Computational Complexity and Compressed Sensing

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Chemnitz, October 4, 2010

Three Talks

(i) Generalized Sampling and Infinite Systems of Equations
(ii) Compressed Sensing in Infinite Dimensions
(iii) Spectral Problems and the Solvability Complexity Index

Computations with the Infinite Matrix

Let

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \qquad T = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

All questions that may be asked about A can be asked about T. For example: How to find the spectrum, how to find potential eigenvectors, how to solve linear systems of equations etc.

Why?

- Computer Science is discrete.
- Physics is continuous.

Let H denote the Schrodinger operator and F the Fourier Transform. In particular,

$$(Hf)(x) = -\Delta f(x) + V(x)f(x), \qquad (Ff)(\omega) = \int f(x)e^{-2\pi i\omega x}$$

- Compute the spectrum of H (Quantum Mechanics).
- Solve the inverse problem $f = Fg$ (Magnetic Resonance Imaging).

Applications

- Mathematical Physics (Quantum Mechanics)
- Signal Processing
- Inverse Problems (MRI, Tomography)
- Compressed Sensing
- Computational Biology

Generalized Sampling and Infinite Systems of Equations

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Motivation

Magnetic Resonance Imaging (MRI)
Let $\mathcal{F}$ denote the Fourier Transform. In particular,

$$(\mathcal{F}g)(\omega) = \int g(x)e^{-2\pi i\omega x}$$

Let

$$f = \mathcal{F}g.$$

We want to recover g (completely) from samples of f.

The Shannon Sampling Theorem

Suppose that

$$f = \mathcal{F}g, \qquad g \in L^2(\mathbb{R}),$$

and $\mathrm{supp}(g) \subset [-T, T]$ for some $T > 0$. If $\epsilon \leq \frac{1}{2T}$ (the Nyquist rate) then

$$f(t) = \sum_{k=-\infty}^{\infty} f(k\epsilon)\mathrm{sinc}\left(\frac{t+k\epsilon}{\epsilon}\right), \quad L^2 \text{ and unif. conv.,} \tag{1}$$

$$g = \epsilon \sum_{k=-\infty}^{\infty} f(k\epsilon)e^{2\pi i \epsilon k \cdot}, \qquad L^2 \text{ convergence.} \tag{2}$$

In practice, one forms the approximations

$$f_N = \sum_{k=-N}^{N} f(k\epsilon)\mathrm{sinc}\left(\frac{t+k\epsilon}{\epsilon}\right), \qquad g_N = \epsilon \sum_{k=-N}^{N} f(k\epsilon)e^{2\pi i \epsilon k \cdot}.$$

Question 1:

Are the approximations

$$f_N = \sum_{k=-N}^{N} f(k\epsilon)\mathrm{sinc}\left(\frac{t+k\epsilon}{\epsilon}\right), \qquad g_N = \epsilon \sum_{k=-N}^{N} f(k\epsilon)e^{2\pi \mathrm{i}\epsilon k\cdot}$$

optimal given the samples

$$\{f(k\epsilon)\}_{k=-N}^{N}.$$

Question 2:

Could there be L^2 functions $\{\varphi_k\}_{k\in\mathbb{N}}$ and coefficients $\{\beta_k\}_{k\in\mathbb{N}}$ such that the series

$$f = \sum_{k\in\mathbb{N}} \beta_k \mathcal{F}\varphi_k, \qquad g = \sum_{k\in\mathbb{N}} \beta_k \varphi_k$$

converge faster than the series

$$f(t) = \sum_{k=-\infty}^{\infty} f(k\epsilon)\mathrm{sinc}\left(\frac{t+k\epsilon}{\epsilon}\right),$$

$$g = \epsilon \sum_{k=-\infty}^{\infty} f(k\epsilon)e^{2\pi\mathrm{i}\epsilon k\cdot}.$$

Question 3:

There are therefore two important questions to ask:

(i) Can one obtain the coefficients $\{\beta_k\}_{k\in\mathbb{N}}$ (or at least approximations to them), based on the same sampling information $\{f(k\epsilon)\}_{k\in\mathbb{N}}$, and will this yield better approximations to f and g?

(ii) Can one subsample from $\{f(\epsilon k)\}_{k\in\mathbb{N}}$ (e.g. not sampling at the Nyquist rate) and still get recovery of $\{\beta_k\}_{k\in\mathbb{N}}$ and hence f and g?

Example:

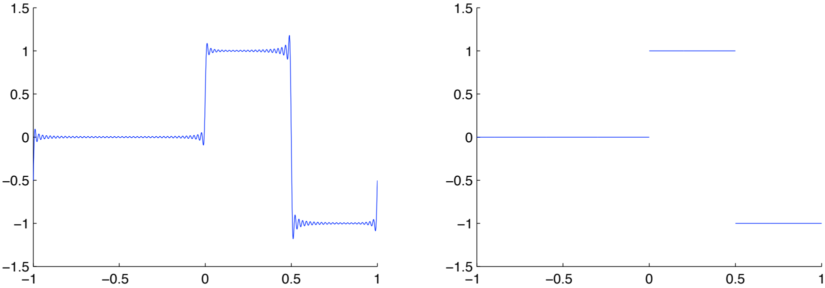


Figure: The figure shows $g_N = \epsilon \sum_{k=-N}^{N} f(k\epsilon) e^{2\pi i \epsilon k \cdot}$ for $N = 500$ and $\epsilon = 0.5$ (left) as well as g (right).

The Abstract Sampling Idea

Let $\mathcal{H}$ be a separable Hilbert space and let $f \in \mathcal{H}$ be an element we would like to reconstruct.

- Given $m \in \mathbb{N}$ linearly independent sampling vectors $\{s_1, \ldots, s_m\}$ that span a subspace $\mathcal{S} \subset \mathcal{H}$, we can access the sampled inner products $c_k = \langle s_k, f \rangle$, $k = 1, \ldots, m$.
- Given linearly independent reconstruction vectors $\{w_1, \ldots, w_m\}$ that span a subspace $\mathcal{W} \subset \mathcal{H}$.
- Construct an approximation $\tilde{f} \in \mathcal{W}$ to f based on the samples $\{c_k\}_{k=1}^m$. Want to find coefficients $\{d_k\}_{k=1}^m$ (that are computed from the samples $\{c_k\}_{k=1}^m$) such that $\tilde{f} = \sum_{k=1}^m d_k w_k$.

The State of the Art

The reconstruction (Unser and Aldrobi) (Eldar) is

$$\tilde{f} = \sum_{k=1}^{m} d_k w_k = W(S^*W)^{-1}S^*f, \qquad (3)$$

where the operators $S, W : \mathbb{C}^m \to \mathcal{H}$ are defined by

$$Sx = x_1 s_1 + \ldots + x_m s_m, \qquad Wy = y_1 w_1 + \ldots + y_m w_m, \qquad (4)$$

and their adjoints $S^*, W^* : \mathcal{H} \to \mathbb{C}^m$ are easily seen to be

$$S^*g = \{\langle s_1, g\rangle, \ldots, \langle s_m, g\rangle\}, \qquad W^*h = \{\langle w_1, h\rangle, \ldots, \langle w_m, h\rangle\}.$$

The State of the Art

From this it is clear that we can express $S^*W : \mathbb{C}^m \to \mathbb{C}^m$ as the matrix

$$\begin{pmatrix} \langle s_1, w_1 \rangle & \dots & \langle s_m, w_1 \rangle \\ \vdots & \vdots & \vdots \\ \langle s_1, w_M \rangle & \dots & \langle s_m, w_m \rangle \end{pmatrix}. \tag{5}$$

Also, S^*W is invertible if and only if

$$\mathcal{W} \cap \mathcal{S}^\perp = \{0\}. \tag{6}$$

Thus, to construct $\tilde{f}$ one simply solves a linear system of equations. The error can now conveniently be bounded from above and below by

$$\|f - P_\mathcal{W} f\| \leq \|f - \tilde{f}\| \leq \frac{1}{\cos(\theta_{\mathcal{W}\mathcal{S}})} \|f - P_\mathcal{W} f\|,$$

where $P_\mathcal{W}$ is the projection onto $\mathcal{W}$,

$$\cos(\theta_{\mathcal{W}\mathcal{S}}) = \inf\{\|P_\mathcal{S} g\| : g \in \mathcal{W}, \|g\| = 1\}.$$

Questions

(i) What if $\mathcal{W} \cap \mathcal{S}^{\perp} \neq \{0\}$ so that S^*W is not invertible?

(ii) What if $\|(S^*W)^{-1}\|$ is large? The stability of the method must clearly depend on $\|(S^*W)^{-1}\|$. Thus, even if $(S^*W)^{-1}$ exists, one may not be able to use the method in practice as there will likely be increased sensitivity to both round-off error and noise.

Example

As for (i), the simplest example is to let $\mathcal{H} = l^2(\mathbb{Z})$ and let $\{e_j\}_{j\in\mathbb{Z}}$ be the natural basis (e_j is the infinite sequence with 1 in its j-th coordinate and zeros elsewhere). For $m \in \mathbb{N}$, let the sampling vectors $\{s_k\}_{k=-m}^{m}$ and the reconstruction vectors $\{w_k\}_{k=-m}^{m}$ be defined by $s_k = e_k$ and $w_k = e_{k+1}$. Then, clearly, $\mathcal{W} \cap \mathcal{S}^{\perp} = \{e_{m+1}\}$.

Example I

For an example of more practical interest, consider the following: Let, for $0 < \epsilon \leq 1$, $\mathcal{H} = L^2([0, 1/\epsilon])$ and, for odd $m \in \mathbb{N}$, define the sampling vectors

$$\{s_{\epsilon,k}\}_{k=-(m-1)/2}^{(m-1)/2}, \qquad s_{\epsilon,k} = e^{-2\pi \mathrm{i} \epsilon k \cdot} \chi_{[0,1/\epsilon]},$$

(this is exactly the type of measurement vector that will be used if one models Magnetic Resonance Imaging) and let the reconstruction vectors $\{w_k\}_{k=1}^{m}$ denote the m first Haar wavelets on $[0, 1]$ (including the constant function, $w_1 = \chi_{[0,1]}$).

Example II

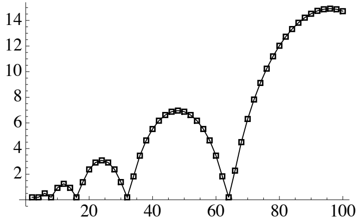


Figure: This figure shows $\log_{10} \|(S_\epsilon^* W)^{-1}\|$ as a function of m and ϵ for $m = 1, 2, \ldots, 150$. The left plot corresponds to $\epsilon = 1$, whereas the right plot corresponds to $\epsilon = 7/8$ (circles), $\epsilon = 1/2$ (crosses) and $\epsilon = 1/8$ (diamonds).

Example III

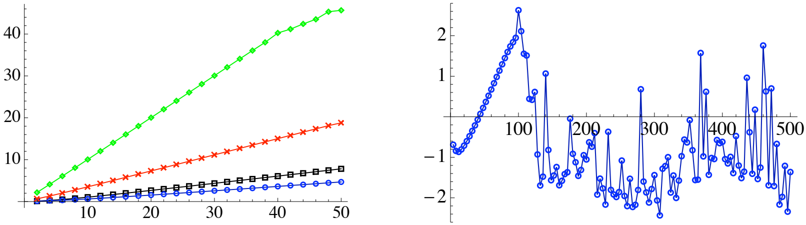


Figure: The left figure shows $\log_{10} \|(S^* W)^{-1}\|$ as a function of m for $m = 2, 4, \ldots, 50$ and $\epsilon = 1, \frac{1}{2}, \frac{7}{8}, \frac{1}{8}$ (squares, circles, crosses and diamonds respectively). The right figure shows $\log_{10} \|f - \tilde{f}\|$ for $m = 4, 8, \ldots, 500$, where $f(x) = \frac{1}{1+16x^2}$.

Connection to the Finite Section Method

Let $\{s_k\}_{k\in\mathbb{N}}$ and $\{w_k\}_{k\in\mathbb{N}}$ be two sequences of linearly independent elements in a Hilbert space $\mathcal{H}$. Define the infinite matrix U by

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots \\ u_{21} & u_{22} & u_{23} & \dots \\ u_{31} & u_{32} & u_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad u_{ij} = \langle s_j, w_i \rangle. \tag{7}$$

Thus, by (5) the operator S^*W is simply the m by m finite section of U. In particular

$$S^*W = P_m U P_m|_{P_m l^2(\mathbb{N})}.$$

The finite section method has been studied extensively in the last decades (Böttcher, Hagen, Lindner, Roch, Silbermann).

Connection to the Finite Section Method II

It is well known that even if U is invertible we may have that

- $P_m U P_m|_{P_m l^2(\mathbb{N})}$ may never be invertible for any m.
- $P_m U P_m|_{P_m l^2(\mathbb{N})}$ is invertible for all $m \in \mathbb{N}$, but

$$\|(P_m U P_m|_{P_m l^2(\mathbb{N})})^{-1}\| \to \infty, \qquad m \to \infty.$$

- There exist x, y such that

$$x = U^{-1} y, \quad x, y \in l^2(\mathbb{N}), \qquad x_m = (P_m U P_m|_{P_m l^2(\mathbb{N})})^{-1} P_m y$$

but

$$x_m \nrightarrow x, \qquad m \to \infty.$$

The New Approach

One would like to have a completely general sampling theory that can be described as follows:

(i) We have a signal $f \in \mathcal{H}$ and a Riez basis $\{w_k\}_{k\in\mathbb{N}}$ that spans some closed subspace $\mathcal{W} \subset \mathcal{H}$, and

$$f = \sum_{k=1}^{\infty} \beta_k w_k, \qquad \beta_k \in \mathbb{C}.$$

(ii) We have sampling vectors $\{s_k\}_{k\in\mathbb{N}}$ that form a Riez basis for a closed subspace $\mathcal{S} \subset \mathcal{H}$, and we can access the sampling values $\{\langle s_k, f\rangle\}_{k\in\mathbb{N}}$.

The New Approach

Goal: To reconstruct the best possible approximation $\tilde{f} \in \mathcal{W}$ based on a finite set of the sampling information $\{\langle s_k, f \rangle\}_{k \in \mathbb{N}}$, say, we are given $m \in \mathbb{N}$ samples $\{\langle s_k, f \rangle\}_{k=1}^{m}$.

The New Approach

Since $\{s_j\}$ and $\{w_j\}$ are Riesz bases, there exist constants $A, B, C, D > 0$ such that

$$A \sum_{k\in\mathbb{N}} |\alpha_k|^2 \leq \left\| \sum_{k\in\mathbb{N}} \alpha_k w_k \right\|^2 \leq B \sum_{k\in\mathbb{N}} |\alpha_k|^2$$

$$C \sum_{k\in\mathbb{N}} |\alpha_k|^2 \leq \left\| \sum_{k\in\mathbb{N}} \alpha_k s_k \right\|^2 \leq D \sum_{k\in\mathbb{N}} |\alpha_k|^2, \quad \forall \{\alpha_1, \alpha_2, \ldots\} \in l^2(\mathbb{N}). \tag{8}$$

The New Approach

Now let U be defined as in (7), choose $n \in \mathbb{N}$ and compute the solution $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\}$ of the following equation:

$$A \begin{pmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \vdots \\ \tilde{\beta}_n \end{pmatrix} = P_n U^* P_m \begin{pmatrix} \langle s_1, f \rangle \\ \langle s_2, f \rangle \\ \vdots \\ \langle s_m, f \rangle \end{pmatrix}, \qquad A = P_n U^* P_m U P_n|_{P_n \mathcal{H}}, \quad (9)$$

provided a solution exists (later we will provide estimates on the size of n, m for (9) to have a unique solution). Finally we let

$$\tilde{f} = \sum_{k=1}^{n} \tilde{\beta}_k w_k. \qquad (10)$$

Theoretical Results

Proposition

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{S}, \mathcal{W} \subset \mathcal{H}$ be closed subspaces. Suppose that $\{s_k\}_{k \in \mathbb{N}}$ and $\{w_k\}_{k \in \mathbb{N}}$ are Riesz bases for $\mathcal{S}$ and $\mathcal{W}$ respectively. In particular, we have constants $A, B, C, D > 0$ such that (8) is satisfied. Define U as in (7). Then U is a bounded operator on $l^2(\mathbb{N})$ with $\|U\| \leq \sqrt{BD}$. Also, U is invertible if and only if $\mathcal{W} \cap \mathcal{S}^{\perp} = \{0\}$ and $\mathcal{W}^{\perp} \cap \mathcal{S} = \{0\}$, with $\|U^{-1}\| \leq \sqrt{(AC)^{-1}}$.

Theoretical Results

Theorem

[Adcock,H'10] Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{S}, \mathcal{W} \subset \mathcal{H}$ be closed subspaces such that $\mathcal{W} \cap \mathcal{S}^{\perp} = \{0\}$. Suppose that $\{s_k\}_{k\in\mathbb{N}}$ and $\{w_k\}_{k\in\mathbb{N}}$ are Riesz bases for $\mathcal{S}$ and $\mathcal{W}$ respectively with constants $A, B, C, D > 0$. Suppose that

$$f = \sum_{k\in\mathbb{N}} \beta_k w_k, \qquad \beta = \{\beta_1, \beta_2, \dots,\} \in l^2(\mathbb{N}).$$

Let $n \in \mathbb{N}$. Then there is an $M \in \mathbb{N}$ such that for all $m \geq M$ then the solution $\{\tilde{\beta}_1, \dots, \tilde{\beta}_n\}$ to (9) is unique. Also, if $\tilde{f}$ is as in (10) then

$$\|f - \tilde{f}\|_{\mathcal{H}} \leq \sqrt{B}(1 + K_{n,m})\|P_n^{\perp}\beta\|_{l^2(\mathbb{N})}, \tag{11}$$

where

$$K_{n,m} = \left\|(P_n U^* P_m U P_n|_{P_n\mathcal{H}})^{-1} P_n U^* P_m U P_n^{\perp}\right\|. \tag{12}$$

Theoretical Results

Corollary

With the same assumptions as in Theorem 2, for fixed $n \in \mathbb{N}$ then

$$\left\|(P_n U^* P_m U P_n|_{P_n\mathcal{H}})^{-1}\right\| \longrightarrow \left\|(P_n U^* U P_n|_{P_n\mathcal{H}})^{-1}\right\| \leq \left\|(U^* U)^{-1}\right\|, \quad (13)$$

as $m \to \infty$. In addition, if U is an isometry onto its range (in particular, when $\{w_k\}_{k\in\mathbb{N}}, \{s_k\}_{k\in\mathbb{N}}$ are orthonormal) then, for fixed $n \in \mathbb{N}$, it follows that

$$K_{n,m} \longrightarrow 0, \qquad m \to \infty.$$

Proposition

Let $\mathcal{F}$ denote the Fourier transform on $L^2(\mathbb{R}^d)$. Suppose that $\{\varphi_j\}_{j\in\mathbb{N}}$ is a Riesz basis with constants A, B (as in (8)) for a subspace $\mathcal{V} \subset L^2(\mathbb{R}^d)$ such that there exists a $T > 0$ with $\mathrm{supp}(\varphi_j) \subset [-T, T]^d$ for all $j \in \mathbb{N}$. For $\epsilon > 0$, let $\rho : \mathbb{N} \to (\epsilon\mathbb{Z})^d$ be a bijection. Define the infinite matrix

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots \\ u_{21} & u_{22} & u_{23} & \dots \\ u_{31} & u_{32} & u_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad u_{ij} = (\mathcal{F}\varphi_j)(\rho(i)).$$

Then, for $\epsilon \leq \frac{1}{2T}$, we have that $U : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is bounded and invertible onto its range with $\|U\| \leq \sqrt{\epsilon^{-d} B}$ and $\|U^{-1}\| \leq \sqrt{\epsilon^d A^{-1}}$.

Theorem

(The Generalized Sampling Theorem)[Adcock,H'10] With the same setup as in Proposition 4, set

$$f = \mathcal{F}g, \quad g = \sum_{j=1}^{\infty} \beta_j \varphi_j \in L^2(\mathbb{R}^d),$$

and let P_n denote the projection onto $\mathrm{span}\{e_1, \ldots, e_n\}$. Then, for every $n \in \mathbb{N}$ there is an $M \in \mathbb{N}$ such that, for all $m \geq M$, the solution to

$$A \begin{pmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \vdots \\ \tilde{\beta}_n \end{pmatrix} = P_n U^* P_m \begin{pmatrix} f(\rho(1)) \\ f(\rho(2)) \\ \vdots \\ f(\rho(m)) \end{pmatrix}, \qquad A = P_n U^* P_m U P_n|_{P_n\mathcal{H}}, \qquad (14)$$

is unique.

Theorem

Also, if

$$\tilde{g}_{n,m} = \sum_{j=1}^{n} \tilde{\beta}_j \varphi_j, \qquad \tilde{f}_{n,m} = \sum_{j=1}^{n} \tilde{\beta}_j \mathcal{F}\varphi_j,$$

then

$$\|g - \tilde{g}_{n,m}\|_{L^2(\mathbb{R}^d)} \leq \sqrt{B}(1 + K_{n,m})\|P_n^{\perp}\beta\|_{l^2(\mathbb{N})}, \quad \beta = \{\beta_1, \beta_2, \ldots\}, \quad (15)$$

and

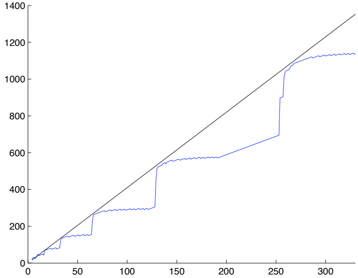
$$\|f - \tilde{f}_{n,m}\|_{L^\infty(\mathbb{R}^d)} \leq (2T)^{d/2}\sqrt{B}(1 + K_{n,m})\|P_n^{\perp}\beta\|_{l^2(\mathbb{N})}, \qquad (16)$$

where $K_{n,m}$ *is given in (12) and satisfies (13). Moreover, when* $\{\varphi_j\}_{j\in\mathbb{N}}$ *is an orthonormal set, it follows that*

$$K_{n,m} \longrightarrow 0, \qquad m \to \infty,$$

for fixed n.

Norm Bounds



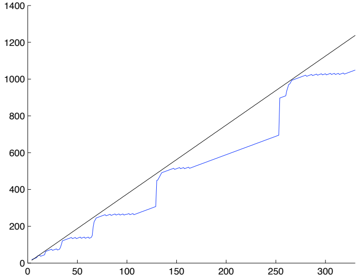


Figure: The left figure shows m (for the Haar wavelet) such that $K_{n,m} \leq 1$ together with the functions (in black) $x \mapsto 4.1x$. The right figure shows m such that $K_{n,m} \leq 2$ together with the function $x \mapsto 3.75x$.

Example (reconstruction from Fourier Transform) I

Suppose now that we consider the function

$$g(t) = \cos(2\pi t)\chi_{[0.5,1]}(t).$$

In this case, due to the discontinuity, forming

$$g_N = \epsilon \sum_{n=-N}^{N} f(n\epsilon)\, e^{2\pi i n \epsilon \cdot}, \qquad \epsilon = \frac{1}{2}, \quad N \in \mathbb{N}, \tag{17}$$

may be less than ideal, since the convergence $g_N \to g$ as $N \to \infty$ may be slow.

Suppose that we are given the finite collection of samples

$$\eta_f = \{f(-N\epsilon), f((-N+1)\epsilon), \ldots, f((N-1)\epsilon), f(N\epsilon)\}, \tag{18}$$

with $N = 900$ and $\epsilon = \frac{1}{2}$. Define $\tilde{\beta} = \{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\}$ by equation (14), and let

$$\tilde{g}_{n,m} = \sum_{j=1}^{n} \tilde{\beta}_j \psi_j, \qquad m = 1801, n = 500.$$

Example (reconstruction from Fourier Transform) II

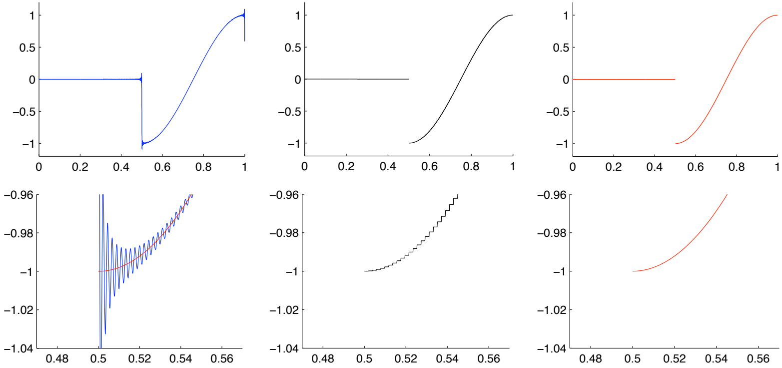


Figure: The upper figures show g_N (left), $\tilde{g}_{n,m}$ (middle) and g (right) on the interval $[0, 1]$. The lower figures show g_N (left), $\tilde{g}_{n,m}$ (middle) and g (right) on the interval $[0.47, 0.57]$.

Example (reconstruction from point samples) I

In this example we consider the following problem. Let $f \in L^2(\mathbb{R})$ such that

$$f = \mathcal{F}g, \qquad g(x) = \sum_{j=1}^{K} \alpha_j \psi_j(x) + \sin(2\pi x)\chi_{[0.3,0.6]}(x),$$

for $K = 400$, where $\{\psi_j\}$ are Haar wavelets on $[0,1]$, and $\{\alpha_j\}_{j=1}^{K}$ are some arbitrarily chosen real coefficients in $[0,10]$. Suppose that we can access the following pointwise samples of f:

$$\eta_f = \{f(-N\epsilon), f((-N+1)\epsilon), \ldots, f((N-1)\epsilon), f(N\epsilon)\},$$

with $\epsilon = \frac{1}{2}$ and $N = 600$. We may form

$$f_N(t) = \sum_{k=-N}^{N} f\left(\frac{k}{2}\right) \mathrm{sinc}(2t-k), \qquad N = 600.$$

Example (reconstruction from point samples) II

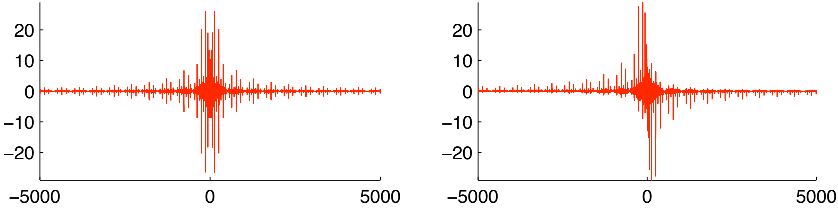


Figure: The figure shows $\mathrm{Re}(f)$ (left) and $\mathrm{Im}(f)$ (right) on the interval $[-5000, 5000]$.

Example (reconstruction from point samples) III

Let

$$\tilde{f}_{n,m} = \sum_{j=1}^{n} \tilde{\beta}_j \mathcal{F}\psi_j, \qquad n = 500, m = 1201$$

where $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\}$ satisfies (14).
In this case we have

$$\begin{aligned} \left\| \left(P_n U^* P_m U P_n|_{P_n\mathcal{H}} \right)^{-1} \right\| &\leq 0.9022, \\ \left\| \left(P_n U^* P_m U P_n|_{P_n\mathcal{H}} \right)^{-1} P_n U^* P_m \right\| &\leq 0.9498. \end{aligned}$$

Example (reconstruction from point samples) IV

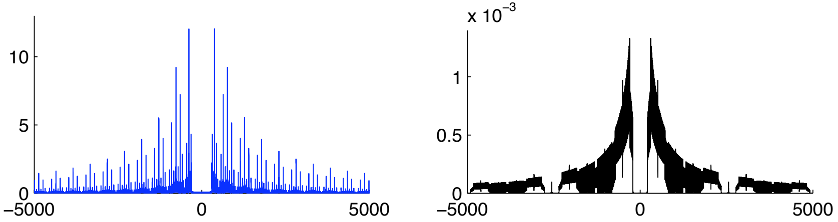


Figure: The figure shows the error $|f - f_N|$ (left) and $|f - \tilde{f}|$ (right) on the interval $[-5000, 5000]$.