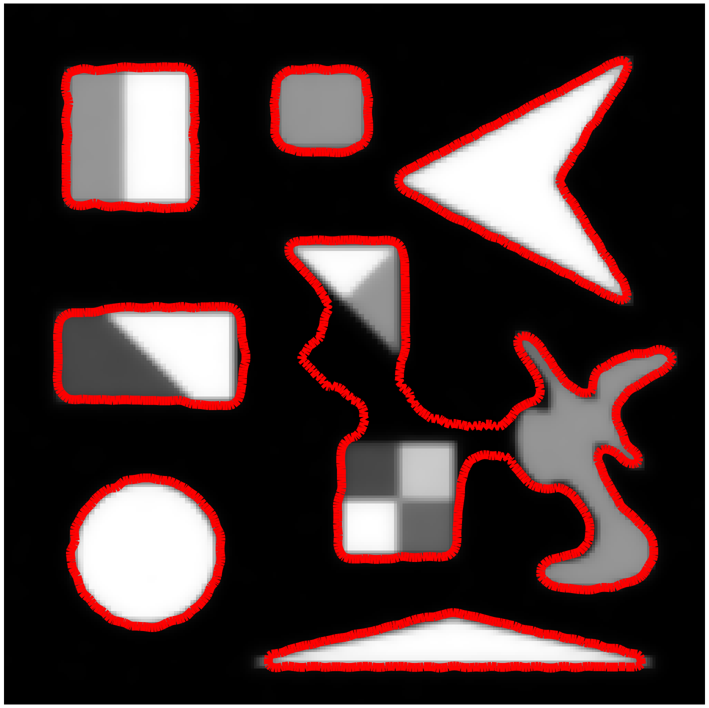
Example 10.



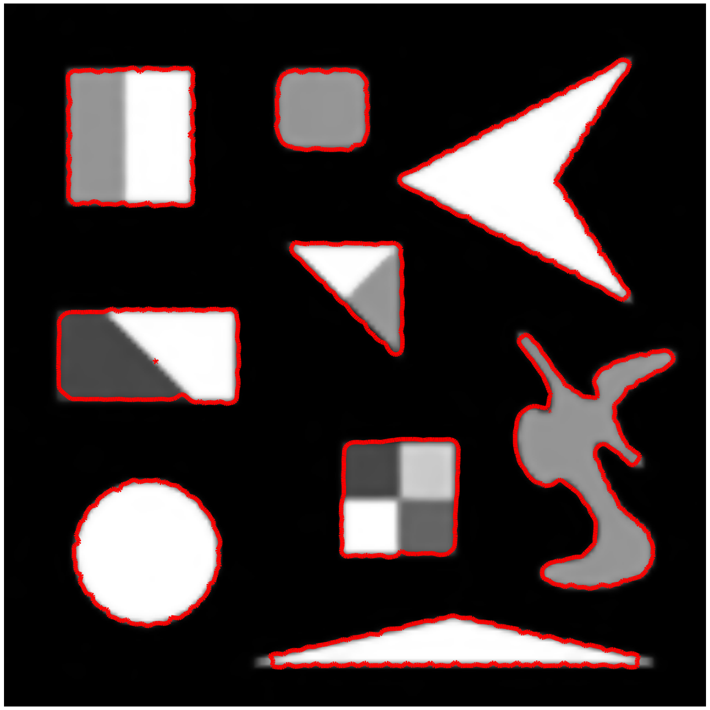
Segmentation of noise- and blur-free image.



Available image with 10% noise and Gaussian blur restored by CGNR and then segmented.



Available contaminated images restored and segmented by 3-level CGNR-based method. Segmentation on level 2.



Segmentation on level 3.

Wavelet-based multilevel methods

Let ϕ and and ψ be scaling and wavelet functions, respectively. Define

$$\Psi_{j,k} := 2^{j/2}\Psi(2^j \cdot -k), \quad \phi_{j,k} := 2^{j/2}\phi(2^j \cdot -k),$$

where j and k the scale and space parameters, respectively. Then

$$x = \sum_{k\in\mathbf{Z}}(x,\phi_{0,k})\phi_{0,k} + \sum_{j\geq 0}\sum_{k\in\mathbf{Z}}(x,\Psi_{j,k})\Psi_{j,k}\ , \quad x \in \mathcal{L}_2(\mathrm{R}).$$

Let $\Psi_{-1,k} := \phi_{0,k}$ and define the subspaces

$$\mathcal{W}_j = \mathrm{span}\{\Psi_{j,k}\}_{|k|\leq k(j)}.$$

Here $k(j)$ finite since we consider a finite interval $[a,b]$.

Define

Subspace:

$$\mathcal{U}_i := \bigoplus_{l=-1}^{i} \mathcal{W}_l \quad \subset \; \mathcal{L}_2([a,b])$$

Orthogonal projector:

$$Q_i : \mathcal{L}_2([a,b]) \to \mathcal{U}_i.$$

Restricted operator $A_i : \mathcal{U}_i \to \mathcal{U}_i$:

$$A_i := Q_i A Q_i, \qquad i = -1, 0, 1, \ldots \; .$$

Prolongation operator $P_i : \mathcal{U}_{i-1} \rightarrow \mathcal{U}_i$:

$$P_i x := \sum_{j=-1}^{i} \sum_{|k| \leq k(j)} \tilde{x}_{j,k} \Psi_{j,k}$$

with

$$\tilde{x}_{j,k} = \begin{cases} x_{j,k}, & \text{for} \quad j \leq i-1, \quad |k| \leq k(i), \\ 0, & \text{for} \quad j = i, \quad |k| \leq k(i). \end{cases}$$

Then

$$\mathcal{R}(P_i|_{\overline{\mathcal{R}(A_{i-1}^*)}}) \subset \overline{\mathcal{R}(A_i^*)} = \mathcal{N}(A_i)^{\perp}.$$

Theorem: Apply multilevel method based on the CGNR or MR-II iterative methods. Then the computed approximate solution on level i lives in $\mathcal{N}(A_i)^\perp$.

Example 11. Consider the integral equation

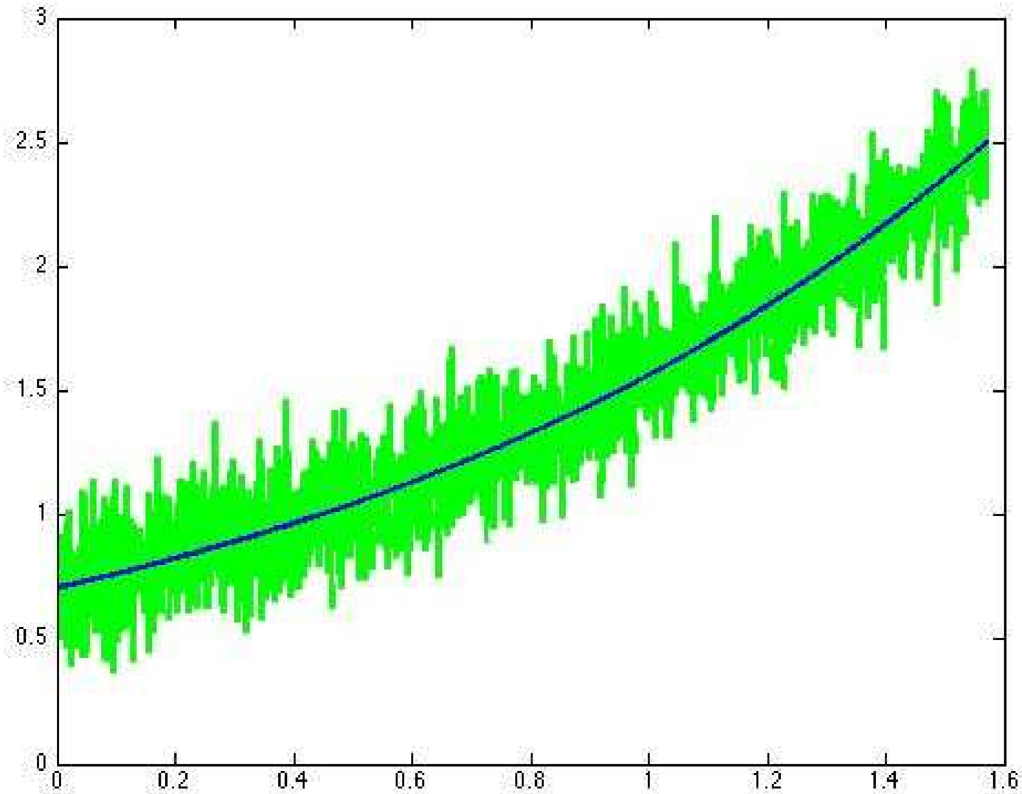
$$\int_0^{\pi/2} \kappa(s,t)x(s)ds = \sin^2(3t) - 0.9t^3 + t^2, \quad 0 \leq t \leq \pi/2,$$

with $\kappa(s,t) := \exp(s\cos(t))$.

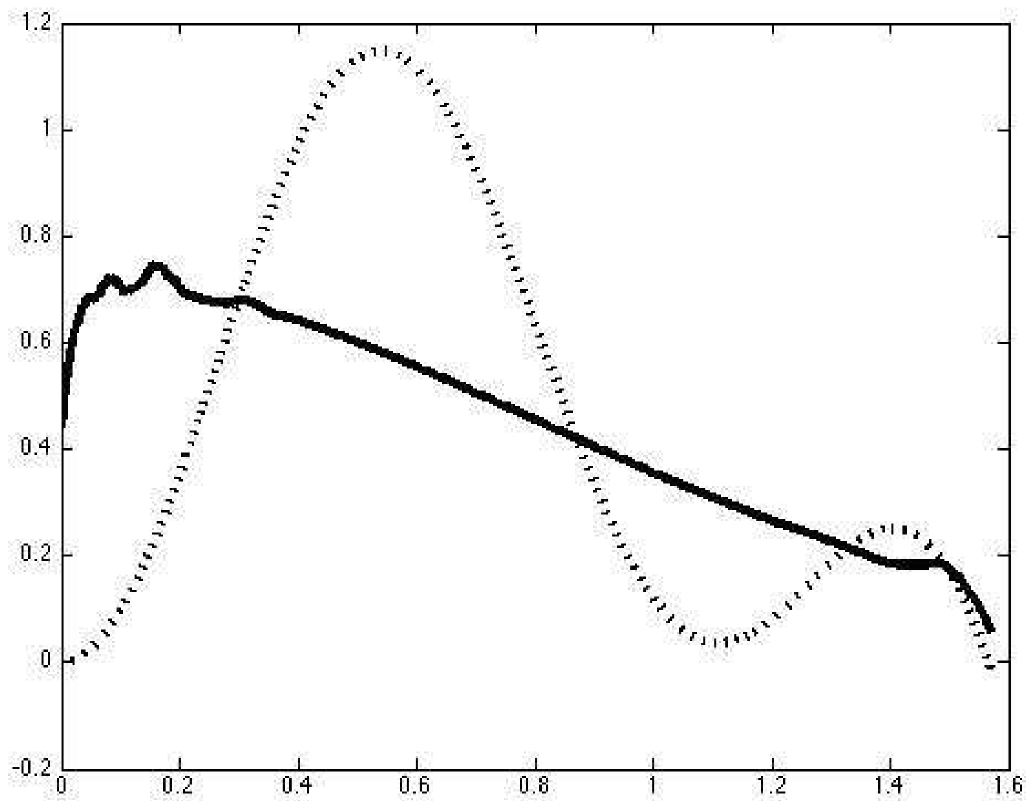
Order n_i of the matrices A_i.

Submatrix	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
n_i	28	50	88	158	292	554	1071	2098

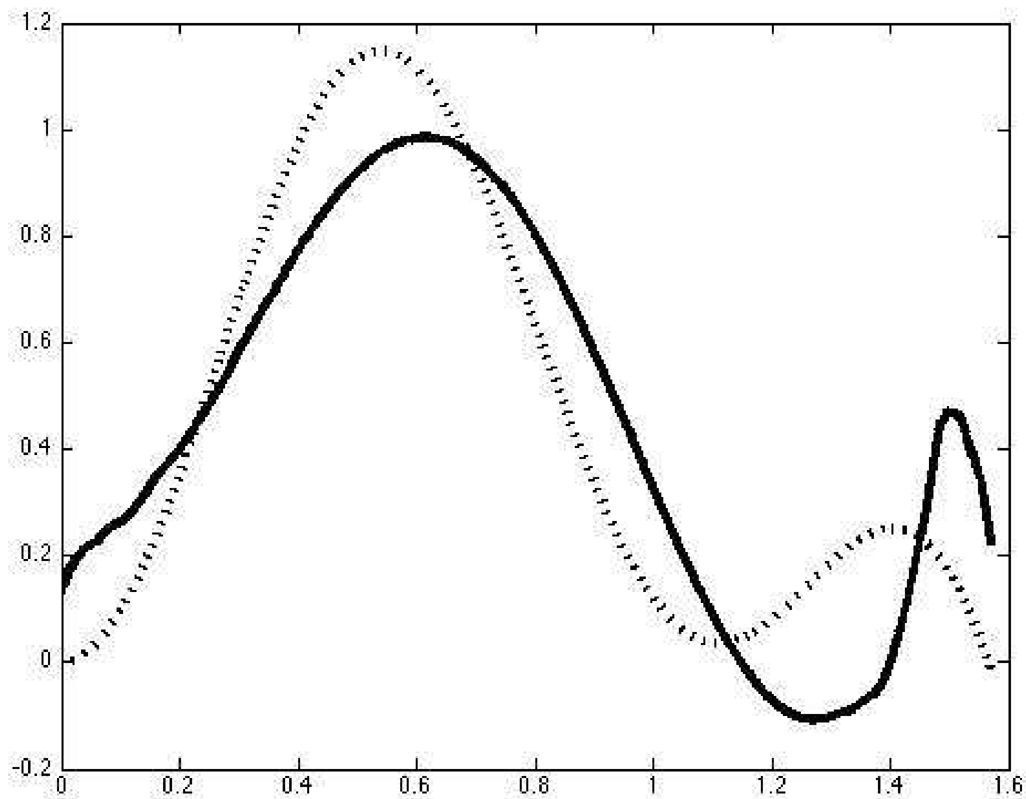
Right-hand sides without noise and with 10% noise



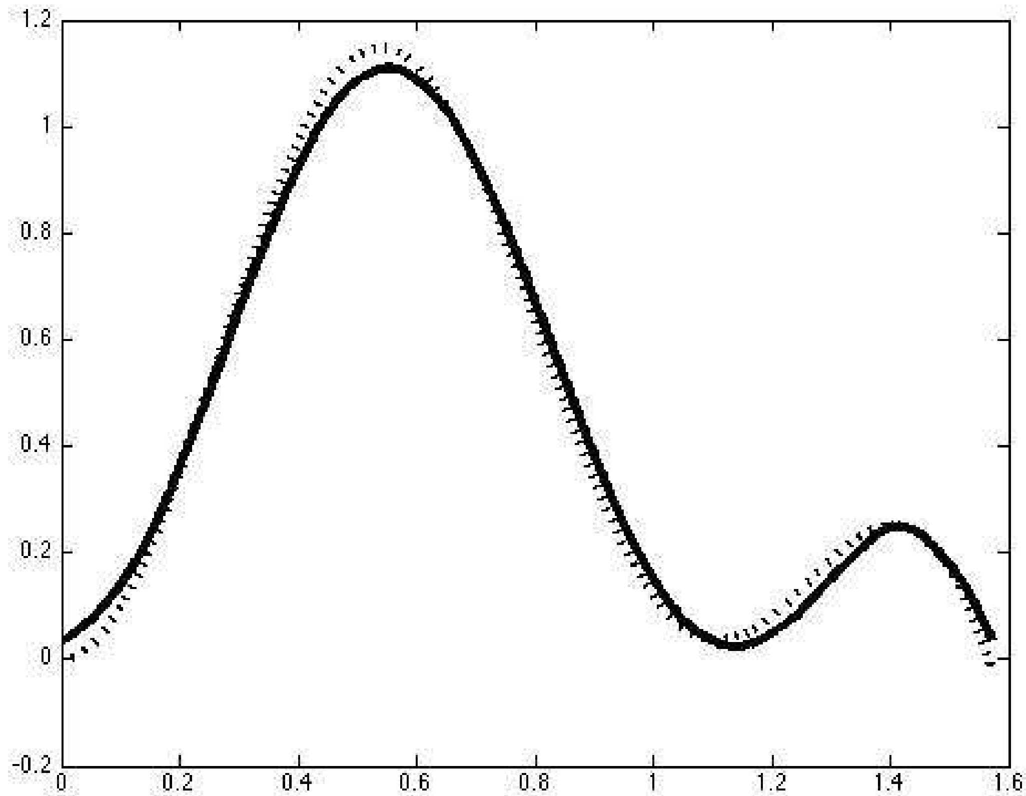
Exact and computed approximate solutions for 10% noise



Exact and computed approximate solutions for 1% noise



Exact and computed approximate solutions for 0.001% noise



	CGNR		ML-CGNR		
noise	k	$\frac{\|x_k - \hat{x}\|}{\|\hat{x}\|}$	k_i	$\frac{\|x_{8,k_8} - \hat{x}\|}{\|\hat{x}\|}$	Accel.
$1 \cdot 10^{-1}$	2	$5.78 \cdot 10^{-1}$	$2, 0, 0, 0, 0, 0, 0, 0$	$5.66 \cdot 10^{-1}$	17.6
$1 \cdot 10^{-2}$	3	$5.35 \cdot 10^{-1}$	$3, 0, 0, 0, 0, 0, 0, 0$	$5.28 \cdot 10^{-1}$	23.5
$1 \cdot 10^{-3}$	5	$3.41 \cdot 10^{-1}$	$5, 0, 0, 0, 0, 0, 0, 0$	$3.24 \cdot 10^{-1}$	35.2
$1 \cdot 10^{-4}$	5	$3.32 \cdot 10^{-1}$	$5, 0, 0, 0, 0, 0, 0, 0$	$3.14 \cdot 10^{-1}$	35.2
$1 \cdot 10^{-5}$	8	$3.49 \cdot 10^{-2}$	$8, 0, 0, 0, 0, 0, 0, 0$	$3.46 \cdot 10^{-2}$	52.6

Alternating iterative methods

Continuous image degradation model

$$f(x) = \int_\Omega h(x,y)\hat{u}(y)dy + \eta(x), \quad x \in \Omega$$

with h the point spread function and f the available blur- and noise- contaminated image.

Determine the blur- and noise-free image $\hat{u}$ by solving

$$\min_{u,w} J(u,w),$$

where

$$
\begin{aligned}
J(u,w) \;:=\; & \int_\Omega \Bigg(\bigg(\int_\Omega h(x,y)u(y)dy - f(x) \bigg)^2 \\
& + \alpha \left(\mathcal{L}(u-w)(x) \right)^2 + \beta \left| \nabla w(x) \right| \Bigg) \, dx,
\end{aligned}
$$

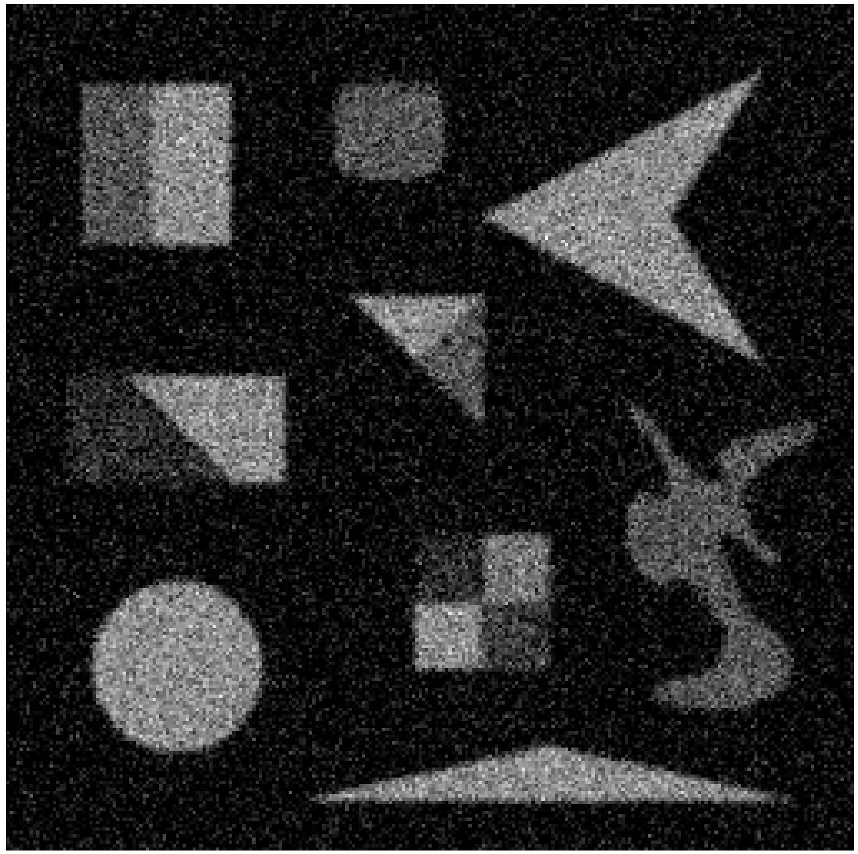
using an alternating iterative method. Here $\mathcal{L}$ is a differential operator, e.g., the Perona-Malik operator and $\alpha > 0$, $\beta > 0$ are regularization parameters.

Discrete setting:

$$
\begin{aligned}
u^{(i)} &= S_h(w^{(i-1)}) := \operatorname{argmin}_{u\in\mathcal{K}_\ell}\{\|Hu - f^\delta\|^2 \\
&\quad +\alpha\|L^{(i-1)}(u - w^{(i-1)})\|^2\}, \\
w^{(i)} &= S_{tv}(u^{(i)}) := \operatorname{argmin}_{w\in\mathcal{K}_\ell}\{\|L^{(i-1)}(w - u^{(i)})\|^2 + \beta\|w\|_{tv}\},
\end{aligned}
$$

for $i = 1, 2, 3, \dots$, where $\|\cdot\|_{tv}$ is a discrete TV-norm, $L^{(i-1)}$ is a discretization of the operator $\mathcal{L}$, and $\mathcal{K}_\ell$ a Krylov subspace.

Available blur- and noise-contaminated image (Gaussian blur, 50% noise.



Restored image.

Available blur- and noise-contaminated image (Gaussian blur, 10% noise.

Restored image.

Vielen Dank!