On the power of function values for the approximation problem in various settings

Erich Novak

University of Jena

joint work with Henryk Woźniakowski

Problem

Assume that you want to approximate functions from a class F, with error in the L_p-sense.

Are algorithms that are based on arbitrary linear functionals (like Fourier coefficients) better than algorithms that are based on function values?

Observation: For many classes F it turns out that the answer is: *no, not much better*.

Can we prove general results?

Worst Case Setting

Let F be a Banach space of functions such that the $f \mapsto f(x)$ are continuous. Assume $F \subset L_p$, continuous embedding.

Approximate $f \in F$ using linear functionals $L \in F^*$ or function values,

$$A_n(f) = \phi_n(L_1(f), L_2(f), \dots, L_n(f)),$$

where $\phi_n : \mathbb{R}^n \to L_p$ and $L_j \in \Lambda$, where $\Lambda \in \{\Lambda^{\mathrm{all}}, \Lambda^{\mathrm{std}}\}$.

Define

$$e_n^{\mathrm{all-wor}}(F, L_p) = \inf_{A_n \text{ with } L_j \in \Lambda^{\mathrm{all}}} \sup_{\|f\|_F \le 1} \big\| f - A_n(f) \big\|_p$$

and

$$e_n^{\mathrm{std-wor}}(F, L_p) = \inf_{A_n \text{ with } L_j \in \Lambda^{\mathrm{std}}} \sup_{\|f\|_F \le 1} \big\| f - A_n(f) \big\|_p.$$

Example: Sobolev Spaces, $p = 2$

a) Standard Sobolev spaces $W_2^s([0,1]^d)$ with $2s > d$, known

$$e_n^{\mathrm{all-wor}}(W_2^s([0,1]^d), L_2) \asymp e_n^{\mathrm{std-wor}}(W_2^s([0,1]^d), L_2) \asymp n^{-s/d}.$$

b) Sobolev spaces $W_2^{r,\mathrm{mix}}([0,1]^d)$ with $r > 1/2$, known

$$e_n^{\mathrm{all-wor}}(W_2^{r,\mathrm{mix}}([0,1]^d), L_2) \asymp n^{-r}(\log n)^{(d-1)r},$$

and

$$e_n^{\mathrm{std-wor}}(W_2^{r,\mathrm{mix}}([0,1]^d), L_2) = \mathcal{O}\left(n^{-r}(\log n)^{(d-1)(r+1/2)}\right).$$

Not known whether this extra power $(d-1)/2$ for log is needed.

Rate of Convergence + Power Function

Assume that (c_n) converges to zero. Define its rate of convergence by

$$r(c_n) = \sup\{ \beta \ge 0 \mid \lim_{n\to\infty} c_n n^{\beta} = 0\}.$$

For $\alpha > 0$ the rate of convergence of $n^{-\alpha}$ is α.

Compare the rates of $e_n^{\mathrm{all-wor}}(F, L_p)$ and of $e_n^{\mathrm{std-wor}}(F, L_p)$.

Define the power function by

$$\ell^{\,\mathrm{wor-x}}(r,p) := \inf_{F:\, r^{\mathrm{all-wor}}(F,L_p)=r} \frac{r^{\mathrm{std-wor}}(F, L_p)}{r},$$

where $\mathrm{x} \in \{H, B\}$ indicates that all Hilbert spaces $(\mathrm{x} = H)$ or all Banach spaces $(\mathrm{x} = B)$ are taken, for which the rate is r when we use linear functionals.

Double Hilbert Case

Wasilkowski, Woźniakowski (2001):

$r^{\text{all}-\text{wor}}(H, L_2) = r > \frac{1}{2}$ implies $r^{\text{std}-\text{wor}}(H, L_2) \geq r^{\text{all}-\text{wor}}(H, L_2) - \frac{1}{2}$.

Improved by Kuo, Wasilkowski, Woźniakowski (2009) to

$r^{\text{std}-\text{wor}}(H, L_2) \geq r - \frac{r}{2r+1} = \frac{2r^2}{2r+1}$.

Case $r \leq 1/2$ was studied in Hinrichs, N., Vybiral (2008): there is a Hilbert space H such that

$r^{\text{all}-\text{wor}}(H, L_2) = r$ and $r^{\text{std}-\text{wor}}(H, L_2) = 0$.

Hence

$$\ell^{\,\text{wor}-\text{H}}(r, 2) = 0 \qquad \text{for all} \quad r \in (0, \tfrac{1}{2}],$$

$$\ell^{\,\text{wor}-\text{H}}(r, 2) \in \left[\frac{2r}{2r+1}, 1\right] \qquad \text{for all} \quad r \in (\tfrac{1}{2}, \infty).$$

Open Problem: Suppose that $r > 1/2$. Is it true that $\ell^{\,\text{wor}-\text{H}}(r, 2) = 1$?

Simple Hilbert Case

Result of Tandetzky: Hilbert spaces and arbitrary $p \in [1, \infty)$. For any $r \in (0, \min(\frac{1}{p}, \frac{1}{2})]$ there exists a Hilbert space H continuously embedded in $L_p = L_p([0,1])$ such that

$$r^{\text{all}-\text{wor}}(H, L_p) = r \qquad \text{and} \qquad r^{\text{std}-\text{wor}}(H, L_p) = 0.$$

Hence the power function is zero over $(0, \min(\frac{1}{p}, \frac{1}{2})]$. We do not know the the power function over $(\min(\frac{1}{p}, \frac{1}{2}), \infty)$. Hence, for $p \neq 2$,

$$\begin{aligned} \ell^{\,\text{wor}-\text{H}}(r,p) &= 0 && \text{for all} \quad r \in (0, \min(\tfrac{1}{p}, \tfrac{1}{2})], \\ \ell^{\,\text{wor}-\text{H}}(r,p) &\in [0,1] && \text{for all} \quad r \in (\min(\tfrac{1}{p}, \tfrac{1}{2}), \infty). \end{aligned}$$

Only for $p = \infty$ we know more, see N. 88.

$$\ell^{\,\text{wor}-\text{H/B}}(r, \infty) \in \left[\frac{r-1}{r}, 1\right] \qquad \text{for all} \quad r > 1.$$

Banach Spaces

We summarize the properties of the power function that can be proved using results from the literature on Sobolev embeddings:

$$\ell^{\,\mathrm{wor-B}}(r,p) = 0 \qquad r \in (0,1] \text{ and } p \in [1,2],$$

$$\ell^{\,\mathrm{wor-B}}(r,p) = 0 \qquad r \in (0, \tfrac{1}{2} + \tfrac{1}{p}] \text{ and } p \in (2,\infty),$$

$$\ell^{\,\mathrm{wor-B}}(r,p) \le 1 - \frac{1}{r}\left(1 - \frac{1}{p}\right) \qquad r > 1 \text{ and } p \in [1,2],$$

$$\ell^{\,\mathrm{wor-B}}(r,p) \le 1 - \frac{1}{2r} \qquad r > 1 \text{ and } p \in [2,\infty),$$

$$1 - \frac{1}{r} \le \ell^{\,\mathrm{wor-B}}(r,\infty) \le 1 - \frac{1}{2r} \qquad r > 1.$$

Randomized Setting, double Hilbert case

Consider randomized algorithms and define the error by

$$e^{\mathrm{ran}}(A_n) = \sup_{\|f\|_F \le 1} \left(\mathbb{E}_\omega \| I(f) - A_n(f,\omega) \|_p^2 \right)^{1/2}.$$

Compare the rates of convergence

$$r^{\mathrm{all-ran}}(F, L_p) = r\left(e_n^{\mathrm{all-ran}}(F, L_p)\right) \text{ and } r^{\mathrm{std-ran}}(F, L_p) = r\left(e_n^{\mathrm{std-ran}}(F, L_p)\right).$$

Result of Wasilkowski, Woźniakowski (2007):

For arbitrary Hilbert spaces $I : H \to L_2(\Omega)$

$$r^{\mathrm{all-ran}}(H, L_2) = r^{\mathrm{std-ran}}(H, L_2).$$

Therefore

$$\ell^{\,\mathrm{ran-H}}(r, 2) = 1 \qquad \text{for all} \quad r > 0.$$

It was known before that also

$$r^{\mathrm{all-ran}}(H, L_2) = r^{\mathrm{all-wor}}(H, L_2).$$

Randomized Setting, other cases

Assume $p > 2$, consider $I : W_2^r([0,1]) \to L_p([0,1])$.
Mathé (1991): With Λ^{all} obtain optimal order n^{-r}.
Heinrich (2008): With Λ^{std} the optimal order is $n^{-r+1/2-1/p}$.

Hence

$$\ell^{\,\mathrm{ran-H}}(r,p) \le \frac{r - 1/2 + 1/p}{r} \qquad \text{if} \quad r \ge 1 \quad \text{and} \quad p > 2.$$

Open Problem:
Study the power function in the randomized setting for the Hilbert case with $p \in [1,2)$ and for the Banach case for all $p \in [1,\infty]$.

Average Case Setting with a Gaussian Measure

Let F be a separable Banach space equipped with a zero mean Gaussian measure μ. Average error

$$e^{\mathrm{avg}}(A) = \left(\int_F \|f - A(f)\|_p^2 \, \mathrm{d}\mu(f) \right)^{1/2}.$$

Define the minimal nth average case errors $e_n^{\mathrm{all-avg}}(F, L_p)$, $e_n^{\mathrm{std-avg}}(F, L_p)$ and the power function $\ell^{\mathrm{avg-H/B}}$.

Results known for $p = 2$.

Wasilkowski, Woźniakowski (2008): Let $I : F \to L_2(\Omega)$ be arbitrary. Then

$$r^{\mathrm{all-avg}}(F, L_2) = r^{\mathrm{std-avg}}(F, L_2).$$

Therefore

$$\ell^{\mathrm{avg-B}}(r, 2) = 1 \qquad \text{for all} \quad r > 0.$$