Phase Retrieval Problems

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Phase retrieval problems arise in various fields of applied physics and engineering, such as crystallography, electron microscopy, astronomy, and optics. An abstract phase retrieval problem is stated as follows: Find a well–localized smooth complex-valued function f for given moduli $|f|$ and $|\hat{f}|$. Here $\hat{f}$ denotes the Fourier transform of f. Thus the problem is to find the phase of a complex–valued function f, where $|f|$ and $|\hat{f}|$ are given. This explains the names *phase retrieval* and *phase reconstruction*, respectively.

R.W. Gerchberg and W.O. Saxton (1972) proposed an error reduction algorithm for the iterative solution of a phase retrieval problem. This algorithm can be considered as projection algorithm of nonconvex sets. Often one can observe that the iterates stagnate away from a solution. But up to now, there exists no iterative algorithm for phase reconstruction without stagnation problems.

Our research is motivated by applications in laser optics. Here one measures ultrashort laser pulses at 128 or 256 points. Thus we are interested in numerical phase reconstruction of a complex–valued function f from *finitely many given data* of $|f|$ and $|\hat{f}|$. We assume additionally that f is a compactly supported, linear spline with $N = 2^J$ complex spline coefficients a_n, where the values $|f(n)|$ $(n = 0, \dots, N-1)$ and $|\hat{f}(k\pi/N)|$ $(k = -N, \dots, N-1)$ are given. Then we reconstruct the phases of the spline coefficients a_n. Note that our approach is not restricted to linear splines.

In these talks, we propose a new method of numerical phase reconstruction. Since our phase retrieval problem is equivalent to the nonlinear inverse ill–posed problem, we can apply *iterative Tikhonov regularization*. Regularization methods like the *iteratively regularized Gauss–Newton method* (IRGN) are attractive for numerically solving ill–posed nonlinear problems, since these methods can be implemented straightforward. We can show that IRGN produces a solution of the nonlinear system. But the restrictive conditions are the crucial point of this convergence result. With other words, IRGN requires a sufficiently good initial guess. Therefore we construct a convenient initial guess by using a *multilevel strategy*. First we compute a solution of a related minimization problem on a coarse grid. Then we interpolate the coarse grid solution to the fine grid and use this as initial guess of IRGN on the fine grid. The multilevel strategy concentrates the main effort of the solution of the phase retrieval problem in the coarse, less expensive levels and provides convenient initial guesses on the next finer level. On each level, the corresponding nonlinear system is solved by IRGN. This method is applicable to a wide range of examples as shown in several numerical tests for noiseless and noisy data.