Spectral Problems and the Solvability Complexity Index

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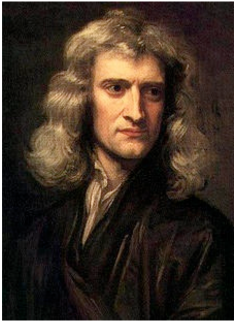
Chemnitz, October 7, 2010

Motivation

Classical Mechanics



Described by Ordinary Differential Equations

$$y'(x) = f(x, y(x)), \qquad y(0) = y_0.$$

Motivation

Question: For $T > 0$, can we always construct approximations to the solution $y(T)$ to

$$y'(x) = f(x, y(x)), \qquad y(0) = y_0 \in \mathbb{R}^d$$

by using finitely many arithmetic operations and radicals of elements in

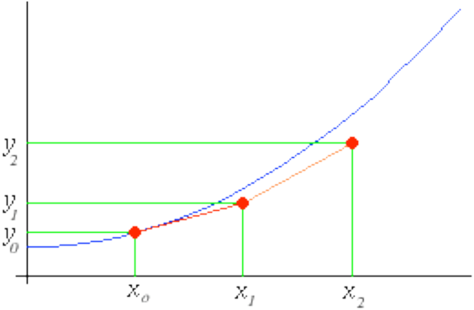
$$\{f(x,z) : x \in \mathbb{R}, z \in \mathbb{R}^d\} \cup \{y_0\}?$$

Motivation

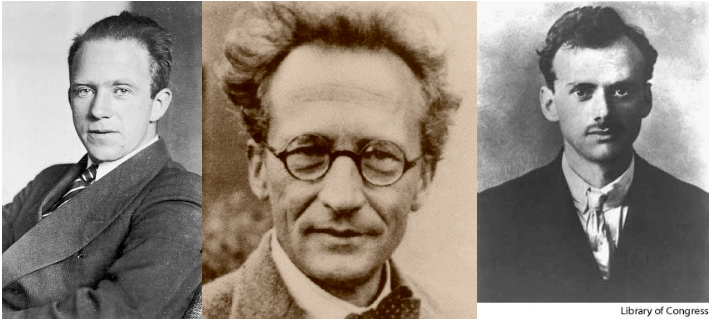
Answer: Yes! Even the simplest method of Euler

$$y_{n+1} = y_n + hf(x_n, y_n)$$

gives a convergent sequence of approximations.



Quantum Mechanics



Described by linear operators on Hilbert spaces, for example the Schrodinger operator:

$$(Hf)(x) = -\Delta f(x) + V(x)f(x)$$

Non-Hermitian Quantum Mechanics

- Open Systems: Then the time evolution operator e^{-itH} cannot be unitary. Thus, H cannot be self-adjoint.
- Closed Systems: Different inner products on the Hilbert space $\mathcal{H}$ could result in H not being self-adjoint.
- Resonances.

Motivation

Question:

Can we always construct approximations to the spectrum $\sigma(H)$ of

$$H = -\Delta + V$$

by using finitely many arithmetic operations and radicals of elements in

$$\{V(x) : x \in \mathbb{R}^d\}?$$

Motivation

Answer: Not Known!

The Finite Matrix

Let

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix},$$

Question:
Can we always construct approximations to the spectrum $\sigma(A)$ by using finitely many arithmetic operations and radicals of elements in

$$\{a_{ij} : 1 \leq i, j \leq n\}?$$

The Finite Matrix

Answer: YES!

The Infinite Matrix

Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$. Let T be a linear operator such that $\mathcal{D}(T) \supset \mathrm{span}(\{e_j\}_{j\in\mathbb{N}})$ such that we can form the infinite matrix

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} & \dots \\ t_{21} & t_{22} & t_{23} & \dots \\ t_{31} & t_{32} & t_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad t_{ij} = \langle Te_j, e_i \rangle$$

Question: Suppose that T is compact. Can we always construct approximations to the spectrum $\sigma(T)$ by using finitely many arithmetic operations and radicals of elements in

$$\{t_{ij} : i, j \in \mathbb{N}\}?$$

The Infinite Matrix

Answer: YES!

How to Compute Spectra of Compact Infinite Matrices

Suppose that

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} & \dots \\ t_{21} & t_{22} & t_{23} & \dots \\ t_{31} & t_{32} & t_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is compact. Let P_m be the projection onto $\mathrm{span}\{e_1, \dots, e_m\}$. Compute $\sigma(P_m T P_m)$. Then $\sigma(P_m T P_m) \to \sigma(T)$ as $m \to \infty$. Note that there are two limits to be taken:

- $\sigma(P_m T P_m) \to \sigma(T), \qquad m \to \infty.$
- $\mathrm{Alg}_n(P_m T P_m) \to \sigma(P_m T P_m), \qquad n \to \infty.$

The General Question

Suppose that

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} & \dots \\ t_{21} & t_{22} & t_{23} & \dots \\ t_{31} & t_{32} & t_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Question: Can we always construct approximations to the spectrum $\sigma(T)$ by using finitely many arithmetic operations and radicals of elements in

$$\{t_{ij} : i, j \in \mathbb{N}\}?$$

The General Question

Answer: We will try to find out.

Discontinuity of the Spectrum

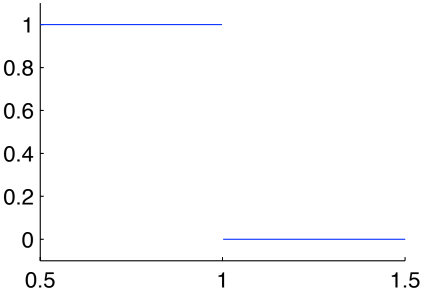
The computational spectral problem in infinite dimensions is much more delicate than the finite-dimensional case. One reason is the possibly discontinuous behavior of the spectrum as the following well known example shows. Let $A_\epsilon : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be defined by

$$(A_\epsilon f)(n) = \begin{cases} \epsilon f(n+1) & n = 0 \\ f(n+1) & n \neq 0. \end{cases}$$

Now for $\epsilon \neq 0$ we have $\sigma(A_\epsilon) = \{z : |z| = 1\}$ but for $\epsilon = 0$ then $\sigma(A_0) = \{z : |z| \leq 1\}$.

Computing with Discontinuous Functions

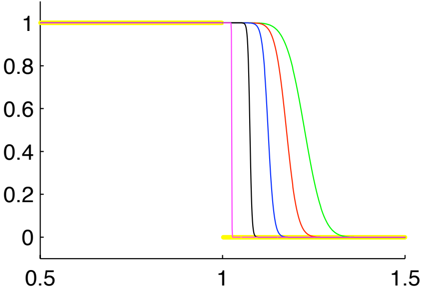
Question: Does it even make sense to compute with discontinuous functions?
Consider the following function f:



Suppose we want to compute $f(1)$, but our input values are $1 + \frac{1}{2}, 1 - \frac{1}{2}, 1 + \frac{1}{3}, 1 - \frac{1}{3}, \ldots$, so that $\lim_{n\to\infty} f(x_n)$ does not exist.
Does that mean that we cannot compute $f(1)$?

Computing with Discontinuous Functions

Suggestion: What if we choose a sequence of continuous functions $\{f_k\}$ such that $f_k \to f$ pointwise, e.g.



Then

$$f(1) = \lim_{k\to\infty} \lim_{n\to\infty} f_k(x_n).$$

The n-pseudospectrum

Definition

Let T be a closed operator on a Hilbert space $\mathcal{H}$ such that $\sigma(T) \neq \mathbb{C}$, and let $n \in \mathbb{Z}_+$ and $\epsilon > 0$. The (n, ϵ)-pseudospectrum of T is defined as the set

$$\sigma_{n,\epsilon}(T) = \sigma(T) \cup \{z \notin \sigma(T) : \|(T - z)^{-2^n}\|^{1/2^n} > \epsilon^{-1}\}.$$

The n-pseudospectrum

Theorem

Let $T \in \mathcal{B}(\mathcal{H})$ and $\epsilon > 0$. Then the following is true:

(i) $\sigma_{n+1,\epsilon}(T) \subset \sigma_{n,\epsilon}(T)$.

(ii) *Let $\omega_\epsilon(\sigma(T))$ denote the ϵ-neighborhood around $\sigma(T)$. Then*

$$d_H\left(\overline{\sigma_{n,\epsilon}(T)}, \overline{\omega_\epsilon(\sigma(T))}\right) \longrightarrow 0, \qquad n \to \infty.$$

(iii) *If $\{T_k\} \subset \mathcal{B}(\mathcal{H})$ and $T_k \to T$ in norm, it follows that*

$$d_H\left(\overline{\sigma_{n,\epsilon}(T_k)}, \overline{\sigma_{n,\epsilon}(T)}\right) \longrightarrow 0, \qquad k \to \infty.$$

Tests with Discontinuity

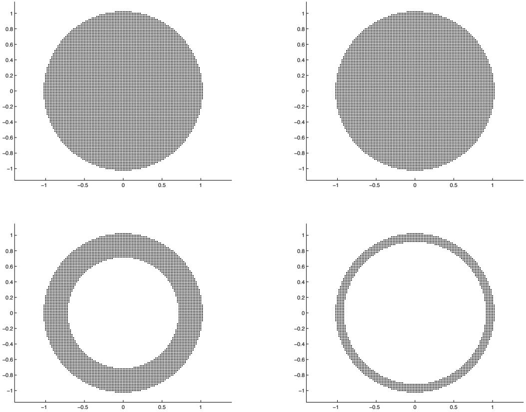


Figure: The figure shows $\sigma_{2,\epsilon}(A_0)$, $\sigma_{2,\epsilon}(A_{10^{-16}})$, $\sigma_{1,\epsilon}(A_{0.005})$, and $\sigma_{2,\epsilon}(A_{0.005})$, for $\epsilon = 0.025$.

The Solvability Complexity Index

The Complexity Index is least amount of limits required to compute $\sigma(T)$.

Estimating Functions

Definition

Let $\mathcal{H}$ be a Hilbert space spanned by $\{e_j\}_{j\in\mathbb{N}}$ and let

$$\Upsilon = \{T \in \mathcal{C}(\mathcal{H}) : \mathrm{span}\{e_j\}_{n\in\mathbb{N}} \subset \mathcal{D}(T)\}. \tag{1}$$

Let $\Delta \subset \Upsilon$ and $\Xi : \Delta \to \Omega$, where Ω denotes the collection of closed subsets of $\mathbb{C}$. Let

$$\Pi_\Delta = \{\{x_{ij}\}_{i,j\in\mathbb{N}} : \exists\, T \in \Delta,\ x_{ij} = \langle Te_j, e_i\rangle\}.$$

A set of estimating functions of order k for Ξ is a family of functions

$$\Gamma_{n_1} : \Pi_\Delta \to \Omega, \Gamma_{n_1,n_2} : \Pi_\Delta \to \Omega, \ldots, \Gamma_{n_1,\ldots,n_{k-1}} : \Pi_\Delta \to \Omega,$$

$$\Gamma_{n_1,\ldots,n_k} : \{\{x_{ij}\}_{i,j\le N(n_1,\ldots,n_k)} : \{x_{ij}\}_{i,j\in\mathbb{N}} \in \Pi_\Delta\} \to \Omega,$$

where $N(n_1,\ldots,n_k) < \infty$ depends on $n_1,\ldots,n_k$,

Estimating Functions

Definition

with the following properties:

(i) The evaluation of $\Gamma_{n_1,\ldots,n_k}(\{x_{ij}\})$ requires only finitely many arithmetic operations and radicals of the elements $\{x_{ij}\}_{i,j\leq N(n_1,\ldots,n_k)}$.

(ii) Also, we have the following relation between the limits

$$\begin{aligned}
\Xi(T) &= \lim_{n_1\to\infty} \Gamma_{n_1}(\{x_{ij}\}), \\
\Gamma_{n_1}(\{x_{ij}\}) &= \lim_{n_2\to\infty} \Gamma_{n_1,n_2}(\{x_{ij}\}), \\
&\vdots \\
\Gamma_{n_1,\ldots,n_{k-1}}(\{x_{ij}\}) &= \lim_{n_k\to\infty} \Gamma_{n_1,\ldots,n_k}(\{x_{ij}\}).
\end{aligned}$$

The limit is defined as follows, for $\omega \in \Omega$ then $\omega = \lim_{n\to\infty} \omega_n$ if and only if, for any compact ball K such that $\omega \cap K^o \neq \emptyset$ we have $d_H(\omega \cap K, \omega_n \cap K) \to 0$, when $n \to \infty$.

Definition

Let $\mathcal{H}$ be a Hilbert space spanned by $\{e_j\}_{j\in\mathbb{N}}$, define Υ as in (1), and let $\Delta \subset \Upsilon$. A set valued function

$$\Xi : \Delta \subset \mathcal{C}(\mathcal{H}) \to \Omega$$

is said to have Solvability Complexity Index k if k is the smallest integer for which there exists a set of estimating functions of order k for Ξ. Also, Ξ is said to have infinite Solvability Complexity Index if no set of estimating functions exists. If there is a function

$$\Gamma : \{\{x_{ij}\} : \exists\, T \in \Delta,\ x_{ij} = \langle Te_j, e_i\rangle\} \to \Omega$$

such that $\Gamma(\{x_{ij}\}) = \Xi(T)$, and the evaluation of $\Gamma(\{x_{ij}\})$ requires only finitely many arithmetic operations and radicals of a finite subset of $\{x_{ij}\}$, then Ξ is said to have Solvability Complexity Index zero. The Solvability Complexity Index of a function Ξ will be denoted by $SC_{\mathrm{ind}}(\Xi)$.

Example

- Let $\mathcal{H}$ be a Hilbert space with basis $\{e_j\}$, $\Delta = \mathcal{B}(\mathcal{H})$ and $\Xi(T) = \sigma(T)$ for $T \in \mathcal{B}(\mathcal{H})$. Suppose that $\dim(\mathcal{H}) \leq 4$. Then Ξ must have Solvability Complexity Index zero, since one can obviously express the eigenvalues of T using finitely many arithmetic operations and radicals of the matrix elements $x_{ij} = \langle Te_j, e_i \rangle$.
- For $\dim(\mathcal{H}) \geq 5$ then obviously $SC_{\mathrm{ind}}(\Xi) > 0$, by the much celebrated theory of Abel on the unsolvability of the quintic using radicals.
- Now, what about compact operators? Suppose for a moment that we can show that $SC_{\mathrm{ind}}(\Xi) = 1$ if $\dim(\mathcal{H}) < \infty$. A standard way of determining the spectrum of a compact operator T is to let P_n be the projection onto $\mathrm{span}\{e_j\}_{j \leq n}$ and compute the spectrum of $P_nA\lceil_{P_n\mathcal{H}}$. This approach is justified since $\sigma(P_nA\lceil_{P_n\mathcal{H}}) \to \sigma(T)$ as $n \to \infty$. By the assumption on the Solvability Complexity Index in finite dimensions, it follows that if Δ denotes the set of compact operators then $SC_{\mathrm{ind}}(\Xi) \leq 2$.

Theoretical Results

Theorem

Let $\{e_j\}_{j\in\mathbb{N}}$ *be a basis for the Hilbert space* $\mathcal{H}$ *and let* $\Delta = \mathcal{B}(\mathcal{H})$*. Define, for* $n \in \mathbb{Z}_+, \epsilon > 0$*, the set valued functions*

$$\Xi_1, \Xi_2, \Xi_3 : \Delta \to \Omega,$$

$$\Xi_1(T) = \overline{\sigma_{n,\epsilon}(T)}, \quad \Xi_2(T) = \overline{\omega_\epsilon(\sigma(T))}, \quad \Xi_3(T) = \sigma(T).$$

Then

$$SC_{\mathrm{ind}}(\Xi_1) \leq 2, \qquad SC_{\mathrm{ind}}(\Xi_2) \leq 3, \qquad SC_{\mathrm{ind}}(\Xi_3) \leq 3.$$

The Estimating Functions I

Let $x_{ij} = \langle Te_j, e_i \rangle$ for $T \in \mathcal{B}(\mathcal{H})$. Also, define the set

$$\Theta_k = \{z \in \mathbb{C} : \Re z, \Im z = r\delta, r \in \mathbb{Z}, |r| \leq k\}, \qquad \delta = \sqrt{\frac{1}{k}}, \quad (2)$$

and define the set of estimating functions Γ_{n_1,n_2} and Γ_{n_1} in the following way. Let

$$\begin{aligned}\Gamma_{n_1,n_2}(\{x_{ij}\}) &= \{z \in \Theta_{n_2} : \nexists\, L \in LT_{\mathrm{pos}}(P_{n_1}\mathcal{H}),\, T_{\epsilon,n_1,n_2}(z) = LL^*\} \\ &\quad \cup \{z \in \Theta_{n_2} : \nexists\, L \in LT_{\mathrm{pos}}(P_{n_1}\mathcal{H}),\, \widetilde{T}_{\epsilon,n_1,n_2}(z) = LL^*\}, \\ \Gamma_{n_1}(\{x_{ij}\}) &= \{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(T_{\epsilon,n_1}(z)) \neq \emptyset\} \\ &\quad \cup \{z \in \mathbb{C} : (-\infty, 0] \cap \sigma(\widetilde{T}_{\epsilon,n_1}(z)) \neq \emptyset\}, \end{aligned} \quad (3)$$

The Estimating Functions II

where $LT_{\mathrm{pos}}(P_m\mathcal{H})$ denotes the set of lower triangular matrices in $\mathcal{B}(P_m\mathcal{H})$ (with respect to $\{e_j\}$) with strictly positive diagonal elements and

$$\begin{aligned} T_{\epsilon,n_1,n_2}(z) &= T_{n_1,n_2}(z) - \epsilon^{2^{n+1}} I, \\ \widetilde{T}_{\epsilon,n_1,n_2}(z) &= \widetilde{T}_{n_1,n_2}(z) - \epsilon^{2^{n+1}} I, \\ T_{\epsilon,n_1}(z) &= T_{n_1}(z) - \epsilon^{2^{n+1}} I, \\ T_{\epsilon,n_1}(z) &= T_{n_1}(z) - \epsilon^{2^{n+1}} I, \end{aligned} \tag{4}$$

$$\begin{aligned} T_m(z) &= P_m((T-z)^*)^{2^n}(T-z)^{2^n}\lceil_{P_m\mathcal{H}} \\ T_{m,k}(z) &= P_m((P_k(T-z)P_k)^*)^{2^n}(P_k(T-z)P_k)^{2^n}\lceil_{P_m\mathcal{H}}, \\ \widetilde{T}_m(z) &= P_m(T-z)^{2^n}((T-z)^*)^{2^n}\lceil_{P_m\mathcal{H}} \\ \widetilde{T}_{m,k}(z) &= P_m(P_k(T-z)P_k)^{2^n}((P_k(T-z)P_k)^*)^{2^n}\lceil_{P_m\mathcal{H}} \end{aligned} \tag{5}$$

Theoretical Results

Theorem

Let $\{e_j\}_{j\in\mathbb{N}}$ be a basis for the Hilbert space $\mathcal{H}$ and let d be a positive integer. Define

$$\Delta = \{T \in \mathcal{B}(\mathcal{H}) : \langle Te_{j+l}, e_j\rangle = \langle Te_j, e_{j+l}\rangle = 0, \quad l > d\}.$$

Let $\epsilon > 0$ and $n \in \mathbb{Z}_+$ and $\Xi_1, \Xi_2, \Xi_3 : \Delta \rightarrow \Omega$ be defined by $\Xi_1(T) = \overline{\sigma_{n,\epsilon}(T)}$, $\Xi_2(T) = \overline{\omega_\epsilon(\sigma(T))}$ and $\Xi_3(T) = \sigma(T)$. Then

$$SC_{\mathrm{ind}}(\Xi_1) = 1, \qquad SC_{\mathrm{ind}}(\Xi_2) \leq 2, \qquad SC_{\mathrm{ind}}(\Xi_3) \leq 2.$$

Spectral Theory for Laurent operators

Given a Laurent operator A_L on $l^2(\mathbb{Z})$

$$A_L = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \dots & a_0 & a_{-1} & a_{-2} & a_{-3} & \dots \\ \dots & a_1 & a_0 & a_{-1} & a_{-2} & \dots \\ \dots & a_2 & a_1 & a_0 & a_{-1} & \dots \\ \dots & a_3 & a_2 & a_1 & a_0 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

A_L is a bounded operator if and only if there is a function $f \in L^\infty(\mathbb{T})$, where $\mathbb{T}$ denotes the circle, such that $\{a_n\}_{n=-\infty}^{\infty}$ is the sequence of Fourier coefficients of f, that is

$$a_n = \frac{1}{2\pi} \int f(e^{i\theta}) e^{-in\theta}\, d\theta, \quad n \in \mathbb{Z}.$$

Also, $\sigma(A_L) = \mathcal{R}(f)$, where $\mathcal{R}(f)$ denotes the essential range of f.

Spectral Theory for Toeplitz operators

Given a Toeplitz operator A_T on $l^2(\mathbb{Z}_+)$

$$A_T = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \dots \\ a_1 & a_0 & a_{-1} & a_{-2} & \dots \\ a_2 & a_1 & a_0 & a_{-1} & \dots \\ a_3 & a_2 & a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that if $t \mapsto f(e^{it})$, $t \in [0, 2\pi]$ is a continuous function, then $\mathcal{R}(f) = f(\mathbb{T})$ is a curve in $\mathbb{C}$, and hence we can assign a winding number to every point $z \in \mathbb{C}$ with respect to the curve. We then have that $\sigma(A_T)$ is equal to this curve together with all complex numbers with non-zero winding number with respect to the curve.

Tests with Laurent and Toeplitz operators

In the examples we have chosen Laurent and Toeplitz operators with symbols

$$f_1(z) = 2z^{-3} - z^{-2} + 2iz^{-1} - 4z^2 - 2iz^3$$

and

$$f_2(z) = z^{-2} + z^{-1} + 1 + 2z,$$

Tests with Toeplitz I

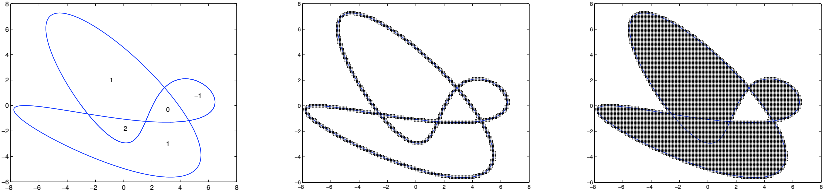


Figure: First figure shows the curve of f_1. Second figure shows $\Gamma_k^{n,\epsilon}(x_{ij}^L)$, and the third figure shows $\Gamma_k^{n,\epsilon}(x_{ij}^T)$ for $n = 2$, $\epsilon = 0.15$ and $k = 3000$

Tests with Toeplitz II

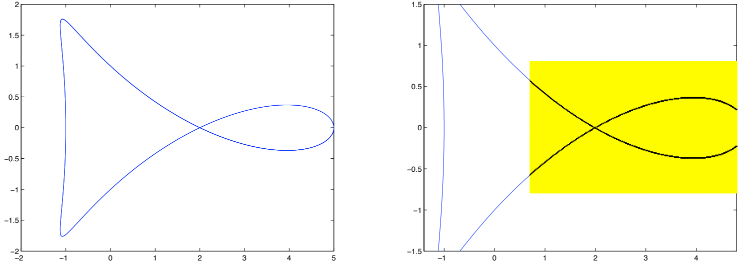


Figure: First figure shows the curve of f_2. Second figure shows $\Gamma_k^{n,\epsilon}(x_{ij}^L)$, for $n = 1$, $\epsilon = 0.01$ and $k = 6000$.

Tests with the Operator $\Psi(Q)$ for $\Psi \in L^\infty(\mathbb{R})$ I

Let

$$\Psi(x) = \frac{i(\exp(-2\pi i x) - 1)}{2\pi x}, \qquad x \in \mathbb{R},$$

and consider the following Gabor basis for $L^2(\mathbb{R})$:

$$e^{2\pi i m x}\chi_{[0,1]}(x-n), \qquad m, n \in \mathbb{Z}.$$

(where χ is the characteristic function) and then chosen some enumeration of $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{N}$ to obtain a basis $\{\psi_j\}$ that is just indexed over $\mathbb{N}$. To get our basis we let $\varphi_j = \mathcal{F}\psi_j$, where $\mathcal{F}$ is the Fourier Transform. Let

$$x_{ij} = \langle \Psi(Q)\varphi_j, \varphi_i \rangle.$$

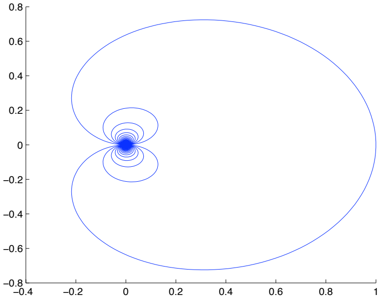
Now we can use

$$\Gamma^{n,\epsilon}_{n_1,n_2}(\{x_{ij}\})$$

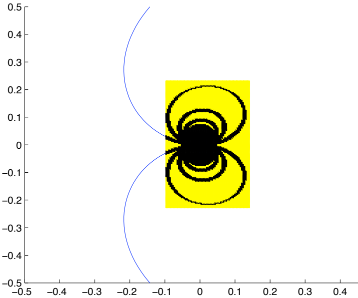
to estimate $\sigma(\Psi(Q))$.

Tests with the Operator $\Psi(Q)$ for $\Psi \in L^{\infty}(\mathbb{R})$ II

$\sigma(\Psi(Q))$



$\Gamma^{n,\epsilon}_{n_1,n_2}(\{x_{ij}\})$



$$n = 0, \qquad \epsilon = 0.01, \qquad n_1 = 10000, \qquad n_2 = 15000.$$

The disadvantage of the n-pseudospectrum is that even though one can estimate the spectrum by taking n very large, n may have to be too large for practical purposes. Thus, since we only have the estimate for $T \in \mathcal{B}(\mathcal{H}), \epsilon > 0$ that

$$\sigma(T) \subset \sigma_{n,\epsilon}(T),$$

it is important to get a "lower" bound on $\sigma(T)$ i.e. we want to find a set $\Omega \subset \mathbb{C}$ such that

$$\Omega \subset \sigma(T).$$

The Residual Pseudospectrum

Definition

Define the function $\Phi : \mathcal{B}(\mathcal{H}) \times \mathbb{C} \to \mathbb{R}$ by

$$\Phi(S, z) = \min \left\{ \lambda^{1/2} : \lambda \in \sigma\left((S - z)^*(S - z)\right) \right\}.$$

and for $T \in \mathcal{B}(\mathcal{H})$ let

$$\zeta_1(z) = \Phi_0(T, z), \qquad \zeta_2(z) = \Phi_0(T^*, \bar{z}).$$

Now let $\epsilon > 0$ and define the ϵ-residual pseudospectrum to be the set

$$\sigma_{\mathrm{res},\epsilon}(T) = \{z : \zeta_1(z) > \epsilon,\, \zeta_2(z) = 0\}$$

and the adjoint ϵ-residual pseudospectrum to be the set

$$\sigma_{\mathrm{res}^*,\epsilon}(T) = \{z : \zeta_1(z) = 0,\, \zeta_2(z) > \epsilon\}.$$

The Residual Pseudospectrum

Theorem

Let $T \in \mathcal{B}(\mathcal{H})$ and let $\{T_k\} \subset \mathcal{B}(\mathcal{H})$ such that $T_k \to T$ in norm, as $k \to \infty$. Then for $\epsilon > 0$ we have the following,

(i) $\sigma(T) \supset \bigcup_{\epsilon>0} \sigma_{\mathrm{res},\epsilon}(T) \cup \sigma_{\mathrm{res}^*,\epsilon}(T)$

(ii) $\mathrm{cl}(\{z \in \mathbb{C} : \zeta_1(z) < \epsilon\}) = \{z \in \mathbb{C} : \zeta_1(z) \leq \epsilon\}$

(iii) $\mathrm{cl}(\{z \in \mathbb{C} : \zeta_2(z) < \epsilon\}) = \{z \in \mathbb{C} : \zeta_2(z) \leq \epsilon\}$

(iv)
$$d_H(\mathrm{cl}(\sigma_{\mathrm{res},\epsilon}(T_k)), \mathrm{cl}(\sigma_{\mathrm{res},\epsilon}(T))) \longrightarrow 0, \qquad k \to \infty.$$

(v)
$$d_H(\mathrm{cl}(\sigma_{\mathrm{res}^*,\epsilon}(T_k)), \mathrm{cl}(\sigma_{\mathrm{res}^*,\epsilon}(T))) \longrightarrow 0, \qquad k \to \infty.$$

Theoretical Results

Theorem

Let $\{e_j\}_{j\in\mathbb{N}}$ *be a basis for* $\mathcal{H}$ *and define* $\Xi_1, \Xi_2 : \mathcal{B}(\mathcal{H}) \to \Omega$*, for* $\epsilon > 0$*, by* $\Xi_1(T) = \mathrm{cl}(\sigma_{\mathrm{res},\epsilon}(T))$ *and* $\Xi_2(T) = \mathrm{cl}(\sigma_{\mathrm{res}^*,\epsilon}(T))$*. Then*

$$SC_{\mathrm{ind}}(\Xi_1) \leq 2, \qquad SC_{\mathrm{ind}}(\Xi_2) \leq 2.$$

Tests with the Residual Pseudospectrum

$$T_1 = \begin{pmatrix} 0 & a & b & c & 0 & 0 & \dots \\ d & 0 & a & b & c & 0 & \dots \\ f & e & 0 & a & b & c & \dots \\ g & f & d & 0 & a & b & \dots \\ 0 & g & f & e & 0 & a & \dots \\ 0 & 0 & g & f & d & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & a & b & c & 0 & 0 & \dots \\ d & 0 & a & b & c & 0 & \dots \\ f & e & 0 & a & b & c & \dots \\ g & f & d & 0 & a & b & \dots \\ \phi_1 & g & f & e & 0 & a & \dots \\ 0 & \psi_1 & g & f & d & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 0 & a & b & c & 0 & 0 & \dots \\ d & 0 & a & b & c & 0 & \dots \\ 0 & e & 0 & a & b & c & \dots \\ g & 0 & d & 0 & a & b & \dots \\ \phi_1 & g & 0 & e & 0 & a & \dots \\ 0 & \psi_1 & g & 0 & d & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $a = 1 + 2i$, $b = -1$, $c = 5 + i$, $d = -2$, $e = 1 + 2i$, $f = -4$, $g = -1 - 2i$, $\phi_j = -2 + \frac{-5+15i}{j^{1/6}}$ and $\psi_j = 1 + 2i + \frac{5+15i}{j^{1/3}}$.

Other Types of Pseudospectra

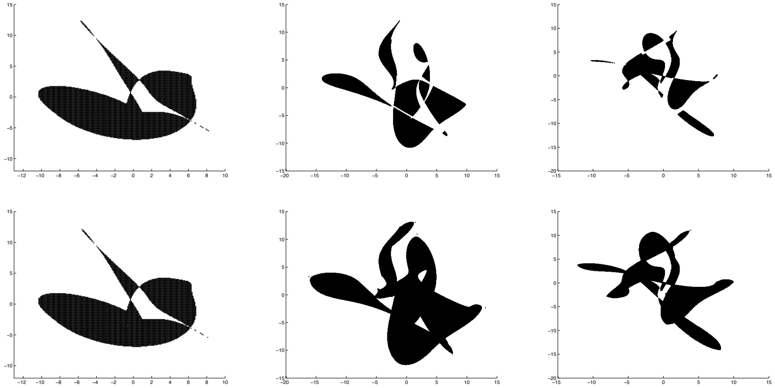


Figure: The upper figures show $\sigma_{\mathrm{res},\epsilon}(T_j) \cup \sigma_{\mathrm{res}^*,\epsilon}(T_j)$ and the lower figures show $\sigma_\epsilon(T_j)$ for $\epsilon = 0.01$ and $j = 1, 2, 3$.

Theorem

Let $\{e_j\}_{j\in\mathbb{N}}$ and $\{\tilde{e}_j\}_{j\in\mathbb{N}}$ be bases for the Hilbert space $\mathcal{H}$ and let

$$\tilde{\Delta} = \{T \in \mathcal{C}(\mathcal{H}\oplus\mathcal{H}) : T = T_1 \oplus T_2, T_1, T_2 \in \mathcal{C}(\mathcal{H}), T_1^* = T_2\}$$

$$\Delta = \{T \in \tilde{\Delta} : \mathrm{span}\{e_j\}_{j\in\mathbb{N}} \text{ is a core for } T_1, \mathrm{span}\{\tilde{e}_j\} \text{ is a core for } T_2\}.$$

Let $\epsilon > 0$, $\Xi_1 : \Delta \to \Omega$ and $\Xi_2 : \Delta \to \Omega$ be defined by $\Xi_1(T) = \overline{\sigma_\epsilon(T_1)}$ and $\Xi_2(T) = \sigma(T_1)$. Then

$$SC_{\mathrm{ind}}(\Xi_1) \leq 2, \qquad SC_{\mathrm{ind}}(\Xi_2) \leq 3.$$

Theoretical Results

Theorem

Let $\{e_j\}_{j\in\mathbb{N}}$ *be a basis for the Hilbert space* $\mathcal{H}$ *and let*

$$\Delta = \{T \in \mathcal{C}(\mathcal{H}) : T = W + A,\ W \in WS(\mathcal{H}),\ A \in \mathcal{B}(\mathcal{H})\ \cap \{T \in \mathcal{C}(\mathcal{H}) : \|R(T,\cdot)^{2^n}\|^{1/2^n} \text{ is never constant for any } n\}. \tag{6}$$

Define, for $n \in \mathbb{Z}_+, \epsilon > 0$, *the set valued functions*

$$\Xi_1, \Xi_2, \Xi_3 : \Delta \to \Omega, \quad \Xi_1(T) = \overline{\sigma_{n,\epsilon}(T)}, \quad \Xi_2(T) = \overline{\omega_\epsilon(\sigma(T))}, \quad \Xi_3(T)$$

Then

$$SC_{\mathrm{ind}}(\Xi_1) \leq 3, \qquad SC_{\mathrm{ind}}(\Xi_2) \leq 4, \qquad SC_{\mathrm{ind}}(\Xi_3) \leq 4.$$

Theoretical Results

Corollary

Let $\{e_j\}_{j\in\mathbb{N}}$ *be a basis for the Hilbert space* $\mathcal{H}$ *and let*

$$\Delta = \{A \in \mathcal{SA}(\mathcal{H}) : \mathrm{span}\{e_j\}_{j\in\mathbb{N}} \text{ is a core for } A\}.$$

Let $\epsilon > 0$ *and* $\Xi_1, \Xi_2 : \Delta \to \Omega$ *be defined by* $\Xi_1(T) = \sigma(T)$ *and* $\Xi_2(T) = \overline{\omega_\epsilon(\sigma(T))}$. *Then*

$$SC_{\mathrm{ind}}(\Xi_1) \leq 3, \qquad SC_{\mathrm{ind}}(\Xi_2) \leq 2.$$

The Final Questions

- What is the Solvability Complexity Index of spectra of different classes of operators?
- Can we compute the spectrum without access to the the matrix elements of the adjoint?

Reference and Acknowledgements

I would like to thank Bill Arveson, Erik Bédos, Albrecht Böttcher, Brian Davies, Percy Deift, Weinan E, Arieh Iserles, Olavi Nevanlinna, Don Sarason, Barry Simon and Nick Trefethen for useful discussions and comments.

Results can be found in [Han11]

A.C. Hansen, *On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators*, J. Amer. Math. Soc. **24** (2011), no. 1, 81–124.