Infinite Dimensional Compressed Sensing

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Compressed Sensing

Let $U \in \mathbb{C}^{n \times n}$, $x_0 \in \mathbb{C}^n$ and consider

$$y = Ux_0.$$

We want to recover x_0 from y. This is obvious if U is invertible and we know y.
What if we do not know y, but rather

$$P_\Omega y,$$

where P_Ω is the projection onto $\mathrm{span}\{e_j\}_{j \in \Omega}$ and $\Omega \subset \{1, \dots, n\}$ with $|\Omega| = m$ and Ω is randomly chosen.
Can we recover x_0 from $P_\Omega y$?

Where

Magnetic Resonance Imaging (MRI)
Let U be the discrete Fourier Transform and x_0 be an image of the brain. The question is now: How to reconstruct x_0 from the measurement vector y. In particular, we have:

$x_0 =$  $y = Ux_0,$

Sparsity

Given $x_0 \in \mathbb{C}^n$ let

$$\Delta = \{k \in \mathbb{N} : \langle x_0, e_j \rangle \neq 0\}.$$

Want to find a strategy so that x_0 can be reconstructed from $P_\Omega U x_0$, where $|\Omega| = m$, with high probability. In particular we would like to know how large m must be as a function of n and $|\Delta|$.

Convex Optimization

Want to recover x_0 from $P_\Omega U x_0$ by finding

$$\inf_x \|x\|_{l^0}, \qquad P_\Omega U x = P_\Omega U x_0 \tag{1}$$

where $\|x\|_{l^0} = |\{j : x_j \neq 0\}|$ or

$$\inf_x \|x\|_{l^1}, \qquad P_\Omega U x = P_\Omega U x_0, \tag{2}$$

where $\|x\|_{l^1} = \sum_{j=1}^n |x_j|$. Note that (1) is a non-convex optimization problem and (2) is a convex optimization problem.

Theoretical Results

Theorem

(Candes, Romberg, Tao) Let $x_0 \in \mathbb{C}^n$ be a discrete signal supported on an unknown set Δ, and choose Ω of size $|\Omega| = m$ uniformly at random. For a given accuracy parameter M there is a constant C_M such that if

$$m \geq C_M \cdot |\Delta| \cdot \log(n)$$

then with probability at least

$$1 - \mathcal{O}(n^{-M}),$$

the minimizer to the problem (2) is unique and is equal to x_0 .

The Model

- Given a separable Hilbert space $\mathcal{H}$ with an orthonormal set $\{\varphi_k\}_{k\in\mathbb{N}}$.
- Given a vector

$$x_0 = \sum_{k=1}^{\infty} \beta_k \varphi_k, \qquad \beta = \{\beta_1, \beta_2, \ldots\}.$$

- Suppose also that we are given a set of linear functionals $\{\zeta_j\}_{j\in\mathbb{N}}$ such that we can "measure" the vector x_0 by applying the linear functionals e.g. we can obtain $\{\zeta_j(x_0)\}_{j\in\mathbb{N}}$.

An Infinite System of Equations

With some appropriate assumptions on the linear functionals $\{\zeta_j\}_{j\in\mathbb{N}}$ we may view the full recovery problem as the infinite dimensional system of linear equations

$$\begin{pmatrix} \zeta_1(x_0) \\ \zeta_2(x_0) \\ \zeta_3(x_0) \\ \vdots \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots \\ u_{21} & u_{22} & u_{23} & \dots \\ u_{31} & u_{32} & u_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \end{pmatrix}, \qquad u_{ij} = \zeta_i(\varphi_j), \tag{3}$$

where we will refer to $U = \{u_{ij}\}_{i,j\in\mathbb{N}}$ as the "measurement matrix".

Solution I

If we for example have that U forms an isometry on $l^2(\mathbb{N})$ we could, for every $K \in \mathbb{N}$, compute an approximation $x = \sum_{k=1}^{K} \tilde{\beta}_k \varphi_j$ by solving

$$A \begin{pmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \tilde{\beta}_3 \\ \vdots \\ \tilde{\beta}_K \end{pmatrix} = P_K U^* P_N \begin{pmatrix} \zeta_1(x_0) \\ \zeta_2(x_0) \\ \zeta_3(x_0) \\ \vdots \end{pmatrix}, \qquad A = P_K U^* P_N U P_K|_{P_K l^2(\mathbb{N})},$$

for some appropriately chosen $N \in \mathbb{N}$ (the number of samples). We would then get the following error:

$$\|x - x_0\|_{\mathcal{H}} \leq (1 + C_{K,N}) \|P_N^{\perp} \beta\|_{l^2(\mathbb{N})}, \quad \beta = \{\beta_1, \beta_2, \ldots\},$$

Solution II

where, for fixed K, the constant $C_{K,N} \to 0$ as $N \to \infty$. Moreover, the constant $C_{K,N}$ is given explicitly by

$$C_{K,N} = \left\| (P_K U^* P_N U P_K|_{P_K l^2(\mathbb{N})})^{-1} P_K U^* P_N U P_K^{\perp} \right\|,$$

and hence we may find, for any $K \in \mathbb{N}$, the appropriate choice of $N \in \mathbb{N}$ (the number of samples) to get the desired error bound. In particular, this can be done numerically, by computing with different sections of the infinite matrix U.

(i) Are there other ways of approximating (3)?

(ii) Could there be ways of reconstructing, with the same accuracy, but using fewer samples from $\{\zeta_j(x_0)\}$?

Infinite Dimensional Compressed Sensing

Let $\Omega \subset \mathbb{N}$ such that $|\Omega| = m < \infty$ be randomly chosen and let P_Ω denote the projection onto $\mathrm{span}\{e_j\}_{j\in\Omega}$. Now consider the convex (infinite-dimensional) optimization problem

$$\inf_{\eta \in l^1(\mathbb{N})} \|\eta\|_{l^1(\mathbb{N})} \; : \; P_\Omega \begin{pmatrix} \zeta_1(x_0) \\ \zeta_2(x_0) \\ \zeta_3(x_0) \\ \vdots \end{pmatrix} = P_\Omega \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots \\ u_{21} & u_{22} & u_{23} & \dots \\ u_{31} & u_{32} & u_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \vdots \end{pmatrix}. \tag{4}$$

(i) How do we randomly choose Ω?It does not make sense to choose Ω uniformly from the whole set $\mathbb{N}$.

(iii) What if we chose an $N \in \mathbb{N}$ and choose $\Omega \subset \{1, \dots, N\}$ uniformly at random with $|\Omega| = m < N$? But how big must N be?

(iii) If η is a solution to (4) (note that we may not have uniqueness) what is the error $\|\eta - \beta\|_{l^2(\mathbb{N})}$, and how does it depend on the choice of Ω? In particular, how big must m be. (Note that we must have the extra assumption that $\beta \in l^1(\mathbb{N})$.)

The solution to problem (4) cannot be computed explicitly because it is infinite-dimensional, and thus an approximation must be computed instead. For $M \in \mathbb{N}$, consider the optimization problem

$$\inf_{\eta \in P_M l^1(\mathbb{N})} \|\eta\|_{l^1(\mathbb{N})} \ : \ P_\Omega \begin{pmatrix} \zeta_1(x_0) \\ \zeta_2(x_0) \\ \zeta_3(x_0) \\ \vdots \end{pmatrix} = P_\Omega \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots \\ u_{21} & u_{22} & u_{23} & \dots \\ u_{31} & u_{32} & u_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} P_M \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_M \end{pmatrix}. \tag{5}$$

(i) If $\tilde{\eta}_M = \{\eta_1, \dots, \eta_M\}$ is a minimizer of (5), what is the behavior of $\tilde{\eta}_M$ as $M \to \infty$? Moreover, what happens to the error $\|\tilde{\eta}_M - \beta\|_{l^2(\mathbb{N})}$ as $M \to \infty$?

(ii) Observe that M cannot be too small, since then (5) may not have a solution. However, since $U = \{u_{ij}\}$ is an isometry up to a constant, it follows that (8) is feasible for all sufficiently large M.

The Semi-Infinite-Dimensional Model

We are given an $M \in \mathbb{N}$ and for $x_0 = \sum_{k=1}^{\infty} \beta_k \varphi_k \in \mathcal{H}$ we have that

$$\mathrm{supp}(x_0) = \{j \in \mathbb{N} : \beta_j \neq 0\} = \Delta \subset \{1, \dots, M\}$$

We will choose only finitely many of the samples $\{\zeta_j(x_0)\}_{j \in \mathbb{N}}$, in particular, we will choose a set $\Omega \subset \{1, \dots, N\}$ of size m uniformly at random.

- How large must N be?
- How large must m be to recover x_0 with high probability? Moreover, if $m = N$ will we then get perfect recovery with probability one?

The Full Infinite-Dimensional Model

In the full infinite dimensional model we consider the problem of recovering a vector $y_0 = \sum_{k=1}^{\infty} \alpha_k \varphi_k \in \mathcal{H}$ where

$$y_0 = x_0 + h, \qquad h = \sum_{k=1}^{\infty} c_k \varphi_k,$$

$$\mathrm{supp}(x_0) = \Delta \subset \{1, \ldots, M\}, \qquad \mathrm{supp}(h) = \{1, \ldots, \infty\},$$

where we have some estimate on $\sum_{k=1}^{\infty} |c_k|$. In other words, we do not know the support of h. In this case we may get in trouble if we try to solve (5) since it may not have a solution. Note however that (5) will have a solution if we replace P_M with a projection $P_{\widetilde{M}}$ and let $\widetilde{M}$ be sufficiently large? But what happens when $\widetilde{M} \to \infty$?

The Generalized Sampling Theorem

Theorem

(Adcock,H'10) Let $\mathcal{F}$ denote the Fourier transform on $L^2(\mathbb{R}^d)$. Suppose that $\{\varphi_j\}_{j\in\mathbb{N}}$ is an orthonormal set in $L^2(\mathbb{R}^d)$ such that there exists a $T > 0$ with $\mathrm{supp}(\varphi_j) \subset [-T, T]^d$ for all $j \in \mathbb{N}$. For $\epsilon > 0$, let $\rho : \mathbb{N} \to (\epsilon\mathbb{Z})^d$ be a bijection. Define the infinite matrix

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots \\ u_{21} & u_{22} & u_{23} & \dots \\ u_{31} & u_{32} & u_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad u_{ij} = (\mathcal{F}\varphi_j)(\rho(i)). \qquad (6)$$

Then, for $\epsilon \leq \frac{1}{2T}$, we have that $\epsilon^{d/2}U$ is an isometry.

Theorem

Also, set

$$f = \mathcal{F}g, \quad g = \sum_{j=1}^{\infty} \beta_j \varphi_j \in L^2(\mathbb{R}^N),$$

and let (for $l \in \mathbb{N}$)P_l denote the projection onto $\mathrm{span}\{e_1, \ldots, e_l\}$*. Then, for every $K \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that, for all $N \geq n$, the solution to*

$$A \begin{pmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \tilde{\beta}_3 \\ \vdots \\ \tilde{\beta}_K \end{pmatrix} = P_K U^* P_N \begin{pmatrix} f(\rho(1)) \\ f(\rho(2)) \\ f(\rho(3)) \\ \vdots \end{pmatrix}, A = P_K U^* P_N U P_K|_{P_K l^2(\mathbb{N})}, \tag{7}$$

is unique.

The Generalized Sampling Theorem

Theorem

If

$$\tilde{g}_{K,N} = \sum_{j=1}^{K} \tilde{\beta}_j \varphi_j, \qquad \tilde{f}_{K,N} = \sum_{j=1}^{K} \tilde{\beta}_j \mathcal{F}\varphi_j,$$

then

$$\|g - \tilde{g}\|_{L^2(\mathbb{R}^d)} \leq (1 + C_{K,N})\|P_K^{\perp}\beta\|_{l^2(\mathbb{N})}, \quad \beta = \{\beta_1, \beta_2, \dots\},$$

and

$$\|f - \tilde{f}\|_{L^\infty(\mathbb{R}^d)} \leq (2T)^{d/2}(1 + C_{K,N})\|P_K^{\perp}\beta\|_{l^2(\mathbb{N})},$$

where, for fixed K*, the constant* $C_{K,N} \to 0$ *as* $N \to \infty$*.*

Generalized Sampling and Compressed Sensing

Consider the optimization problem

$$\inf_{\eta \in P_M l^1(\mathbb{N})} \|\eta\|_{l^1(\mathbb{N})} \, : \, P_\Omega \begin{pmatrix} f(\rho(1)) \\ f(\rho(2)) \\ f(\rho(3)) \\ \vdots \end{pmatrix} = P_\Omega \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots \\ u_{21} & u_{22} & u_{23} & \dots \\ u_{31} & u_{32} & u_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} P_M \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_M \end{pmatrix}. \tag{8}$$

Experiment

Consider the function $f \in L^2(\mathbb{R})$ defined by $f = \mathcal{F}g$

$$g(t) = \sum_{k=1}^{L} \alpha_k \psi_k(t) + \cos(2\pi t)\chi_{[\frac{1}{2},\frac{9}{16}]}(t), \qquad L = 200,$$

where $|\{\alpha_k : \alpha_k \neq 0\}| = 25$, and the task is to reconstruct f from its point samples. Define, for $N \in \mathbb{N}$ and N odd, the function

$$f_N(t) = \sum_{k=-(N-1)/2}^{(N-1)/2} f(k\epsilon)\mathrm{sinc}\left(\frac{t+k\epsilon}{\epsilon}\right), \qquad \epsilon = 0.5$$

Experiment

Define also the functions

$$\tilde{f}_{N,K}(t) = \sum_{k=1}^{K} \tilde{\beta}_k \mathcal{F}\psi_k(t), \qquad \gamma_{N,m,M}(t) = \sum_{k=1}^{M} \eta_k \mathcal{F}\psi_k(t), \quad (9)$$

where $\tilde{\beta} = \{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\}$ is the solution to equation (7), and U is defined as in (6) (with the Haar wavelets $\{\psi_k\}_{k\in\mathbb{N}}$ as the basis). And also let $\eta = \{\eta_1, \ldots, \eta_M\}$ be a solution to (8) where $\Omega \subset \{1, \ldots, N\}$ is chosen uniformly at random with $|\Omega| = m$. Note that if we express f and g in series expansions then

$$f = \sum_{k=1}^{\infty} \beta_k \mathcal{F}\psi_k, \quad g = \sum_{k=1}^{\infty} \beta_k \psi_k, \qquad |\{\beta_k : k \in \mathbb{N}, \beta_k \neq 0\}| = \infty,$$

Results

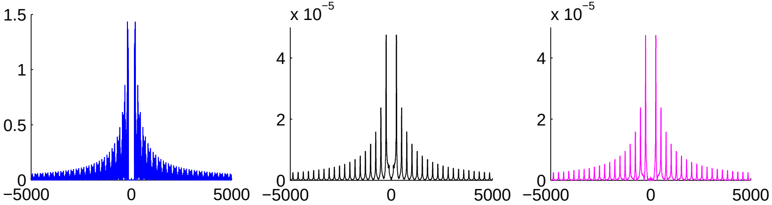


Figure: The figure displays the errors $|f_N - f|$ (left), $|\tilde{f}_{N,K} - f|$ (middle), $|\gamma_{N,m,M} - f|$ (right), for $N = 601, K = 200, m = 230$, $M = 650$. Note that $\gamma_{N,m,M}$ requires only thirty eight percent of the samples.

Results

N	$\|f_N - f\|_{L^\infty(\mathbb{R})}$	$\|\tilde{f}_{N,K} - f\|_{L^\infty(\mathbb{R})}$	$\|\gamma_{N,m,M} - f\|_{L^\infty(\mathbb{R})}$ (avg. 20 trials)
601	1.43	$4.74 \cdot 10^{-4}, (K = 200)$	$4.73 \cdot 10^{-5}, (m = 230, M = 550)$
1201	0.85	$2.36 \cdot 10^{-5}, (K = 400)$	$2.38 \cdot 10^{-5}, (m = 460, M = 1400)$

Table: The table shows the error corresponding to the reconstruction functions f_N, $\tilde{f}_{N,K}$ and $\gamma_{N,m,M}$ for different values of N, K, m, M.

Results

N	$E_{N,m,M} = \|\gamma_{N,m,M} - f\|_{L^\infty(\mathbb{R})}$ (avg. 20 trials)			
601	$E_{N,230,200} = \infty$	$E_{N,230,350} = \infty$	$E_{N,230,550} = 4.759 \cdot 10^{-5}$	$E_{N,230,850} = 4.727 \cdot 10^{-5}$
1201	$E_{N,460,400} = \infty$	$E_{N,460,500} = \infty$	$E_{N,460,1000} = 2.384 \cdot 10^{-5}$	$E_{N,460,1300} = 2.392 \cdot 10^{-5}$

Table: The table shows the error $\|\gamma_{N,m,M} - f\|_{L^\infty(\mathbb{R})}$ for different values of N, m and M.

Theorem

(H'10) Let $U \in \mathcal{B}(\mathcal{H})$ be an isometry. Suppose that for $M \in \mathbb{N}$ we have $\Delta \subset \{1, \ldots, M\}$, and $x_0 \in l^1(\mathbb{N})$ such that $\mathrm{supp}(x_0) = \Delta$. Let, for $\epsilon > 0$, the integers m and N be chosen such that

$$\|P_M U^* P_N U P_M - P_M\| \leq \left(4\sqrt{\log_2\left(4N\sqrt{|\Delta|}/m\right)}\right)^{-1}, \qquad (10)$$

$$\max_{|\Gamma|=|\Delta|, \Gamma \subset \{1,\ldots,M\}} \|P_M P_\Gamma^\perp U^* P_N U P_\Gamma\| \leq \frac{1}{8\sqrt{|\Delta|}}, \qquad (11)$$

$$m \geq C \cdot N \cdot \mu^2(U) \cdot |\Delta| \cdot \left(\log\left(\epsilon^{-1}\right) + 1\right) \cdot \log\left(MN\sqrt{|\Delta|}/m\right), \qquad \mu(U) = \sup_{i,j \in \mathbb{N}} |\langle Ue_j, e_i\rangle|. \qquad (12)$$

for some universal constant C. Let $\Omega \subset \{1, \ldots, N\}$ be chosen uniformly at random with $|\Omega| = m$. If $\xi \in \mathcal{H}$ satisfies

$$\|\xi\|_{l^1} = \inf_{\eta \in \mathcal{H}} \{\|\eta\|_{l^1} : P_\Omega U P_M \eta = P_\Omega U x_0\},$$

then, with probability exceeding $1 - \epsilon$, we have that ξ is unique and $\xi = x_0$. If $m = N$ then ξ is unique and $\xi = x_0$ with probability one.

Corollary

Let $U \in \mathbb{C}^{n\times n}$ be an isometry. Let $\Delta \subset \{1,\ldots,n\}$, and $x_0 \in \mathbb{C}^n$ such that $\mathrm{supp}(x_0) = \Delta$. Let, for $\epsilon > 0$, the integer m be chosen such that

$$m \geq C \cdot n \cdot \mu^2(U) \cdot |\Delta| \cdot \left(\log\left(\epsilon^{-1}\right) + 1\right) \cdot \log(n), \qquad (13)$$

for some universal constant C. Let $\Omega \subset \{1,\ldots,n\}$ be chosen uniformly at random with $|\Omega| = m$. If $\xi \in \mathcal{H}$ satisfies

$$\|\xi\|_{l^1} = \inf_{\eta\in\mathbb{C}^n}\{\|\eta\|_{l^1} : P_\Omega U\eta = P_\Omega U x_0\},$$

then, with probability exceeding $1-\epsilon$, we have that ξ is unique and $\xi = x_0$.

Theory

Theorem

(H'10) Let $U \in \mathcal{B}(\mathcal{H})$ *be an isometry. Suppose that for* $M \in \mathbb{N}$ *we have* $\Delta \subset \{1, \dots, M\}$, *and* $x_0, h \in l^1(\mathbb{N})$ *such that* $\mathrm{supp}(x_0) = \Delta$ *and* $\mathrm{supp}(h) \subset \{1, \dots, M\}$. *Define* $y_0 = x_0 + h$. *Let, for* $\epsilon > 0$, *the integers* m *and* N *be chosen according to (10), (11) and (12) (with a possibly different, however universal* C*). Let* $\Omega \subset \{1, \dots, N\}$ *be chosen uniformly at random with* $|\Omega| = m$. *If* $\xi \in \mathcal{H}$ *satisfies*

$$\|\xi\|_{l^1} = \inf_{\eta \in \mathcal{H}} \{\|\eta\|_{l^1} : P_\Omega U P_M \eta = P_\Omega U y_0\},$$

then, with probability exceeding $1 - \epsilon$, *we have that*

$$\|\xi - y_0\| \leq \left(\frac{20N}{m} + 11 + \frac{m}{N}\right) \|h\|_{l^1}. \tag{14}$$

If $m = N$ *then (14) is true with probability one.*

Theory

Theorem

(H'10) Let $U \in \mathcal{B}(\mathcal{H})$ be an isometry. Suppose that for $M \in \mathbb{N}$ we have $\Delta \subset \{1, \ldots, M\}$, and $x_0, h \in l^1(\mathbb{N})$ such that $\mathrm{supp}(x_0) = \Delta$. Define $y_0 = x_0 + h$. Let, for $\epsilon > 0$, the integers m and N be chosen according to (10) and also

$$\max_{|\Gamma|=|\Delta|, \Gamma \subset \{1,\ldots,M\}} \|P_\Gamma^\perp U^* P_N U P_\Gamma\| \leq \frac{1}{8\sqrt{|\Delta|}},$$

$$m \geq C \cdot N \cdot \mu^2(U) \cdot |\Delta| \cdot \left(\log\left(\epsilon^{-1}\right) + 1\right) \cdot \log\left(\frac{\Theta N \sqrt{|\Delta|}}{m}\right),$$

$$\Theta = \left| \left\{ i \in \mathbb{N} : \max_{\substack{\Gamma_1 \subset \{1,\ldots,M\}, |\Gamma_1| = |\Delta| \\ \Gamma_2 \subset \{1,\ldots,N\}}} \|P_{\Gamma_1} U^* P_{\Gamma_2} U e_i\| > \frac{m}{4N\sqrt{|\Delta|}} \right\} \right|.$$

for some universal constant C. Let $\Omega \subset \{1, \ldots, N\}$ be chosen uniformly at random with $|\Omega| = m$.

Theory

Theorem

If $\xi \in \mathcal{H}$ satisfies

$$\|\xi\|_{l^1} = \inf_{\eta \in \mathcal{H}} \{\|\eta\|_{l^1} : P_\Omega U \eta = P_\Omega U y_0\},$$

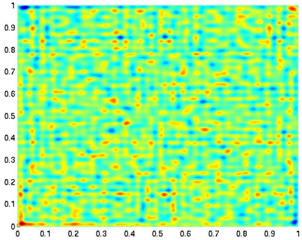
then, with probability exceeding $1 - \epsilon$, we have that

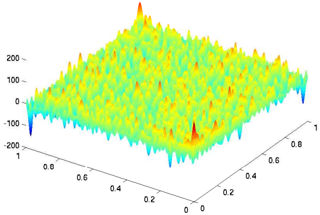
$$\|\xi - y_0\| \leq \left(\frac{20N}{m} + 11 + \frac{m}{N} \right) \|h\|_{l^1}. \tag{15}$$

If $m = N$ then (15) is true with probability one.

Infinite Resolution Image

Let x_0 denote the infinite resolution image:

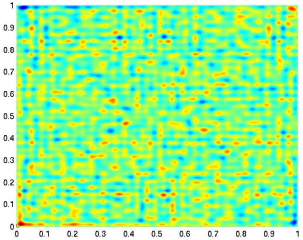


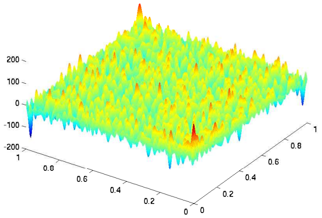


In particular,

$$g = \sum_{j=1}^{\infty} \alpha_j \varphi_j, \quad \varphi_j(x,y) = \sin(kx)\sin(ly), \quad x_0 = \{\alpha_1, \alpha_2, \dots\}$$

Infinite Resolution Image





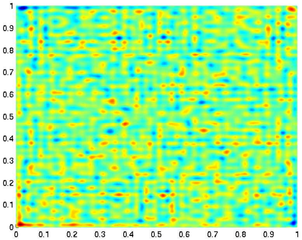
$$|\{\alpha_j : \alpha_j \neq 0\}| = 70,$$

$$\alpha_j = 0, \qquad j > 700.$$

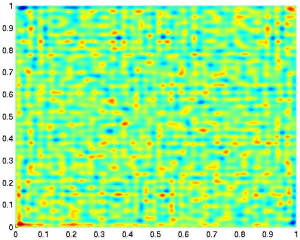
Classical MRI Reconstruction

$$g(t) = \epsilon \sum_{n=-\infty}^{\infty} (Fg)(n\epsilon)\, e^{2\pi i n\epsilon t}, \qquad g_N(t) = \epsilon \sum_{n=-N}^{N} (Fg)(n\epsilon)\, e^{2\pi i n\epsilon t}$$

Original



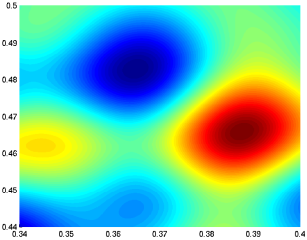
Reconstruction (501 by 501)



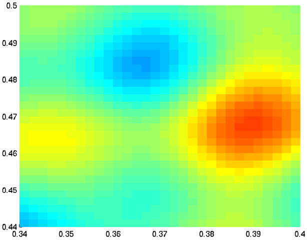
Classical MRI Reconstruction (enlarged)

$$g(t) = \epsilon \sum_{n=-\infty}^{\infty} (Fg)(n\epsilon)\, e^{2\pi i n\epsilon t}, \qquad g_N(t) = \epsilon \sum_{n=-N}^{N} (Fg)(n\epsilon)\, e^{2\pi i n\epsilon t}$$

Original



Reconstruction (501 by 501)

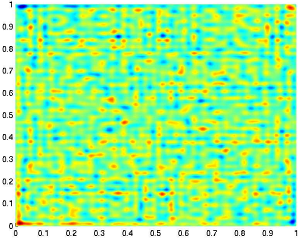


Finite Dim Comp Sens Reconstruction

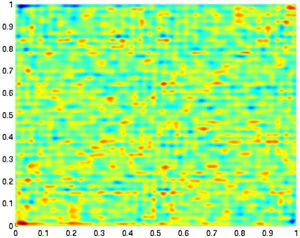
Solve

$$\min_x \|x\|_{TV}, \qquad P_\Omega U_{dft} x = P_\Omega y, \quad |\Omega| = 501^2/2$$

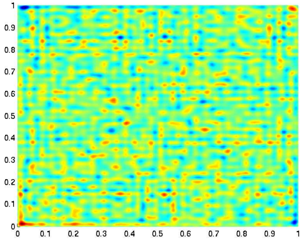
Original

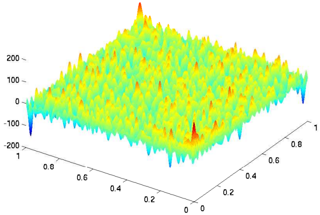


Reconstruction



Let x_0 denote the infinite resolution image:

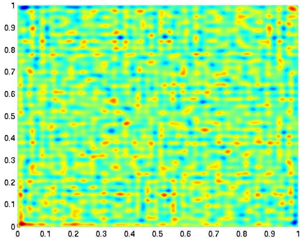


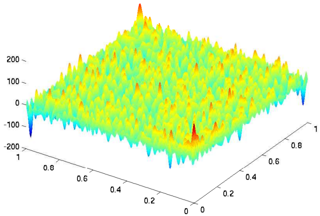


In particular,

$$g = \sum_{j=1}^{\infty} \alpha_j \varphi_j, \quad \varphi_j(x,y) = \sin(kx)\sin(ly), \quad x_0 = \{\alpha_1, \alpha_2, \ldots\}$$

Infinite Resolution Image





$$|\{\alpha_j : \alpha_j \neq 0\}| = 70,$$

$$\alpha_j = 0, \qquad j > 700.$$

Sampling

Choose $\epsilon > 0$ ($\epsilon = 0.5$), and consider the grid

$$\epsilon\mathbb{Z} \times \epsilon\mathbb{Z}.$$

Choose a bijection $\rho : \mathbb{N} \to \epsilon\mathbb{Z} \times \epsilon\mathbb{Z}$. Form the infinite matrix

$$U = \begin{pmatrix} F\varphi_1(\rho(1)) & F\varphi_2(\rho(1)) & F\varphi_3(\rho(1)) & \dots \\ F\varphi_1(\rho(2)) & F\varphi_2(\rho(2)) & F\varphi_3(\rho(2)) & \dots \\ F\varphi_1(\rho(3)) & F\varphi_2(\rho(3)) & F\varphi_3(\rho(3)) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

Choose $N \in \mathbb{N}$ ($N = 15000$). Randomly choose a set $\Omega = \{\omega_1, \dots \omega_m\} \subset \{1, \dots, N\}$ such that $|\Omega| = m = 500$. Let $y = \{Fg(\rho(\omega_1), \dots, Fg(\rho(\omega_m))\}$. Then

$$y = P_\Omega U x_0.$$

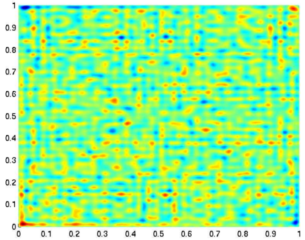
Recovery

Solve

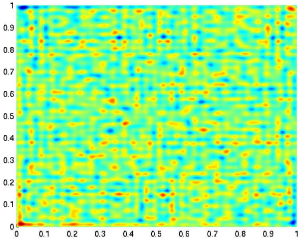
$$\inf_x \|x\|_{l^1}, \qquad P_\Omega U x = P_\Omega U x_0,$$

```
>> norm(x - x_0) = 3.2959e-08
```

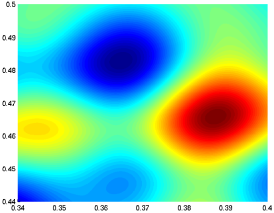
Original

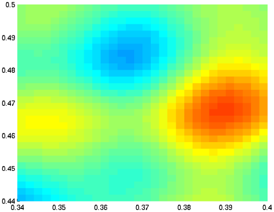


Reconstruction



Comparison


0.5
0.49
0.48
0.47
0.46
0.45
0.44
0.34
0.35
0.36
0.37
0.38
0.39
0.4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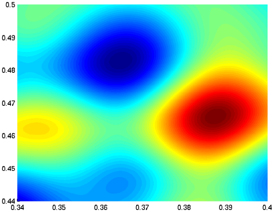
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