The Smolyak Algorithm and Tensor Products of Function Spaces

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Abstract

The Smolyak algorithm represents one possible approach to the approximation of functions of many variables. The natural domains of definition are given by tensor products of function spaces defined on $\mathbb{R}$ or on some interval $I \subset \mathbb{R}$. Here Besov as well as Sobolev spaces of dominating mixed smoothness come into play. They are tensor products of Besov and Sobolev spaces defined on $\mathbb{R}$.

1 Introduction

The material presented in these four lectures is mainly based on results published in [34], [35], [44], and [45].
However, there are some monographs and many papers dealing with similar problems. We refer to the monographs of Delvos and Schempp [9], Temlyakov [38] and Trigub and Belinsky [42], where hyperbolic cross type approximation is studied.

2 An example from quantum chemistry

To begin with we describe one example from quantum chemistry: the electronic Schrödinger equation. We follow Yserentant [49, 50, 51].
Atoms and molecules are physically described by the Schrödinger equation for a system of charged particles that interact by Coulomb attraction and repulsion forces. As the nuclei are much heavier than the electrons, the electrons almost instantaneously follow their motion. Therefore it is usual in quantum chemistry to separate the motion of the nuclei from the motion of the electrons and to start from the electronic Schrödinger

equation, that is to look for the eigenvalues and eigenfunctions of the electronic Hamilton operator

$$H := -\frac{1}{2}\sum_{i=1}^{N}\Delta_i - \sum_{i=1}^{N}\sum_{\nu=1}^{K}\frac{Z_\nu}{|x_i - a_\nu|} + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{|x_i - x_j|}\,. \tag{1}$$

Here $x_i := (x_i^1, x_i^2, x_i^3)$ represents the coordinates of the i-th electron in space. The positions $a_1, \ldots, a_K$ of the nuclei are kept fixed. The positive values Z_ν are the charges of the nuclei in multiples of the electron charge. Δ_i denotes the Laplace operator with respect to x_i.

The operator H will be considered on the Hilbert space $H^1(\mathbb{R}^{3N})$, which is the set of all functions $u : R^{3N} \to \mathbb{R}$ s.t. $u \in L_2(\mathbb{R}^{3N})$ and u possesses first order weak derivatives also belonging to $L_2(\mathbb{R}^{3N})$. This set is euipped with the norm

$$\| \, u \,|H^1(\mathbb{R}^{3N})\| := \| \, u \,|L_2(\mathbb{R}^{3N})\| + \| \, \nabla u \,|L_2(\mathbb{R}^{3N})\| \,.$$

In fact, one considers the bilinear form

$$a(u,v) := \langle (H + \mu I)u, v\rangle\,, \qquad u,\ v \in H^1(\mathbb{R}^{3N})$$

for μ sufficiently large (s.t. the form becomes positive definite). Now one studies existence of eigenvalues and regularity of the associated eigenfunctions. Not all eigenfunctions are of physical relevance. Those, which are, satisfy additional symmetry relations, we refer to [51] for details. We concentrate on these eigenfunctions. Then one can prove:

(i) The eigenfunctions u have exponential decay in the H^1-sense, i.e. there exists some $\delta = \delta(\lambda) > 0$ s.t.

$$e^{\delta\,|x|}\,(|u(x)| + |\nabla u(x)|) \in L_2(\mathbb{R}^{3N})\,. \tag{2}$$

(ii) The Fourier transform $\mathcal{F}u$ of the eigenfunction u satisfies

$$\int_{\mathbb{R}^{3N}} \Big(\sum_{i=1}^{N}|\xi_i|^2\Big)\prod_{i=1}^{N}\Big(1 + |\xi_i|^2\Big)|\mathcal{F}u(\xi)|^2\,d\xi < \infty\,, \qquad \xi_i := (\xi_i^1, \xi_i^2, \xi_i^3)\,. \tag{3}$$

(iii) In general the eigenfunctions are not infinitely differentiable, see [51] for an explicit example.

Remark 1 One can characterize the space H^1 in terms of the Fourier transform. In fact, one has

$$\| \, u \,|H^1(\mathbb{R}^{3N})\|^2 \asymp \int_{\mathbb{R}^{3N}} \sum_{i=1}^{N}(1 + |\xi_i|^2)\,|\mathcal{F}u(\xi)|^2\,d\xi\,.$$

Comparing this with formula (3) we see that the Fourier transform of the eigenfunctions has some additional decay, i.e. the eigenfunction itself has some additional smoothness.

Summary: We need to approximate functions u of $3N$ variables and N can be even larger than 100. Carbon has six electrons, oxygen 8, Neon has 10 electrons, argon 18, iron 26, ...
These functions u do not belong to $C^\infty(\mathbb{R}^{3N})$, but they have some additional regularity.

Say, we are interested in the case $N = 10$. The number of variables is 30. Our eigenfunctions are exponentially decaying near infinity, so we can concentrate on a ball or a cube with centre in the origin. For simplicity we take $Q := [-1,1]^{30}$. To approximate a function f of one variable on $[-1,1]$ with a reasonable precision, say, we simply want to produce a plot, it will be enough to evaluate this functions at 100 sample points. We may take

$$f(x_k)\,, \quad x_k := \frac{k}{50}\,, \quad k = -49, \ldots, 50\,.$$

Turning back to our original function u the same procedure (with respect to each dimension) would require to evaluate u on the grid

$$\Big(\frac{k_1}{50}, \ldots, \frac{k_{30}}{50}\Big)\,, \qquad k_1, \ldots, k_{30} = -49, \ldots\ , 50\,.$$

Alltogether this means we have to evaluate u at $100^{30} = 10^{60}$ sample points. Of course, this is impossible.
Similarly, in case of carbon we would end up with 10^{36} which is still a very large number.

Linear widths

For two quasi-Banach spaces X, Y such that $X \hookrightarrow Y$ we define

$$\lambda_n(I,X,Y) := \inf\Big\{\|\, I - L\,|\mathcal{L}(X,Y)\| : \quad L \in \mathcal{L}(X,Y),\ \operatorname{rank} L \le n\Big\}\,.$$

The numbers $\lambda_n(I,X,Y)$ are called the linear widths of the embedding operator $I : X \to Y$. We are looking for the optimal approximation with respect linear operators L of $\operatorname{rank} L \le n$.

Theorem 1 *Let Q be a cube in $\mathbb{R}^d$. Then*

$$\lambda_n(I, H^1(Q), L_2(Q)) \asymp n^{-1/d}\,, \qquad n \in \mathbb{N}\,. \tag{4}$$

Proof A proof in the periodic setting may be found in [38, 1.4]. ∎

Remark 2 If we want to approximate our eigenfunction u in $[-1,1]^d$ with $d = 30$ with an error at most $\varepsilon > 0$ then we need a linear operator of rank $n \geq (c\,\varepsilon^{-1})^d$. Taking $\varepsilon = 10^{-2}$, then we end up with $n \asymp 10^{60}$ (ignoring c for the moment) which reflects again the fact that we can not do it in this way.

3 Wavelets and function spaces

When the first constructions of wavelets became known to us, it has been around 1986, we have been interested in, but we have not been aware of the enormous impact of this development to the theory of function spaces.

3.1 Wavelets

For the aim of my lecture it will be enough to use the following notion. We call a function $\psi \in L_2(\mathbb{R})$ a wavelet if the family

$$\widetilde{\psi}_{j,k}(t) := 2^{j/2}\,\psi(2^j t - k)\,, \qquad j,k \in \mathbb{Z}\,, \quad t \in \mathbb{R}\,, \tag{5}$$

is an orthonormal basis of $L_2(\mathbb{R})$. The most simple one is the Haar wavelet given by

$$\psi(t) := \begin{cases} 1 & \text{if} \quad 0 \leq t < 1/2\,, \\ -1 & \text{if} \quad 1/2 \leq t < 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Here we shall need two generalizations of the Haar wavelet, namely the Daubechies wavelets and the spline wavelets. If we define

$$W_j := \overline{\operatorname{span}\{\psi_{j,k}:\ k \in \mathbb{Z}\}}^{\|\cdot|L_2(\mathbb{R})\|}\,, \qquad j \in \mathbb{Z}\,,$$

then these closed subspaces satisfy $W_j \perp W_k$, $j \neq k$. They generate an orthogonal decomposition of $L_2(\mathbb{R})$, i.e.

$$L_2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j\,.$$

For us it will be more convenient to work with the (nonhomogeneous) decomposition into

$$L_2(\mathbb{R}) \quad = \quad \underbrace{\Big(\bigoplus_{j=-\infty}^{-1} W_j\Big)}_{=:\,V} \oplus \Big(\bigoplus_{j=0}^{\infty} W_j\Big)$$

There is a simple argument to obtain an appropriate orthonormal basis. Usually the construction of wavelets starts with the construction of an appropriate multiresolution analysis.

Definition 1 *A multiresolution analysis is a sequence* $\{V_j\}_{j=-\infty}^{\infty}$ *of closed subspaces of* $L_2(\mathbb{R})$ *s.t.*

(i) $\ldots V_j \subset V_{j+1} \subset V_{j+2} \ldots$;

(ii) *the set* $\bigcup_{j=-\infty}^{\infty} V_j$ *is dense in* $L_2(\mathbb{R})$*;*

(iii) $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$*;*

(iv) $f \in V_j$ *if and only if* $f(2t) \in V_{j+1}$*;*

(v) *there exists a function* $\varphi \in V_0$ *s.t. the family* $\{\varphi(t-m) : \; m \in \mathbb{Z}\}$ *is an orthonormal basis (ONB) of* V_0*.*

Such a function φ *is called orthogonal scaling function.*

If one has found an orthogonal scaling function φ then the associated multiresolution analysis is given by

$$V_j := V_j(\varphi) = \overline{\operatorname{span}\{\varphi(2^j t - k) : \; k \in \mathbb{Z}\}}^{\| \cdot |L_2(\mathbb{R})\|}, \qquad j \in \mathbb{Z},$$

see properties (vi) and (v). Having a multiresolution analysis $\{V_j\}_{j=-\infty}^{\infty}$ one obtains W_0 as the orthogonal complement of V_0 in V_1, i.e.

$$V_1 = V_0 \oplus W_0 \,.$$

Hence, an associated wavelet ψ, see [48, Thm. 2.20] for explicit formulas, generates an orthonormal basis of W_0 by taking $\{\psi(t-m) : \quad m \in \mathbb{Z}\}$. From property (iv) and the definition of W_j we derive

$$V_{j+1} = V_j \oplus W_j \,, \qquad j \in \mathbb{Z} \,.$$

Iterating this decomposition we obtain

$$V_{j+1} = V_0 \oplus W_0 \oplus W_1 \oplus \ldots \oplus W_j \,, \qquad j \in \mathbb{N}_0 \,,$$

and consequently

$$L_2(\mathbb{R}) = V_0 \oplus \Big(\bigoplus_{j=0}^{\infty} W_j \Big) \,.$$

Interpreting this decomposition we have found that also the family

$$\varphi(t-k),\ \widetilde{\psi}_{j,k}(t)\,, \qquad j \in \mathbb{N}_0\,,\ k \in \mathbb{Z}\,,$$

represents an orthonormal basis of $L_2(\mathbb{R})$. We shall work with this family. For brevity we put

$$\psi_{j,k}(t) := \begin{cases} 2^{(j-1)/2}\,\psi(2^{(j-1)}t-k) & \text{if} \quad j \in \mathbb{N}, \quad k \in \mathbb{Z}\,, \\ \varphi(t-k) & \text{if} \quad j=0\,, \quad k \in \mathbb{Z}\,. \end{cases} \tag{6}$$

As mentioned above later we shall discuss two families of examples, namely Daubechies wavelets and spline wavelets.

3.2 Function spaces related to wavelet systems

Any function $f \in L_2(\mathbb{R})$ can be represented by a wavelet series

$$f = \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k}\rangle\, \psi_{j,k}$$

(convergence in $L_2(\mathbb{R})$) and

$$\|\, f\,|L_2(\mathbb{R})\| = \Big(\sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} |\langle f, \psi_{j,k}\rangle|^2\Big)^{1/2}\,.$$

All information about f is contained in the coefficients $\langle f, \psi_{j,k}\rangle$. The mapping

$$J:\ f \rightarrow (\langle f, \psi_{j,k}\rangle)_{j,k}$$

is an isomorphism of $L_2(\mathbb{R})$ onto $\ell_2(\mathbb{N}_0 \times \mathbb{Z})$. Now we shall use the mapping J to introduce a two parameter scale of function spaces. This needs a preparation.

Definition 2 *Let $s \ge 0$.*
(i) *Let $0 < p < \infty$. Then b^s_p is the collection of all sequences $a := (a_{j,k})_{j,k}$ of complex numbers such that*

$$\|\, a\,|b^s_p\| := \Big(\sum_{j=0}^{\infty} 2^{j(s+\frac12-\frac1p)p} \sum_{k=-\infty}^{\infty} |a_{j,k}|^p\Big)^{1/p} < \infty\,.$$

(ii) *Let $p=\infty$. Then b^s_∞ is the collection of all sequences $a := (a_{j,k})_{j,k}$ of complex numbers such that*

$$\|\, a\,|b^s_\infty\| := \sup_{j=0,1,\ldots} 2^{j(s+\frac12)} \sup_{k\in\mathbb{Z}} |a_{j,k}| < \infty\,.$$

(iii) *Let $p=\infty$. Then $\mathring{b}^s_\infty$ denotes the closure of the finite sequences with respect to the norm $\|\cdot|b^s_\infty\|$.*

Remark 3 The normalization is chosen in such a way s.t. $b_2^0 = \ell_2$.

Lemma 1 (i) *The classes b_p^s are quasi-Banach spaces.*
(ii) *The classes b_p^s are monotone with respect to s, i.e. $b_p^{s_1} \hookrightarrow b_p^{s_0}$, $s_0 < s_1$, and*

$$\| \, a \, | b_p^{s_0} \| \le \| \, a \, | b_p^{s_1} \| \, , \qquad a \in b_p^{s_1} \, .$$

(iii) *Let $p_0 < p_1$ and*

$$s_0 - \frac{1}{p_0} \ge s_1 - \frac{1}{p_1} \, . \tag{7}$$

Then $b_{p_0}^{s_0} \hookrightarrow b_{p_1}^{s_1}$ and

$$\| \, a \, | b_{p_1}^{s_1} \| \le \| \, a \, | b_{p_0}^{s_0} \| \, .$$

holds for all $a \in b_{p_0}^{s_0}$.

Proof Parts (i) and (ii) are more or less obvious. We omit details. By (ii) it will be sufficient to deal with the case of equality in (7). Then, for fixed $k \in \mathbb{Z}$, we find by using the monotonicity of the ℓ_p-norms

$$\begin{aligned} \Big(\sum_{j=0}^\infty 2^{j(s_1 + \frac{1}{2} - \frac{1}{p_1}) p_1} \, |a_{j,k}|^{p_1} \Big)^{1/p_1} & = \Big(\sum_{j=0}^\infty 2^{j(s_0 + \frac{1}{2} - 1/p_0) p_1} \, |a_{j,k}|^{p_1} \Big)^{1/p_1} \\ & \le \Big(\sum_{j=0}^\infty 2^{j(s_0 + \frac{1}{2} - 1/p_0) p_0} \, |a_{j,k}|^{p_0} \Big)^{1/p_0} . \end{aligned}$$

Hence, using again the monotonicity of the ℓ_p-norms

$$\begin{aligned} \Big(\sum_{k \in \mathbb{Z}} \sum_{j=0}^\infty 2^{j(s_1 + \frac{1}{2} - 1/p_1) p_1} \, |a_{j,k}|^{p_1} \Big)^{1/p_1} & \le \Big(\sum_{k \in \mathbb{Z}} \Big(\sum_{j=0}^\infty 2^{j(s_0 + \frac{1}{2} - 1/p_0) p_0} \, |a_{j,k}|^{p_0} \Big)^{p_1/p_0} \Big)^{1/p_1} \\ & \le \| \, a \, | b_{p_0}^{s_0} \| \, . \end{aligned}$$

The proof is complete. ■

Remark 4 Lemma 1 is a discrete counterpart of the Sobolev embedding.

Let $\mathcal{B} = (e_{j,k})_{j,k}$ denote the canonical orthonormal basis of ℓ_2. Now we consider the properties of the projections

$$P_N a := \sum_{j=0}^N \sum_{k=-\infty}^\infty a_{j,k} \, e_{j,k} \, , \qquad a = (a_{j,k})_{j,k} \, , \quad N \in \mathbb{N}_0 \, .$$

Lemma 2 *Let $s > 0$. We have*

$$\| \, I - P_N \, | b_p^s \to b_p^0 \| \asymp 2^{-sN} \, , \qquad N \in \mathbb{N}_0 \, .$$

Proof Since

$$\| (I - P_N)a \,|b_p^0\| \;=\; \Big(\sum_{j=N+1}^{\infty} \sum_{k=-\infty}^{\infty} 2^{j(\frac{1}{2}-1/p)p} \, |a_{j,k}|^p \Big)^{1/p}$$
$$\leq \; 2^{-sN} \, \| a \,|b_p^s\|$$

the estimate from above follows. For the estimate from below we test P_N on $e_{N+1,k}$ and obtain

$$\| (I - P_N)e_{N+1,k} \,|b_p^0\| = 2^{(N+1)(\frac{1}{2}-\frac{1}{p})} = 2^{-(N+1)s} \, \| e_{N+1,k} \,|b_p^s\|$$

which implies

$$\| I - P_N \,|b_p^s \to b_p^0\| \geq 2^{-s(N+1)} \, .$$

This proves the claim. ■

Definition 3 *Let* $0 < p \leq \infty$ *and* $s \geq 0$. *Let* φ *be an orthogonal scaling function and* ψ *be an associate wavelet. Then we define*

$$B_p^s(\varphi, \psi) := J^{-1} b_p^s$$

and

$$\| f \,|B_p^s(\varphi, \psi)\| := \| (\langle f, \psi_{j,k} \rangle)_{j,k} \,|b_p^s\| \, .$$

Remark 5 Of course, $B_2^0(\varphi, \psi) = L_2(\mathbb{R})$. Furthermore, the results obtained in Lemma 1, carry over to these spaces.

The question is, of course, how are these classes related to known function spaces, or with other words, what type of smoothness do they describe. Let me first recall some well-known concepts in describing smoothness.

3.3 Hölder-Zygmund spaces

Let $m \in \mathbb{N}$. Then $C^m(\mathbb{R})$ denotes the set of all m-times continuously differentiable functions equiped with the norm

$$\| f \,|C^m(\mathbb{R})\| := \sum_{\ell=0}^{m} \sup_{t \in \mathbb{R}} |f^{(\ell)}(t)| \, .$$

Let $0 < s < 1$. Then the Hölder class $C^s(\mathbb{R})$ is the collection of all complex-valued functions f such that

$$\sup_{h \neq 0} \sup_{t \in \mathbb{R}} \frac{|f(t+h) - f(t)|}{|t|} < \infty \, .$$

We equipp this set with the norm

$$\| \, f \,|C^s(\mathbb{R})\| := \sup_{t\in\mathbb{R}} |f(t)| \, + \, \sup_{h\neq 0} \, \sup_{t\in\mathbb{R}} \, \frac{|f(t+h)-f(t)|}{|h|} < \infty \, .$$

We get a complete scale by defining in case $s = m + r$, $0 < r < 1$, $m \in \mathbb{N}$, the space $C^s(\mathbb{R})$ using the norm

$$\| \, f \,|C^s(\mathbb{R})\| := \| \, f \,|C^m(\mathbb{R})\| + \| \, f^{(m)} \,|C^r(\mathbb{R})\| \, .$$

It was known to Riemann but has been made popular by Zygmund that in case $s = 1$ the following modification is useful. Instead the first derivative one uses a second order difference. We define

$$\| \, f \,|\mathcal{C}^1(\mathbb{R})\| := \sup_{t\in\mathbb{R}} |f(t)| \, + \, \sup_{h\neq 0} \, \sup_{t\in\mathbb{R}} \, \frac{|f(t+2h)-2f(t+h)+f(t)|}{|h|} \, .$$

For $m = 2, 3, \ldots$ we put

$$\| \, f \,|\mathcal{C}^m(\mathbb{R})\| := \| \, f \,|C^{m-1}(\mathbb{R})\| \, + \, \| \, f^{(m-1)} \,|\mathcal{C}^1(\mathbb{R})\| \, .$$

The following family of functions can be used to understand this type of smoothness. Let

$$f_\alpha(t) := \varrho(t) \, |t|^\alpha \, , \qquad t \in \mathbb{R} \, , \quad \alpha > 0 \, . \tag{8}$$

Here ϱ denotes a smooth cut-off function supported around the origin and such that $\varrho(0) > 0$. Since we are interested in finite smoothness we exclude the case that α is an even natural number for the moment. Elementary calculations yield

- Let $m \in \mathbb{N}$. Then $f_\alpha \in C^m(\mathbb{R})$ if and only if $\alpha > m$.
- Let $m \in \mathbb{N}$. Then $f_\alpha \in \mathcal{C}^m(\mathbb{R})$ if and only if $\alpha \geq m$.
- Let $s > 0$, $s \not\in \mathbb{N}$. Then $f_\alpha \in C^s(\mathbb{R})$ if and only if $\alpha \geq s$.

Let

$$\mathcal{C}^s(\mathbb{R}) := C^s(\mathbb{R}) \qquad \text{if} \qquad s > 0 \, , \quad s \not\in \mathbb{N} \, .$$

Now we have a second scale. We shall call these spaces Hölder-Zygmund classes. They will be more useful in our context than the first one.

3.4 Besov spaces

Let $M \in \mathbb{N}$. Then

$$\Delta_h^M f(t) := \sum_{j=0}^{M} (-1)^j \binom{M}{j} f(t+(M-j)h)\,, \qquad t,\, h \in \mathbb{R}\,.$$

defines a difference of order M. Then a corresponding L_p-modulus of smoothness is obtained by

$$\omega_M(f,t)_p := \sup_{|h|<t} \| \,\Delta_h^M f(\cdot)\,|L_p(\mathbb{R})\|\,, \quad t>0\,.$$

Definition 4 *Let $0<p,q\le\infty$ and $0<s<M$. Then the Besov space $B^s_{p,q}(\mathbb{R})$ is the set of all L_p-functions f such that*

$$\|\, f\,|B^s_{p,q}(\mathbb{R})\| := \|\, f\,|L_p(\mathbb{R})\| + \Big(\int_0^\infty (t^{-s}\,\omega_M(f,t)_p)^q \frac{dt}{t}\Big)^{1/q}\,.$$

Remark 6 These classes $p,q\ge 1$ have been introduced by Besov in his dissertation around 1960. The classes $B^s_{p,\infty}(\mathbb{R})$ have been considered about ten years earlier by his advisor S.M. Nikol'skij.

Remark 7 It is not difficult to see that

$$B^s_{\infty,\infty}(\mathbb{R}) = \mathcal{C}^s(\mathbb{R})\,, \qquad s>0\,,$$

holds in the sense of equivalent norms.

We return to our family of test functions f_α and extend the definition to negative values of α. Again elementary calculations yield $f_\alpha \in B^s_{p,q}(\mathbb{R})$ if and only if either $s<\alpha+1/p$ or $s=\alpha+1/p$ and $q=\infty$ (again excluding the case of α being an even natural number or zero). By means of the microscopic index q one can see singularities of logarithmic order. Let

$$f_{\alpha,\beta}(t) := \varrho(t)\,|t|^\alpha\,(-\log|t|)^{-\beta}\,, \qquad t\in\mathbb{R}\,, \quad \beta>0\,. \tag{9}$$

Then

$$f_{\alpha,\beta} \in B^{\alpha+1/p}_{p,q}(\mathbb{R}) \qquad \Longleftrightarrow \qquad \beta\, q>1\,,$$

($\alpha \not\in \{0,2,4,\ldots\}$), see [31, 2.3.1].

The Nikol'skij-Besov spaces $B^s_{p,\infty}(\mathbb{R})$ and its periodic counterpart $B^s_{p,\infty}(\mathbb{T})$ play a central role in approximation theory. E.g. in case $1<p<\infty$ one has the following equivalence:

- A 2π-periodic function $f \in L_p(\mathbb{T})$ satisfies

$$\sup_{n\in\mathbb{N}} n^s \, \| \, f - S_n f \, | L_p(\mathbb{T}) \| < \infty \, ,$$

  where $S_n f$ is the partial sum of order n of the Fourier series of f.

- $f \in B^s_{p,\infty}(\mathbb{T})$.

3.5 Sobolev spaces

Let $1 \le p \le \infty$ and $m \in \mathbb{N}$. Then the Sobolev space $W^m_p(\mathbb{R})$ consists of all functions $f \in L_p(\mathbb{R})$ such that all weak derivatives of orders $0 < \ell \le m$ exist and belong to $L_p(\mathbb{R})$. The norm is defined to be

$$\| \, f \, | W^m_p(\mathbb{R}) \| := \sum_{\ell=0}^{m} \| \, f^{(\ell)} \, | L_p(\mathbb{R}) \| \, .$$

For $p < \infty$ elementary calculations yield $f_\alpha \in W^m_p(\mathbb{R})$ if and only if $s < \alpha + 1/p$ as well as $f_{\alpha,\beta} \in W^m_p(\mathbb{R})$ if and only if either $s < \alpha + 1/p$ or $s = \alpha + 1/p$ and $\beta \, p > 1$ ($\alpha \notin \{0, 2, 4, \ldots\}$), see [31, 2.3.1].

Lemma 3 *We have $W^m_p(\mathbb{R}) \neq B^s_{p,q}(\mathbb{R})$ except the case $W^m_2(\mathbb{R}) = B^m_{2,2}(\mathbb{R})$ (in the sense of equivalent norms).*

Proof We refer to [40, 2.3.9]. ■

3.6 Wavelets and Besov spaces

Next we quote a deep result which connects the spaces $B^s_{p,p}(\mathbb{R})$ with $B^s_{p,p}(\varphi, \psi)$.

Theorem 2 *([6, Thm. 3.7.7]) Let φ be an orthogonal scaling function and let ψ be an associated wavelet. Further we assume that $\varphi \in B^s_{p,p}(\mathbb{R})$ and that there exist a natural number n and constant c s.t.*

$$|\varphi(t)| \le \frac{c}{(1 + |t|)^{n+\varepsilon}} \, , \qquad t \in \mathbb{R} \, .$$

Finally, we assume that

$$\int_{-\infty}^{\infty} t^\ell \, \varphi(t) \, dt = 0 \, , \qquad \ell = 1, \ldots , n-1.$$

Then, for $0 < p \le \infty$ *and*

$$\max\Big(0, \frac{1}{p} - 1\Big) < t < \min(s, n)$$

we have

$$\| \, f \, | B^s_{p,p}(\mathbb{R}) \| \asymp \| \, (\langle f, \psi_{j,k} \rangle)_{j,k} \, | b^s_p \|$$

for all $f \in B^s_{p,p}(\mathbb{R})$.

Remark 8 The characterization of Besov spaces by wavelets has a certain history, in particular with respect to the Haar basis. We refer to Ropela [30] and Triebel [39]. Earlier versions of Theorem 2 may be found in the books of Meyer [21] and Wojtasczczyk [48].

3.6.1 Daubechies wavelets

Around 1988 Ingrid Daubechies proved the existence of compactly supported wavelets. At this time it was a certain surprise. I do not want to describe the construction itself, it is nontrivial. For detailed descriptions we refer to the monographs of Daubechies [7] and Wojtasczczyk [48].

Proposition 1 *(Thm. 4.7 in [48]) There exists a constant* C *such that for each* $N \in \mathbb{N}$ *there exists an orthogonal scaling function* φ *and an associated wavelet* ψ *s.t.*

(i) φ *and* ψ *are* C^N *functions,*

(ii) φ *and* ψ *are compactly supported and both* $\operatorname{supp} \varphi$ *and* $\operatorname{supp} \psi$ *are contained in* $[-CN, CN]$.

Remark 9 It follows from Prop. 3.1 in [48] that ψ satisfies a moment condition

$$\int_{-\infty}^{\infty} t^\ell \, \psi(t) \, dt = 0 \qquad \text{if} \quad \ell = 0, 1, \ldots, N \, .$$

Another variant of Theorem 2 can be found in [41].

Theorem 3 *Let* φ *be an orthogonal scaling function and let* ψ *be an associated wavelet. Further we assume that* φ *and* ψ *have compact support and belong to* $C^N(\mathbb{R})$ *for some* $N \in \mathbb{N}$*. Then, for* $0 < p \le \infty$ *and*

$$\max\Big(0, \frac{1}{p} - 1\Big) < t < N$$

we have

$$\| \, f \, | B^s_{p,p}(\mathbb{R}) \| \asymp \| \, (\langle f, \psi_{j,k} \rangle)_{j,k} \, | b^s_p \|$$

for all $f \in B^s_{p,p}(\mathbb{R})$.

3.6.2 Spline wavelets

Let $m \in \mathbb{N}$. Let $\mathcal{X}$ be the characteristic function of the interval $(0,1)$. Then the normalized cardinal B-spline of order $m+1$ is given by

$$\mathcal{N}_{m+1}(x) := \mathcal{N}_m * \mathcal{X}(x)\,, \qquad x \in \mathbb{R}\,, \quad m \in \mathbb{N}\,,$$

beginning with $\mathcal{N}_1 = \mathcal{X}$. Let $\mathcal{F}$ denote the Fourier transform and $\mathcal{F}^{-1}$ its inverse transform. We normalize these transformations by

$$\mathcal{F}f(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, f(x)\, dx\,, \qquad \xi \in \mathbb{R}\,, \quad f \in L_1(\mathbb{R})\,.$$

By

$$\varphi_m(x) := \frac{1}{\sqrt{2\pi}}\, \mathcal{F}^{-1}\Big[\frac{\mathcal{F}\mathcal{N}_m(\xi)}{\Big(\sum\limits_{k=-\infty}^{\infty} |\mathcal{F}\mathcal{N}_m(\xi + 2\pi k)|^2\Big)^{1/2}} \Big](x)\,, \qquad x \in \mathbb{R}\,,$$

we obtain an orthonormal scaling function which is again a spline of order m. Finally, by

$$\psi_m(x) := \sum_{k=-\infty}^{\infty} \langle\, \varphi_m(t/2), \varphi_m(t-k)\rangle\, (-1)^k\, \varphi_m(2x+k+1)$$

we obtain the generator of an orthonormal wavelet system. For $m = 1$ it is easily checked that $-\psi_1(x-1)$ is the Haar wavelet. In general these functions ψ_m have the following properties:

(i) ψ_m restricted to intervals $[k, k+1]$, $k \in \mathbb{Z}$, is a polynomial of degree at most $m-1$.

(ii) $\psi_m \in C^{m-2}(\mathbb{R})$ if $m \geq 2$.

(iii) $\psi_m^{(m-2)}$ is uniformly Lipschitz continuous on $\mathbb{R}$ if $m \geq 2$.

(iv) There exist positive numbers τ_m and sequences $(c_k)_k$ and $(d_k)_k$ such that

$$\psi_m(x) = \sum_{k=-\infty}^{\infty} c_k\, \mathcal{N}_m(2x-k)\,, \qquad \mathcal{N}_m(x) = \sum_{k=-\infty}^{\infty} d_k\, \varphi_m(x-k)\,, \qquad x \in \mathbb{R}\,,$$

and

$$\sup_{k \in \mathbb{Z}} (|c_k| + |d_k|)\, e^{\tau_m |k|} < \infty \qquad \text{and} \qquad \max_{0 \leq \ell \leq m-2}\, \sup_{x \in \mathbb{R}} |\psi_m^{(\ell)}(x)|\, e^{\tau_m |x|} < \infty\,. \quad (10)$$

(v) The functions ψ_m satisfy a moment condition of order $m-1$, i.e.

$$\int_{-\infty}^{\infty} x^\ell\, \psi_m(x)\, dx = 0\,, \qquad \ell = 0, 1, \ldots, m-1\,.$$

It will be convenient for us to use the following abbreviations:

$$\psi^m_{0,k}(x) := \varphi_m(x-k) \qquad \text{and} \qquad \psi^m_{j+1,k}(x) := 2^{j/2}\,\psi_m(2^j x - k)\,, \qquad x \in \mathbb{R}\,, \quad k \in \mathbb{Z}$$

and $j \in \mathbb{N}_0$.
Let us now concentrate on the mapping

$$R_m : \quad f \mapsto (\langle f, \psi^m_{j,k}\rangle)_{j,k}\,.$$

Proposition 2 *Let $m \in \mathbb{N}$.*
(i) *Let either $1 \le p < \infty$ and $0 < s < m - 1 + 1/p$ or $0 < p < 1$ and $\frac{1}{p} - 1 < r < m$. Then the mapping R_m generates an isomorphism of $B^s_{p,p}(\mathbb{R})$ onto b^s_p.*
(ii) *Let $0 < r < m-1$. The mapping R_m generates an isomorphism of $\mathring{B}^s_{\infty,\infty}(\mathbb{R})$ onto $\mathring{b}^s_\infty$.*

Remark 10 (a) The content of this proposition is well-known, see Bourdaud [4], Frazier and Jawerth [14], Meyer [21], Lemarie and Kahane [18] or DeVore [10].
(b) Wavelet characterizations of Besov spaces with $p < 1$ are investigated e.g. in Bourdaud [4], Cohen [6], Kyriazis and Petrushev [17] and Triebel [41]. Cohen and Triebel concentrate on biorthogonal (orthogonal) wavelets with compact support.

Corollary 1 *Let $0 < p_0 < p_1 < \infty$, $s_1 > \max(0, \frac{1}{p_1} - 1)$ and*

$$s_0 - \frac{1}{p_0} \ge s_1 - \frac{1}{p_1}\,.$$

Then $B^{s_0}_{p_0,p_0}(\mathbb{R}) \hookrightarrow B^{s_1}_{p_1,p_1}(\mathbb{R})$ follows.

Proof Combine Lemma 1 with Theorem 2. ■

4 The Smolyak algorithm

4.1 A first example

Let $(a_j)_{j=1}^\infty$ be a convergent series of complex numbers. We define $a_0 := 0$. Then

$$\sum_{j=0}^{M} (a_{j+1} - a_j) = a_{M+1} \xrightarrow[M\to\infty]{} a$$

where a is the limit of the sequence. Hence

$$a = \sum_{j=0}^{\infty} (a_{j+1} - a_j)\,.$$

Now we take a second sequence $(b_j)_{j=1}^{\infty}$ s.t. $\lim_{j\to\infty} b_j = b$. Again we put $b_0 := 0$. Then, assuming

$$\sum_{j=0}^{\infty} |a_{j+1} - a_j| < \infty \qquad \text{as well as} \qquad \sum_{j=0}^{\infty} |b_{j+1} - b_j| < \infty \,,$$

we conclude

$$a \cdot b = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} (a_{j_1+1} - a_{j_1})\,(b_{j_2+1} - b_{j_2})\,.$$

We would like to approximate this product by using information from the single sequences. Let $r > 0$ be given. Then we suppose that for all $j \in \mathbb{N}_0$ we have

$$\max(|a - a_j|, |b - b_j|) \le 2^{-jr}\,. \tag{11}$$

Now we consider two different types of partial sums: for $M \in \mathbb{N}_0$ let

$$\begin{aligned} S_M &:= \sum_{j_1+j_2\le M} (a_{j_1+1} - a_{j_1})\,(b_{j_2+1} - b_{j_2})\,, \\ \widetilde{S}_M &:= \sum_{j_1=0}^{M} \sum_{j_2=0}^{M} (a_{j_1+1} - a_{j_1})\,(b_{j_2+1} - b_{j_2})\,. \end{aligned}$$

Observe that S_M uses

$$\sum_{j_1=0}^{M} \sum_{j_2=0}^{M-j_1} 1 = \sum_{j_1=0}^{M} (M - j_1 + 1) = \sum_{\ell=1}^{M+1} \ell = \frac{(M+1)(M+2)}{2}$$

summands, whereas $\widetilde{S}_M$ uses $(M+1)^2$ of them. Using (11) we find

$$\begin{aligned} |a \cdot b - S_M| &\le \sum_{j_1+j_2>M} |a_{j_1+1} - a_{j_1}|\,|b_{j_2+1} - b_{j_2}| \\ &\le 4 \sum_{j_1=M+1}^{\infty} \sum_{j_2=0}^{\infty} 2^{-r(j_1+j_2)} + \sum_{j_1=0}^{M} \sum_{j_2=M-j_1+1}^{\infty} 2^{-r(j_1+j_2)} \\ &= 4 \left(\frac{2^r}{2^r-1}\right)^2 2^{-r(M+1)} + 4\,\frac{2^r}{2^r-1} \sum_{j_1=0}^{M} 2^{-rj_1}\, 2^{-r(M-j_1+1)} \\ &= 4\,\frac{2^r}{2^r-1}\, 2^{-r(M+1)} \left(\frac{2^r}{2^r-1} + \sum_{j_1=0}^{M} 1\right) \\ &= \frac{2^{r+2}}{2^r-1}\, 2^{-r(M+1)} \left(\frac{2^r}{2^r-1} + M + 1\right) \\ &\asymp M\, 2^{-rM}\,. \end{aligned} \tag{12}$$

On the other hand

$$
\begin{aligned}
|a \cdot b - \widetilde{S}_M| &\leq 4 \sum_{j_1=M+1}^{\infty} \sum_{j_2=0}^{\infty} 2^{-r(j_1+j_2)} + 4 \sum_{j_1=0}^{M} \sum_{j_2=M+1}^{\infty} 2^{-r(j_1+j_2)} \\
&\leq 8 \left(\frac{2^r}{2^r - 1}\right)^2 2^{-r(M+1)} \\
&\asymp 2^{-rM} . \qquad (13)
\end{aligned}
$$

Now we compare the effectiveness of these approximations. Therefore, let K denote the number of summands (information) which are used. Then, because of

$$
\begin{aligned}
M\, 2^{-rM} &\leq \sqrt{(M+1)(M+2)}\, 2^{-r(M+2)}\, 2^{2r} \leq 2^{2r} \sqrt{2\,K}\, 2^{-r\sqrt{2K}} \\
&\asymp \sqrt{K}\, 2^{-r\sqrt{2K}} ,
\end{aligned}
$$

we find

$$
|a \cdot b - S_M| \leq c \sqrt{K}\, 2^{-r\sqrt{2K}} , \qquad (14)
$$

see (12), where c is independent of K. Furthermore, (13) and $K = (M+1)^2$ lead to

$$
|a \cdot b - \widetilde{S}_M| \leq c\, 2^{-r(M+1)} \leq c\, 2^{-r\sqrt{K}} \qquad (15)
$$

where again c does not depend on K. Of course, for K sufficiently large the approximation in (14) is more effective than that one described in (15).

Taking S_M instead of $\widetilde{S}_M$ is the main idea of the construction of Smolyak !

Above we have treated the case $d = 2$. Now we turn to the general case. We consider d convergent sequences $(a^i_j)_{j=1}^{\infty}$ with limits a^i.

Lemma 4 *Let $d \geq 2$ and $r > 0$.*
(i) *Suppose, that there exist constants c_i s.t.*

$$
\max\{|a^i - a^i_j| : \quad i = 1, \ldots, d\} \leq c_i\, 2^{-jr} , \qquad j \in \mathbb{N}_0 , \quad i = 1, \ldots, d . \qquad (16)
$$

Then, with

$$
S_M := \sum_{j_1+\ldots+j_d \leq M} (a^1_{j_1+1} - a^1_{j_1}) \ldots (a^d_{j_d+1} - a^d_{j_d}) , \qquad M \in \mathbb{N}_0 ,
$$

there exists a constant $C = C(d,r)$ s.t.

$$
|a^1 \cdot \ldots a^d - S_M| \leq C \left(\prod_{i=1}^{d} c_i\right) M^{d-1}\, 2^{-rM} \qquad (17)
$$

holds for all $M \in \mathbb{N}$. Here C is also independent of the sequences $(a^i_j)_{j=1}^\infty$, $i=1,\ldots,d$.
(ii) *Let $s>0$. Suppose, that there exist constants c_i s.t.*

$$\max\{|a^i - a^i_j| : \quad i=1,\ldots,d\} \le c\,(1+j)^s\,2^{-jr}\,, \qquad j\in\mathbb{N}_0\,, \quad i=1,\ldots,d\,. \tag{18}$$

Then there exists a constant $C=C(d,r,s)$ s.t.

$$|a^1\,\cdot\,\ldots\,a^d - S_M| \le C\,\Big(\prod_{i=1}^d c_i\Big)\,M^{ds+d-1}\,2^{-rM} \tag{19}$$

holds for all $M\in\mathbb{N}$. As above, C is also independent of the sequences $(a^i_j)_{j=1}^\infty$, $i=1,\ldots,d$.

Proof Observe

$$\begin{aligned}
\sum_{j_1+\ldots+j_d>M} \ldots &= \sum_{j_1=M+1}^\infty \sum_{j_2=0}^\infty \ldots \sum_{j_d=0}^\infty \\
&+ \sum_{j_1=0}^M \sum_{j_2=M-j_1+1}^\infty \sum_{j_3=0}^\infty \ldots \sum_{j_d=0}^\infty \\
&+ \ldots \\
&+ \sum_{j_1=0}^M \sum_{j_2=0}^{M-j_1} \ldots \sum_{j_i=0}^{M-(j_1+\ldots+j_{i-1})} \quad \sum_{j_{i+1}=M-(j_1+\ldots+j_i)+1}^\infty \quad \sum_{j_{i+2}=0}^\infty \ldots \sum_{j_d=0}^\infty \\
&+ \ldots \\
&+ \sum_{j_1=0}^M \sum_{j_2=0}^{M-j_1} \ldots \sum_{j_{d-1}=0}^{M-(j_1+\ldots+j_{d-2})} \quad \sum_{j_d=M-(j_1+\ldots+j_{d-1})+1}^\infty \,.
\end{aligned}$$

We introduce the abbreviations

$$\begin{aligned}
\sum\nolimits^1 &:= \sum_{j_1=M+1}^\infty \sum_{j_2=0}^\infty \ldots \sum_{j_d=0}^\infty (1+j_1)^s\,2^{-rj_1}\ldots(1+j_d)^s\,2^{-rj_d} \\
\sum\nolimits^{i+1} &:= \sum_{j_1=0}^M \sum_{j_2=0}^{M-j_1} \ldots \sum_{j_i=0}^{M-(j_1+\ldots+j_{i-1})} \quad \sum_{j_{i+1}=M-(j_1+\ldots+j_i)+1}^\infty \\
&\quad \times \sum_{j_{i+2}=0}^\infty \ldots \sum_{j_d=0}^\infty (1+j_1)^s\,2^{-rj_1}\ldots(1+j_d)^s\,2^{-rj_d}\,, \qquad i=1,\ldots,d-2\,, \\
\sum\nolimits^d &= \sum_{j_1=0}^M \sum_{j_2=0}^{M-j_1} \ldots \sum_{j_{d-1}=0}^{M-(j_1+\ldots+j_{d-2})} \quad \sum_{j_d=M-(j_1+\ldots+j_{d-1})+1}^\infty (1+j_1)^s\,2^{-rj_1}\ldots(1+j_d)^s\,2^{-rj_d}
\end{aligned}$$

and

$$A := \sum_{j=0}^{\infty} (1+j)^s \, 2^{-rj} < \infty \,.$$

We have

$$\sum_{j=M+1}^{\infty} (1+j)^s \, 2^{-rj} \le B \, (M+1)^s \, 2^{-r(M+1)}$$

for some positive B independent of M. This yields

$$\sum\nolimits^1 \le A^{d-1} \, B \, (M+1)^s \, 2^{-r(M+1)} \,. \tag{20}$$

Furthermore, we find

$$\begin{aligned} \sum_{j_{i+1}=M-(j_1+\ldots+j_i)+1}^{\infty} \quad \sum_{j_{i+2}=0}^{\infty} \ldots \sum_{j_d=0}^{\infty} & (1+j_{i+1})^s \, 2^{-rj_{i+1}} \, \ldots \, (1+j_d)^s \, 2^{-rj_d} \\ \le \quad & A^{d-(i+1)} \, B \, (M+1-(j_1+\ldots+j_i))^s \, 2^{-r(M+1-(j_1+\ldots+j_i))} \end{aligned}$$

Hence

$$\begin{aligned} \sum\nolimits^{i+1} \quad \le \quad & A^{d-(i+1)} \, B \, 2^{-r(M+1)} \sum_{j_1=0}^{M} (1+j_1)^s \sum_{j_2=0}^{M-j_1} (1+j_2)^s \, \ldots \\ \times & \sum_{j_i=0}^{M-(j_1+\ldots+j_{i-1})} (1+j_i)^s \, (M+1-(j_1+\ldots+j_i))^s \,. \end{aligned}$$

To estimate the last sum on the right-hand side observe

$$\sum_{j=0}^{T} (1+j)^s \, (T+1-j)^s \le D_{1,0} \, (1+T)^{2s+1}$$

with some constant $D_{1,0}$ independent of $T \in \mathbb{N}_0$. This implies

$$\begin{aligned} \sum\nolimits^{i+1} \quad \le \quad & A^{d-(i+1)} \, B \, D \, 2^{-r(M+1)} \sum_{j_1=0}^{M} (1+j_1)^s \sum_{j_2=0}^{M-j_1} (1+j_2)^s \, \ldots \\ & \sum_{j_{i-1}=0}^{M-(j_1+\ldots+j_{i-2})} (1+j_{i-1})^s \, (M+1-(j_1+\ldots+j_{i-1}))^{2s+1} \,. \end{aligned}$$

Let $k \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$. Since there exist constants $D_{k,\ell}$ s.t.

$$\sum_{j=0}^{T} (1+j)^s \, (T+1-j)^{ks+\ell} \le D_{k,\ell} \, (1+T)^{(k+1)s+\ell+1}$$

for all $T \in \mathbb{N}_0$ we finally get

$$\sum\nolimits^{i+1} \le A^{d-(i+1)} \, B \, D \, 2^{-r(M+1)} \, (M+1)^{(i+1)s+i} \,. \tag{21}$$

with an appropriate constant D. Also the last summand can be estimated in this way. We obtain

$$\sum^{d} \le B\, D\, 2^{-r(M+1)}\, (M+1)^{ds+d-1} \,. \tag{22}$$

Summarizing (20)-(22) we obtain the desired inequality (19). ■

Now we will deal with a first application of these estimates. For this reason we consider the approximation of periodic Lipschitz funcions by the partial sums of their Fourier series. As usual, for $f \in L_1(\mathbb{T})$ let

$$c_k(f) = (2\pi)^{-1} \int_{\mathbb{T}} f(t)\, e^{-ikt}\, dt \,, \qquad k \in \mathbb{Z} \,,$$

denote the Fourier coefficient of $f \in L_1(\mathbb{T})$. The n-th order partial sum is given by

$$S_n f(t) := \sum_{k=-n}^{n} c_k(f)\, e^{ikt} \,, \qquad n \in \mathbb{N}_0 \,, \tag{23}$$

We recall the following:

Lemma 5 *Let $r > 0$. Then there exists a constant c such that*

$$\sup_{-\pi < t < \pi} |f(t) - S_n f(t)| \le c\, (n+1)^{-r}\, \log(n+1)\ \| \, f\, | C^r(\mathbb{T}) \|$$

holds for all $n \in \mathbb{N}$ and all $f \in C^r(\mathbb{T})$.

We switch to a dyadic subsequence and obtain

$$\sup_{-\pi < t < \pi} |f(t) - S_{2^j} f(t)| \le c\, 2^{-rj}\, (j+1) \| \, f\, | C^r(\mathbb{T}) \| \,.$$

Obviously, $\lim_{n\to\infty} S_n f(t) = f(t)$ and hence

$$f(t) = S_1 f(t) + \sum_{j=0}^{\infty} (S_{2^{j+1}} f(t) - S_{2^j} f(t)) \,.$$

For fixed t we put $a_0^i := 0$ and

$$a_{j+1}^i := S_{2^j} f_i(t)) \,, \qquad j \in \mathbb{N}_0 \,, \quad i = 1, \ldots\, d \,.$$

For brevity we put

$$\Delta_j f_i(t) := \begin{cases} S_{2^j} f_i(t) - S_{2^{j-1}} f_i(t) & \text{if} \quad j \ge 1 \,, \\ S_1 f_i(t) & \text{if} \quad j = 0 \,. \end{cases}$$

Now, Lemma 4 yields the error estimate

$$\Big| f_1(t_1) \cdot \ldots \cdot f_d(t_d) - \sum_{j_1+\ldots+j_d \le M} \Delta_{j_1} f_1(t_1) \cdot \ldots \cdot \Delta_{j_d} f_d(t_d) \Big|$$

$$\le \; c\, 2^{-rM}\, (M+1)^{dr+d-1} \prod_{i=1}^{d} \| \, f_i \, | C^r(\mathbb{T}) \|$$

independent of $t_1, \ldots, t_d$. Consequently

$$\Big\| f_1 \cdot \ldots \cdot f_d - \sum_{j_1+\ldots+j_d \le M} \Delta_{j_1} f_1 \cdot \ldots \cdot \Delta_{j_d} f_d \Big| C(\mathbb{T}^d) \Big\|$$

$$\le \; c\, 2^{-rM}\, (M+1)^{dr+d-1} \prod_{i=1}^{d} \| \, f_i \, | C^r(\mathbb{T}) \| \, .$$

There are several questions around this estimate.

- Can we formulate such an estimate for a slightly more general class of functions, preferably a Banach space ?
- What about effectiveness of this approximation ?
- What about error estimates for different norms, e.g. for $\| \cdot | L_2 \|$?

Let me comment a bit on the second question. How much information do we use of the functions $f_1, \ldots, f_d$? The information is given by the Fourier coefficients of these functions. Define for $M \in \mathbb{N}$ the dyadic hyperbolic cross of order M by

$$H(M,d) := \Big\{ \ell \in \mathbb{Z}^d : \quad \exists\, u_1, \ldots, u_d \in \mathbb{N}_0 \, , \;\; \text{s.t.} \;\; |\ell_k| \le 2^{u_k} \;\text{and}\; \sum_{k=1}^{d} u_k = M \Big\} \, . \qquad (24)$$

Let

$$f(x) := \prod_{j=1}^{d} f_j(x_j) \, , \qquad x = (x_1, \ldots x_d) \, .$$

Then

$$c_k(f) := (2\pi)^{-d} \int_{[-\pi,\pi]^d} f(x)\, e^{-ikx}\, dx = \prod_{j=1}^{d} c_{k_j}(f_j) \, , \qquad k = (k_1, \ldots, k_d) \, .$$

We claim that

$$\sum_{j_1+\ldots+j_d \le M} \Delta_{j_1} f_1(x_1) \cdot \ldots \cdot \Delta_{j_d} f_d(x_d) = \sum_{k \in H(M,d)} c_k(f)\, e^{ikx} \, . \qquad (25)$$

Here it will be more convenient to switch to the language of operators. So, we interpret

$$Q_M := \sum_{j_1+\ldots+j_d \le M} \Delta_{j_1} \cdot \ldots \cdot \Delta_{j_d} \tag{26}$$

as a linear operator defined on the trigonometric polynomials

$$f(x) = e^{ikx} = e^{ik_1x_1} \cdot \ldots \cdot e^{ik_dx_d}\,.$$

We need a special covering of $\mathbb{R}^d$. Let

$$\begin{aligned} P_0 &:= [-1,1]\,, \qquad P_j := \{x \in \mathbb{R} \;:\; 2^{j-1} < |x| \le 2^j\}\,, \quad j \in \mathbb{N}\,, \\ \mathcal{P}_{j_1,\ldots,j_d} &:= P_{j_1} \times \ldots \times P_{j_d}\,, \qquad j \in \mathbb{N}_0^d\,. \end{aligned} \tag{27}$$

Then

$$\mathbb{R}^d = \bigcup_{j \in \mathbb{N}_0^d} \mathcal{P}_j \qquad \text{and} \qquad \mathcal{P}_{j_1,\ldots,j_d} \cap \mathcal{P}_{\ell_1,\ldots,\ell_d} = \emptyset \quad \text{if} \quad j \neq \ell\,. \tag{28}$$

Moreover

$$\Delta_{j_1} e^{ik_1x_1} \cdot \ldots \cdot \Delta_{j_d} e^{ik_dx_d} = \begin{cases} e^{ikx} & \text{if} \quad k \in \mathcal{P}_{j_1,\ldots,j_d}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Because of

$$H(M,d) = \bigcup_{j_1+\ldots+j_d \le M} \mathcal{P}_{j_1,\ldots,j_d} \tag{29}$$

this implies

$$(Q_M e^{ik\cdot})(x) = \begin{cases} e^{ikx} & \text{if} \quad k \in H(M,d)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

This proves the claim (25).

Lemma 6 *We have*

$$|H(M,d)| \asymp M^{d-1}\, 2^M\,.$$

Proof From (28) and (29) we conclude

$$|H(M,d)| = \sum_{j_1+\ldots+j_d \le M} |\mathcal{P}_{j_1,\ldots,j_d}|\,,$$

where

$$|\mathcal{P}_{j_1,\ldots,j_d}| := \Big|\{\ell \in \mathbb{Z}^d : \quad \ell \in \mathcal{P}_{j_1,\ldots,j_d}\}\Big|\,.$$

Since $|P_j| = 2^j$, $j \geq 1$, and $|P_0| = 3$ the following rough estimate

$$\begin{aligned} |H(M,d)| &\leq \sum_{j_1+\ldots+j_d \leq M} 6^d \, 2^{j_1} \ldots 2^{j_d} \\ &= 6^d \sum_{m=0}^{M} \sum_{j_1+\ldots+j_d=m} 2^m \\ &\leq c_d \sum_{m=0}^{M} 2^m \, m^{d-1} \asymp M^{d-1} \, 2^M \end{aligned}$$

follows. ■

The operator Q_M is the model case for the Smolyak algorithm.

4.2 The definition of the Smolyak algorithm

Given two functions f, g their tensor product is defined to be

$$(f \otimes g)(x,y) := f(x) \cdot g(y) \, .$$

Now, let P and Q be linear operators defined on some function spaces X_1 and X_2 with values in Y_1 and Y_2, respectively. For simplicity let us assume that also Y_1 and Y_2 are function spaces. Then the tensor product $P \otimes Q$ of these two operators is defined to be

$$(P \otimes Q)\,(f \otimes g) := Pf \otimes Qg \, , \qquad f \in X_1 \, , \quad g \in X_2 \, .$$

This definition is extended to linear combinations by requiring linearity of $P \otimes Q$, i.e.,

$$(P \otimes Q) \Big(\sum_{j=0}^{n} f_j \otimes g_j \Big) := \sum_{j=0}^{n} (P f_j) \otimes (Q g_j) \, , \qquad f_j \in X_1 \, , \quad g_j \in X_2 \, , j = 1, \ldots , n \, .$$

Let $L_j : X \to Y$ be a sequence of continuous linear operators, denoted by L. Then we put

$$\Delta_j(L) := \begin{cases} L_j - L_{j-1} & \text{if} \quad j \in \mathbb{N} \, , \\ L_0 & \text{if} \quad j = 0 \, . \end{cases}$$

Definition 5 *Let $m \in \mathbb{N}_0$. The Smolyak-Algorithm $A(m,d,\vec{L})$ relative to the d sequences $L^1 := (L_j^1)_{j=0}^\infty, \ldots , L^d := (L_j^d)_{j=0}^\infty$, is the linear operator*

$$A(m,d,\vec{L}) := \sum_{j_1+\ldots+j_d \leq m} \Delta_{j_1}(L^1) \otimes \ldots \otimes \Delta_{j_d}(L^d) \, .$$

Remark 11 *Originally introduced in [36] there are now hundreds of references dealing with this construction. A few basics and some references can be found in [25] and [47].*

4.3 Two examples

To begin with we deal with examples originating from the classical periodic situation.

4.3.1 Approximation with respect to hyperbolic crosses

We chooes $L_j^i := S_{2^j}$ for all $i = 1, 2, \ldots, d$. Then

$$A(m,d,S)f(x) = \sum_{k \in H(m,d)} c_k(f)\, e^{ikx}\,. \tag{30}$$

Formally this operator is defined on trigonometric polynomials only, but the domain of definition can be extended immediatly to $L_1(\mathbb{T}^d)$.

4.3.2 Approximation on sparse grids

We need a preparation. For $\bar{j} = (j_1, \ldots j_d) \in \mathbb{N}_0^d$ we put $|\bar{j}|_1 = j_1 + \ldots + j_d$.

Lemma 7 *(see [47]) We have the identity*

$$A(m,d,\vec{L}) = \sum_{m-d+1 \le |\bar{j}|_1 \le m} (-1)^{m-|\bar{j}|_1} \binom{d-1}{m-|\bar{j}|_1} L_{j_1}^1 \otimes \ldots \otimes L_{j_d}^d\,. \tag{31}$$

Proof Use mathematical induction with respect to m. ■

Now we turn to the classical trigonometric interpolation. Let

$$\mathcal{D}_m(t) := \sum_{|k| \le m} e^{ikt}\,, \qquad t \in \mathbb{T}, \quad m \in \mathbb{N}_0\,,$$

be the Dirichlet kernel of order m and let

$$I_m f(t) := \frac{1}{2m+1} \sum_{\ell=0}^{2m} f(t_\ell)\, \mathcal{D}_m(t - t_\ell)\,, \qquad t_\ell = \frac{2\pi\ell}{2m+1}\,. \tag{32}$$

Then I_m is the unique trigonometric polynomial of degree less than or equal to m which interpolates f at the nodes t_ℓ. Let

$$J_n := \Big\{ \frac{2\pi\ell}{n} : \quad -\frac{n}{2} \le \ell < \frac{n}{2} \Big\}\,.$$

Obviously, the cardinality $|J_n|$ of J_n is equal to n. Define

$$\mathcal{G}(m,d) := \bigcup_{m-d+1 \le |\bar{j}|_1 \le m} J_{2^{j_1+1}+1} \times \ldots \times J_{2^{j_d+1}+1}$$

Let $I = (I_{2^j})_j$. We choose

$$L_j^1 = \ldots = L_j^d = I_{2^j}\,, \qquad j \in \mathbb{N}_0,$$

and define

$$A(m,d,D)f(x) := \sum_{j_1+\ldots+j_d\le m} \Delta_{j_1}(I)\otimes \ldots \otimes \Delta_{j_d}(I)\,. \qquad (33)$$

Then the Smolyak algorithm $A(m,d,D)$ uses only samples from the grid $\mathcal{G}(m,d)$, see Lemma 7.

Lemma 8 *(see [34]) We have*

$$|\mathcal{G}(m,d)| \asymp m^{d-1}\, 2^m\,.$$

Remark 12 Let us compare the cardinality of $\mathcal{G}(m,d)$ with the cardinality of the full tensor product grid $J_{2^{m+1}+1}\times \ldots \times J_{2^{m+1}+1}$ used by the operator $I_{2^m}\otimes \ldots \otimes I_{2^m}$. Of course

$$|J_{2^{m+1}+1}\times \ldots \times J_{2^{m+1}+1}| \asymp 2^{md}\,.$$

This explaines why $\mathcal{G}(m,d)$ is called sparse.

Summary: We hope for an increase of efficiency by using the Smolyak algorithm. However, we can not hope for this in general. Our example above works for tensor products of functions. For this reason it seems to be natural to deal with function spaces obtained as the closure of tensor products of functions.

5 Tensor products of function and sequence spaces

5.1 Some abstract definitions

We follow [19]. Let X and Y be Banach spaces of functions or sequences, respectively. Then the set

$$X\otimes Y := \Big\{ h : \exists n\in \mathbb{N}, f_i \in X\,,\ g_i\in Y \text{ s.t. } h = \sum_{i=1}^n f_i\otimes g_i \Big\}$$

is called algebraic tensor product of X and Y. Next we recall three well-known constructions of tensor norms, namely the injective, the projective and the p-nuclear norm.

Definition 6 *Let X and Y be Banach spaces of functions or sequences, respectively.*
(i) *Let $h\in X\otimes Y$ be given by*

$$h = \sum_{j=1}^n f_j\otimes g_j\,, \qquad f_j\in X\,, \quad g_j \in Y\,.$$

Then the injective tensor norm $\lambda(\cdot, X, Y)$ *is defined as*

$$\lambda(h, X, Y) = \sup \Big\{ \Big\| \sum_{j=1}^{n} \psi(f_j) \cdot g_j \Big| Y \Big\| : \quad \psi \in X', \, \|\psi|X'\| \leq 1 \Big\} .$$

(ii) *The projective tensor norm* $\gamma(\cdot, X, Y)$ *is defined by*

$$\gamma(h, X, Y) = \inf \Big\{ \sum_{j=1}^{n} \|f_j|X\| \, \|g_j|Y\| : \quad f_j \in X, \, g_j \in Y, \, h = \sum_{j=1}^{n} f_j \otimes g_j \Big\} .$$

(iii) *Let* $1 \leq p \leq \infty$ *and let* $1/p + 1/p' = 1$. *Then the p-nuclear tensor norm* $\alpha_p(\cdot, X, Y)$ *is given by*

$$\alpha_p(h, X, Y) :=$$
$$\inf \left\{ \Big(\sum_{i=1}^{n} \|f_i|X\|^p \Big)^{1/p} \cdot \sup \Big\{ \Big(\sum_{i=1}^{n} |\psi(g_i)\Big|^{p'} \Big)^{1/p'} : \psi \in Y', \|\psi|Y'\| \leq 1 \Big\} \right\} ,$$

where the infimum is taken over all representations of h *(as in* (ii)*).*

Remark 13 (i) All three expressions define norms, we refer to [19, Chapt. 1]. In particular, λ is independent of the representation of h.
(ii) In Definition 6(iii) one can replace

$$\sup \Big\{ \Big(\sum_{i=1}^{n} |\psi(g_i)|^{p'} \Big)^{1/p'} : \psi \in Y', \|\psi|Y'\| \leq 1 \Big\} \tag{34}$$

by

$$\sup \Big\{ \Big\| \sum_{i=1}^{n} \lambda_i g_i \Big| Y \Big\| : \Big(\sum_{i=1}^{n} |\lambda_i|^p \Big)^{1/p} \leq 1 \Big\} , \tag{35}$$

see [19, Lem. 1.44].

A part of the motivation for such complicated constructions originates from the following lemma.

Lemma 9 *Let* α *be either the injective, projective or p-nuclear norm. Let* $P \in \mathcal{L}(X_1, Y_1)$ *and* $Q \in \mathcal{L}(X_2, Y_2)$. *Then there is a unique linear extension of* $P \otimes Q$, *originally defined on the algebraic tensor product* $X_1 \otimes X_2$, *to the space* $X_1 \otimes_\alpha X_2$, *s.t.*

$$\| P \otimes Q \, | \mathcal{L}(X_1 \otimes_\alpha X_2, Y_1 \otimes_\alpha Y_2)\| = \| P | \mathcal{L}(X_1, Y_1)\| \, \| Q \, | \, \mathcal{L}(X_2, Y_2)\| \, .$$

For later use we extend the projective norm to quasi-Banach spaces.

Definition 7 *Let $0 < p < 1$. Let X and Y be quasi-Banach spaces of functions or sequences, respectively. Then we define the projective tensor p-norm γ_p by*

$$\gamma_p(h, X, Y) := \inf \Big\{ \Big(\sum_{j=1}^n \|f_j|X\|^p \, \|g_j|Y\|^p \Big)^{1/p} : \quad f_j \in X,\, g_j \in Y,\, h = \sum_{j=1}^n f_j \otimes g_j \Big\}.$$

Remark 14 (i) γ_p defines an uniform quasi-norm on $X \otimes Y$. The inequality

$$\gamma_p(h_1 + h_2, X, Y)^p \le \gamma_p(h_1, X, Y)^p + \gamma_p(h_2, X, Y)^p$$

as well as the uniformness are obvious.
(ii) Different attempts to introduce tensor products of quasi-Banach spaces have been undertaken by Turpin [43] and Nitsche [24]. In particular the approach of Nitsche applies to so-called placid q-Banach spaces. Let us mention that ℓ_q and $B^r_{q,q}(\mathbb{R})$ (as well as $S^{r_1,\cdots,r_d}_{q,q}(\mathbb{R}^d)$, see below) are placid q-quasi-Banach spaces if $0 < q < 1$.

Lemma 10 *Let X_1, X_2, Y_1, Y_2 be quasi-Banach spaces of functions or sequences. Further we suppose that $T_1 \in \mathcal{L}(X_1, Y_1)$ and $T_2 \in \mathcal{L}(X_2, Y_2)$ are linear isomorphisms.*
(i) *Let X_1, X_2, Y_1, Y_2 be Banach spaces of functions or sequences. Then the operator $T_1 \otimes T_2$ is a linear isomorphism from $X_1 \otimes_{\alpha_p} X_2$ onto $Y_1 \otimes_{\alpha_p} Y_2$ as well as from $X_1 \otimes_\lambda X_2$ onto $Y_1 \otimes_\lambda Y_2$.*
(ii) *Let $0 < p < 1$. Let $P \in \mathcal{L}(X_1, Y_1)$ and $Q \in \mathcal{L}(X_2, Y_2)$. Then there is a unique linear extension of $P \otimes Q$ to the space $X_1 \otimes_{\gamma_p} X_2$, s.t.*

$$\| P \otimes Q \,|\mathcal{L}(X_1 \otimes_{\gamma_p} X_2, Y_1 \otimes_{\gamma_p} Y_2)\| = \| P|\mathcal{L}(X_1, Y_1)\| \; \| Q \,|\, \mathcal{L}(X_2, Y_2)\| \,.$$

Furthermore, the operator $T_1 \otimes T_2$ is a linear isomorphism from $X_1 \otimes_{\gamma_p} X_2$ onto $Y_1 \otimes_{\gamma_p} Y_2$.

5.2 Tensor products of weighted sequence spaces

Let I be a countable index set. Let $w = (w(j))_{j \in I}$ be a sequence of positive real numbers. Let $0 < p < \infty$. Then $\ell_p(w, I)$ consists of all sequences $a = (a_j)_{j \in I}$ of complex numbers such that

$$\| a \,|\ell_p(w, I)\| := \Big(\sum_{j \in I} |\, a_j \, w(j) \,|^p \Big)^{1/p} < \infty \,.$$

Clearly, $\ell_p(w_1 \otimes w_2, \mathbb{N}^2)$ means the collection of all sequences $(a_{j,k})_{j,k \in \mathbb{N}}$ such that

$$\| a \,|\ell_p(w_1 \otimes w_2, \mathbb{N}^2)\| := \Big(\sum_{j=1}^\infty \sum_{k=1}^\infty |\, a_{j,k} \, w_1(j) \, w_2(k) \,|^p \Big)^{1/p} < \infty \,.$$

Furthermore, $c_0(w,I)$ denotes the closure of the set of finite sequences with respect to the norm

$$\| \, a \,|c_0(w,I)\| := \| \, a \,|\ell_\infty(w,I)\| = \sup_{j\in I} |\, a_j \, w(j) \,| \, .$$

Proposition 3 *Let* $1 < p < \infty$. *Then*

$$\ell_p(w_1,\mathbb{N}) \otimes_{\alpha_p} \ell_p(w_2,\mathbb{N}) = \ell_p(w_1 \otimes w_2, \mathbb{N}^2). \tag{36}$$

The norms on the left-hand side and on the right-hand side coincide.

Remark 15 Formula (36) represents a special case of the more general formula

$$L_p(\mu_1) \otimes_{\alpha_p} L_p(\mu_2) = L_p(\mu_1 \otimes \mu_2).$$

valid for arbitrary measures μ_1 and μ_2, see Defant and Floret [8, 7.2, p. 79, 186].

We shall give an elementary proof of Proposition 3. The only nontrivial fact we will use within this proof is $(\ell_p)' = \ell_{p'}$.

Proof of Proposition 3. *Step 1.* We shall prove

$$\ell_p(w_1,\mathbb{N}) \otimes_{\alpha_p} \ell_p(w_2,\mathbb{N}) \hookrightarrow \ell_p(w_1 \otimes w_2, \mathbb{N}^2) \, .$$

Let $h \in \ell_p(w_1,\mathbb{N}) \otimes \ell_p(w_2,\mathbb{N})$ be given by

$$h = (h_{k,\ell})_{k,\ell} \, , \qquad h_{k,\ell} = \sum_{i=1}^n a_k^i \, b_\ell^i \, , \qquad k,\ell \in \mathbb{N} \, ,$$

where $(a_k^i)_k \in \ell_p(w_1,\mathbb{N})$, $(b_\ell^i)_\ell \in \ell_p(w_2,\mathbb{N})$, $i = 1,\ldots,n$. Then, using $(\ell_p)' = \ell_{p'}$ and Hölder's inequality, we obtain

$$\begin{aligned}
\sum_{k=1}^\infty \sum_{\ell=1}^\infty w_1^p(k)\, w_2^p(\ell) \Big| \sum_{i=1}^n a_k^i \, b_\ell^i \Big|^p &= \sum_{k=1}^\infty w_1^p(k) \Big\| \Big(\sum_{i=1}^n w_2(\ell)\, a_k^i \, b_\ell^i \Big)_\ell \Big| \ell_p \Big\|^p \\
&= \sum_{k=1}^\infty w_1^p(k) \sup_{\|\delta|\ell_{p'}\|\le 1} \Big| \sum_{i=1}^n \sum_{\ell=1}^\infty w_2(\ell)\, \delta_\ell \, a_k^i \, b_\ell^i \Big|^p \\
&\le \sum_{k=1}^\infty w_1^p(k) \Big(\sum_{i=1}^n |a_k^i|^p \Big) \sup_{\|\delta|\ell_{p'}\|\le 1} \Big(\sum_{i=1}^n \Big| \sum_{\ell=1}^\infty w_2(\ell)\, \delta_\ell \, b_\ell^i \Big|^{p'} \Big)^{p/p'} \\
&= \Big(\sum_{i=1}^n \| \, a^i \, |\ell_p(w_1,\mathbb{N})\|^p \Big) \sup_{\|\delta|\ell_{p'}\|\le 1} \sup_{\|\lambda|\ell_p\|\le 1} \Big| \sum_{i=1}^n \lambda_i \sum_{\ell=1}^\infty w_2(\ell)\, \delta_\ell \, b_\ell^i \Big|^p \\
&= \Big(\sum_{i=1}^n \| \, a^i \, |\ell_p(w_1,\mathbb{N})\|^p \Big) \sup_{\|\lambda|\ell_p\|\le 1} \Big\| \sum_{i=1}^n \lambda_i \, b^i \Big| \ell_p(w_2,\mathbb{N}) \Big\|^p \\
&\le \alpha_p(h, \ell_p(w_1,\mathbb{N}), \ell_p(w_2,\mathbb{N}))^p \, .
\end{aligned}$$

This proves

$$\| h \,|\ell_p(w_1 \otimes w_2, \mathbb{N}^2)\| \le \| h \,|\ell_p(w_1, \mathbb{N}) \otimes_{\alpha_p} \ell_p(w_2, \mathbb{N})\| \,.$$

Step 2. Now we deal with

$$\ell_p(w_1 \otimes w_2, \mathbb{N}^2) \hookrightarrow \ell_p(w_1, \mathbb{N}) \otimes_{\alpha_p} \ell_p(w_2, \mathbb{N})\,.$$

Therefore, let $h = (h_{k,\ell})_{k,\ell} \in \ell_p(w_1 \otimes w_2, \mathbb{N}^2)$ such that only finitely many components are not vanishing. Let $h_{k,\ell} = 0$ if either $k > M$ or if $\ell > N$. Then

$$h = \sum_{k=1}^{M} \sum_{\ell=1}^{N} h_{k,\ell}\, (e_k \otimes e_\ell) = \sum_{\ell=1}^{N} \Big(\sum_{k=1}^{M} h_{k,\ell}\, w_2(\ell)\, e_k \Big) \otimes (1/w_2(\ell))\, e_\ell\,,$$

where e_k denotes the elements of the canonical basis. Let $a_\ell := \sum_{k=1}^{M} h_{k,\ell}\, e_k$. It follows

$$\begin{aligned}
\| h \ \ & |\ell_p(w_1, \mathbb{N}) \otimes_{\alpha_p} \ell_p(w_2, \mathbb{N})\|^p \\
& \le \Big(w_2(\ell)^p \sum_{\ell=1}^{N} \| a_\ell \,|\ell_p(w_1, \mathbb{N})\|^p \Big) \sup_{\|\lambda|\ell_p\| \le 1} \Big\| \sum_{\ell=1}^{N} \lambda_\ell \, \frac{e_\ell}{w_2(\ell)} \Big| \ell_p(w_2, \mathbb{N}) \Big\|^p \\
& = \sum_{\ell=1}^{N} \sum_{k=1}^{M} w_2^p(\ell)\, |h_{k,\ell}\, w_1(k)|^p \ .
\end{aligned}$$

Hence

$$\| h \,|\ell_p(w_1, \mathbb{N}) \otimes_{\alpha_p} \ell_p(w_2, \mathbb{N})\| \le \| h \,|\ell_p(w_1 \otimes w_2, \mathbb{N}^2)\| \,.$$

A density argument completes the proof. ■

Proposition 4 *Let* $0 < p \le 1$. *Then*

$$\ell_p(w_1, \mathbb{N}) \otimes_{\gamma_p} \ell_p(w_2, \mathbb{N}) = \ell_p(w_1 \otimes w_2, \mathbb{N}^2)\,. \tag{37}$$

The quasi-norms on the left-hand side and on the right-hand side coincide.

Remark 16 (i) Formula (37) with $p = 1$ is well-known. We refer to [19, Cor. 1.16].
(ii) In the framework of a more general concept of tensor products of quasi-Banach spaces Nitsche [23] has proved a similar result for the unweighted case.

Proof *Step 1.* We shall prove

$$\ell_p(w_1, \mathbb{N}) \otimes_{\gamma_p} \ell_p(w_2, \mathbb{N}) \hookrightarrow \ell_p(w_1 \otimes w_2, \mathbb{N}^2)\,.$$

Let $h \in \ell_p(w_1, \mathbb{N}) \otimes \ell_p(w_2, \mathbb{N})$ be given by

$$h = (h_{k,\ell})_{k,\ell}\,, \qquad h_{k,\ell} = \sum_{i=1}^{n} a_k^i\, b_\ell^i\,, \qquad k, \ell \in \mathbb{N}\,,$$

where $(a^i_k)_k \in \ell_p(w_1,\mathbb{N})$, $(b^i_\ell)_\ell \in \ell_p(w_2,\mathbb{N})$, $i=1,\ldots,n$.

$$\begin{aligned}
\|h|\ell_p(w_1\otimes w_2,\mathbb{N}^2)\|^p &= \sum_{k=1}^\infty\sum_{\ell=1}^\infty w_1(k)^p w_2(\ell)^p \Big|\sum_{i=1}^n a^i_k b^i_\ell\Big|^p \\
&\le \sum_{k=1}^\infty\sum_{\ell=1}^\infty w_1(k)^p w_2(\ell)^p \sum_{i=1}^n |a^i_k|^p\cdot|b^i_\ell|^p \\
&= \sum_{i=1}^n\sum_{k=1}^\infty\sum_{\ell=1}^\infty w_1(k)^p w_2(\ell)^p |a^i_k|^p\cdot|b^i_\ell|^p \\
&= \sum_{i=1}^n \|a^i|\ell_p(w_1,\mathbb{N})\|^p\cdot\|b^i|\ell_p(w_2,\mathbb{N})\|^p \\
&= \gamma_p(h,\ell_p(w_1,\mathbb{N}),\ell_p(w_2,\mathbb{N}))^p\,.
\end{aligned}$$

Step 2. It remains to prove

$$\ell_p(w_1\otimes w_2,\mathbb{N}^2)\hookrightarrow \ell_p(w_1,\mathbb{N})\otimes_{\gamma_p}\ell_p(w_2,\mathbb{N})\,.$$

We follow the arguments from Step 2 in the proof of the previous Proposition. By a density argument it will be enough to deal with finite sequences. Therefore, let

$$h=\sum_{k=1}^M\sum_{\ell=1}^N h_{k,\ell}\,(e_k\otimes e_\ell)=\sum_{\ell=1}^N\Big(\sum_{k=1}^M h_{k,\ell}\,w_2(\ell)\,e_k\Big)\otimes(1/w_2(\ell))\,e_\ell\,.$$

Then we obtain

$$\begin{aligned}
\|\,h\,|\ell_p(w_1,\mathbb{N}) \quad \otimes_{\gamma_p} \quad & \ell_p(w_2,\mathbb{N})\|^p \\
\le \quad & \sum_{\ell=1}^N\Big\|\sum_{k=1}^M h_{k,\ell}\,w_2(\ell)\,e_k\,\Big|\ell_p(w_1,\mathbb{N})\Big\|^p\,\|\,e_\ell/w_2(\ell)\,|\ell_p(w_2,\ell)\|^p \\
= \quad & \sum_{\ell=1}^N\sum_{k=1}^M|w_2(\ell)|^p\,|w_1(k)|^p\,|h_{k,\ell}|^p \\
= \quad & \|\,h\,|\ell_p(w_1\otimes w_2,\mathbb{N}^2)\|^p\,.
\end{aligned}$$

This proves the claim. ■

Proposition 5 *Let $p=\infty$. Then*

$$c_0(w_1,\mathbb{N})\otimes_\lambda c_0(w_2,\mathbb{N})=c_0(w_1\otimes w_2,\mathbb{N}^2)\,. \tag{38}$$

The norms on the left-hand side and on the right-hand side coincide.

Proof Let $h \in c_0(w_1,\mathbb{N}) \otimes c_0(w_2,\mathbb{N})$ be given by

$$h = (h_{k,\ell})_{k,\ell}\,, \qquad h_{k,\ell} = \sum_{i=1}^n a_k^i\, b_\ell^i\,, \qquad k,\ell \in \mathbb{N}\,,$$

where $a^i := (a_k^i)_k \in c_0(w_1,\mathbb{N})$, $b^i := (b_\ell^i)_\ell \in c_0(w_2,\mathbb{N})$, $i = 1,\ldots,n$. Here we suppose $a_k^i = b_\ell^i = 0$ if $k,\ell > n$. Let $X = c_0(w_1,\mathbb{N})$. Obviously,

$$\begin{aligned}
\|\, h\,|c_0(w_1\otimes w_2,\mathbb{N}^2)\| &= \sup_{k,\ell\in\mathbb{N}} w_1(k)\, w_2(\ell)\,\Big|\sum_{i=1}^n a_k^i b_\ell^i\Big| \\
&= \sup_{\ell=1,\ldots,n} w_2(\ell)\,\Big\|\Big(\sum_{i=1}^n a_k^i b_\ell^i\Big)_k \Big| c_0(w_1,\mathbb{N})\Big\| \\
&= \sup_{\ell=1,\ldots,n} w_2(\ell) \sup_{\|\psi|X'\|\le 1} \Big|\psi\Big(\Big(\sum_{i=1}^n a_k^i b_\ell^i\Big)_k\Big)\Big| \\
&= \sup_{\|\psi|X'\|\le 1}\ \sup_{\ell=1,\ldots,n} w_2(\ell)\,\Big|\sum_{i=1}^n \psi(a^i)\, b_\ell^i\Big| \\
&= \lambda(h, c_0(w_1,\mathbb{N}), c_0(w_2,\mathbb{N}))\,.
\end{aligned}$$

Vice versa, if

$$h \;=\; \sum_{k=1}^M \sum_{\ell=1}^N h_{k,\ell}\,(e_k\otimes e_\ell) = \sum_{\ell=1}^N \Big(\sum_{k=1}^M h_{k,\ell}\, w_2(\ell)\, e_k\Big) \otimes (1/w_2(\ell))\, e_\ell\,.$$

(we put $h_{k,\ell}=0$ if either $\ell > N$ or $k > M$) and using the abbreviations

$$a^\ell := \sum_{k=1}^M h_{k,\ell}\, w_2(\ell)\, e_k \qquad \text{and} \qquad b^\ell := (1/w_2(\ell))\, e_\ell\,,$$

we obtain

$$\begin{aligned}
\|\, h\,|c_0(w_1,\mathbb{N})\otimes_\lambda c_0(w_2,\mathbb{N})\| &= \sup_{\|\psi|X'\|\le 1} \Big\|\sum_{\ell=1}^N \psi(a^\ell)\, b^\ell \Big| c_0(w_2,\mathbb{N})\Big\| \\
&= \sup_{\|\psi|X'\|\le 1}\ \sup_{\ell=1,\ldots,N} |\,\psi(a^\ell)\,| \\
&= \sup_{\ell\in\mathbb{N}} \Big\|\sum_{k=1}^M h_{k,\ell}\, w_2(\ell)\, e_k \Big| c_0(w_1,\mathbb{N})\Big\| \\
&= \sup_{\ell\in\mathbb{N}} w_2(\ell) \sup_{k\in\mathbb{N}} w_1(k)\, |h_{k,\ell}| \\
&= \|\, h\,|c_0(w_1\otimes w_2,\mathbb{N}^2)\|\,.
\end{aligned}$$

A density argument completes the proof of (38). ■

5.3 Tensor products of the spaces b^s_p

We formulate a few consequences of Propositions 3, 4, 5 for the more specialized sequence spaces b^s_p. Of course, the spaces b^s_p can be interpreted as weighted ℓ_p-spaces with respect to the index set $\mathbb{N}_0 \times \mathbb{Z}$. The tensor product of $b^{s_1}_p$ and $b^{s_2}_p$ is then a weighted ℓ_p-space with respect to the index set $(\mathbb{N}_0 \times \mathbb{Z}) \times (\mathbb{N}_0 \times \mathbb{Z})$. In view of this we introduce the following sequence spaces.

Definition 8 *Let $0 < p \le \infty$ and let $r_1, \ldots , r_d \in \mathbb{R}$.*
(i) *We define*

$$
\begin{aligned}
s^{r_1,\cdots,r_d}_p b \ &:= \ \Big\{ (a_{\bar{j},\bar{k}})_{\bar{j},\bar{k}} \subset \mathbb{C} : \\
&\qquad \| \, a \, | s^{r_1,\cdots,r_d}_p b \| := \Big(\sum_{\bar{j}\in\mathbb{N}_0^d} \sum_{\bar{k}\in\mathbb{Z}^d} 2^{(j_1(r_1+\frac{1}{2}-\frac{1}{p})+\ldots+j_d(r_d+\frac{1}{2}-\frac{1}{p}))p} \, |a_{\bar{j},\bar{k}}|^p \Big)^{1/p} < \infty \Big\}
\end{aligned}
$$

(modification if $p = \infty$).
(ii) *By $\mathring{s}^{r_1,\cdots,r_d}_\infty b$ we denote the closure of the finite sequences with respect to the norm $\| \cdot | s^{r_1,\cdots,r_d}_\infty b \|$.*

As a consequence of an iteration process we obtain the following.

Corollary 2 *Let $r_1, \ldots , r_d, r_{d+1} \in \mathbb{R}$.*
(i) *Let $1 < p < \infty$. Then*

$$
b^{r_1}_p \otimes_{\alpha_p} s^{r_2,\cdots,r_{d+1}}_p b = s^{r_1,\cdots,r_d}_p b \otimes_{\alpha_p} b^{r_{d+1}}_p = s^{r_1,r_2,\cdots,r_{d+1}}_p b \, .
$$

(ii) *Let $0 < p \le 1$. Then*

$$
b^{r_1}_p \otimes_{\gamma_p} s^{r_2,\cdots,r_{d+1}}_p b = s^{r_1,\cdots,r_d}_p b \otimes_{\gamma_p} b^{r_{d+1}}_p = s^{r_1,r_2,\cdots,r_{d+1}}_p b \, .
$$

(iii) *Let $p = \infty$. Then*

$$
\mathring{b}^{r_1}_\infty \otimes_\lambda \mathring{s}^{r_2,\cdots,r_{d+1}}_\infty b = \mathring{s}^{r_1,\cdots,r_d}_\infty b \otimes_\lambda \mathring{b}^{r_{d+1}}_\infty = \mathring{s}^{r_1,r_2,\cdots,r_{d+1}}_\infty b \, .
$$

In (i)-(iii) all quasi-norms coincide.

5.4 Tensor products of Besov spaces

In one of the previous subsections we have seen that the mapping

$$
J : \ f \to (\langle f, \psi_{j,k} \rangle)_{j,k}
$$

is an isomorphism between a Besov space $B^s_{p,p}(\mathbb{R})$ and the sequence space b^s_p. Now we turn to the d-fold tensor product of J. Let us have a look at the situation for $d = 2$. Obviously

$$(J \otimes J)(f \otimes g) = (\langle f \otimes g, \psi_{j_1,k_1} \otimes \psi_{j_2,k_2}\rangle)_{j_1,j_2,k_1,k_2} .$$

In view od this identity we consider the tensor product wavelet system

$$\psi_{\bar{j},\bar{k}}(x) := \prod_{i=1}^{d} \psi_{j_i,k_i}(x_i) , \qquad \bar{j} \in \mathbb{N}_0^d , \quad \bar{k} \in \mathbb{Z}^d .$$

Then

$$(J \otimes \ldots \otimes J)(f_1 \otimes \ldots \otimes f_d) = (\langle f_1 \otimes \ldots \otimes f_d, \psi_{\bar{j},\bar{k}}\rangle)_{\bar{j},\bar{k}}$$

follows.

Definition 9 *Let $0 \le r_1, \ldots r_d$ and $0 < p < \infty$. Let J be a wavelet isomorphism satisfying the conditions in Theorem 2 for all tupels (r_i, p), $i = 1, \ldots d$. We define*

$$S^{r_1,\ldots r_d}_{p,p} B(\mathbb{R}^d) := (J \otimes \ldots \otimes J)^{-1} s^{r_1,\ldots r_d}_p b .$$

For $p = \infty$ we put

$$\mathring{S}^{r_1,\cdots,r_d}_{\infty,\infty} B(\mathbb{R}^d) := \mathring{s}^{r_1,\cdots,r_d}_{\infty} b .$$

Combining Lemma 10 with Theorem 2 then we end up with the following theorem.

Theorem 4 *Let $d \ge 1$ and let $r_1, \ldots , r_{d+1} \in \mathbb{R}$.*
(i) *Let $1 < p < \infty$. Then the following holds true*

$$\begin{aligned} B^{r_1}_{p,p}(\mathbb{R}) \otimes_{\alpha_p} S^{r_2,\cdots,r_{d+1}}_{p,p} B(\mathbb{R}^d) &= S^{r_1,\cdots,r_d}_{p,p} B(\mathbb{R}^d) \otimes_{\alpha_p} B^{r_{d+1}}_{p,p}(\mathbb{R}) \\ &= S^{r_1,r_2,\cdots,r_{d+1}}_{p,p} B(\mathbb{R}^{d+1}) \end{aligned} \tag{39}$$

in the sense of equivalent norms.
(ii) *Let $0 < p \le 1$. Then the following formula*

$$\begin{aligned} B^{r_1}_{p,p}(\mathbb{R}) \otimes_{\gamma_p} S^{r_2,\cdots,r_{d+1}}_{p,p} B(\mathbb{R}^d) &= S^{r_1,\cdots,r_d}_{p,p} B(\mathbb{R}^d) \otimes_{\gamma_p} B^{r_{d+1}}_{p,p}(\mathbb{R}) \\ &= S^{r_1,r_2,\cdots,r_{d+1}}_{p,p} B(\mathbb{R}^{d+1}) \end{aligned}$$

holds true in the sense of equivalent quasi-norms.

A little bit more care is needed in case $p = \infty$.

Theorem 5 *Let $p = \infty$. Then we have*

$$\begin{aligned} \mathring{B}^{r_1}_{\infty,\infty}(\mathbb{R}) \otimes_\lambda \mathring{S}^{r_2,\dots,r_{d+1}}_{\infty,\infty}B(\mathbb{R}^d) &= \mathring{S}^{r_1,\dots,r_d}_{\infty,\infty}B(\mathbb{R}^d) \otimes_\lambda \mathring{B}^{r_{d+1}}_{\infty,\infty}(\mathbb{R}) \\ &= \mathring{S}^{r_1,r_2,\dots,r_{d+1}}_{\infty,\infty}B(\mathbb{R}^{d+1}) \end{aligned}$$

in the sense of equivalent norms.

Again it is not clear how to understand the regularity of a function belonging to $S^{r_1,\dots r_d}_{p,p}B(\mathbb{R}^d)$. However, there is a property which helps with this respect. Taking a tensor product of functions

$$f_1 \otimes \ldots \otimes f_d \,, \qquad f_i \in B^{r_i}_{p,p}(\mathbb{R})\,, i = 1, \ldots d,$$

then we immediately get

$$\| \, f_1 \otimes \ldots \otimes f_d \, | S^{r_1,\dots,r_d}_{p,p}(\mathbb{R}^d)\| \asymp \prod_{i=1}^{d} \| \, f_i \, | B^{r_i}_{p,p}(\mathbb{R})\| \, .$$

5.5 Besov and Sobolev spaces of dominating mixed smoothness

Definition 10 *Let $r_1, \dots, r_d \in \mathbb{N}$. Then the Sobolev space $S^{r_1,\dots,r_d}_p W(\mathbb{R}^d)$ of dominating mixed smoothness is the collection of all functions $f \in L_p(\mathbb{R}^d)$ s.t. the weak derivatives*

$$D^\alpha f \in L_p(\mathbb{R}^d)\,, \qquad \alpha_i \le r_i\,, \quad i = 1, \dots, d\,.$$

We put

$$\| \, f \, | S^{r_1,\dots,r_d}_p W(\mathbb{R}^d)\| := \sum_{\alpha_1 \le r_1} \ldots \sum_{\alpha_d \le r_d} \| \, D^\alpha f \, | L_p(\mathbb{R}^d)\| < \infty.$$

Theorem 6 *Let $1 < p < \infty$. Then*

$$S^{r_1,\dots,r_d}_p W(\mathbb{R}^d) = W^{r_1}_p(\mathbb{R}) \otimes_{\alpha_p} \ldots \otimes_{\alpha_p} W^{r_1}_p(\mathbb{R})\,.$$

Proof A first proof in the periodic situation with $d = 2$ has been given by Sprengel in [37], for the nonperiodic case see [35]. ■

Remark 17 For $p = 2$ we conclude

$$S^{r_1,\dots,r_d}_2 W(\mathbb{R}^d) = S^{r_1,\dots,r_d}_{2,2} B(\mathbb{R}^d)$$

in the sense of equivalent norms.

Also in the general case these tensor products of Besov spaces can be interpreted as spaces of dominating mixed smoothness. Up to know there are two monographs dealing with this subject: Amanov [1] and Schmeißer and Triebel [33]. There and in [44] characterization by differences can be found. A characterization by Daubechies wavelets has been established in Vybiral [46]. This allows the identification with the tensor products discussed here. For simplicity we concentrate on $d = 2$ and $p = \infty$. For $(m_1, m_2) \in \mathbb{N}^2$ we define

$$\Delta_{h_1}^{m_1}(\Delta_{h_2}^{m_2} f(x_1, x_2)) := \sum_{j=0}^{m_1} (-1)^j \binom{M}{j} \sum_{k=0}^{m_2} (-1)^k \binom{M}{k} f(x_1+(m_1-j)h_1, x_2+(m_2-k)h_2)\,.$$

We select $(m_1, m_2) \in \mathbb{N}^2$ such that $0 < r_i < m_i$, $i = 1, 2$. Then a continuous function f belongs to $S^{r_1,r_2}_{\infty,\infty}B(\mathbb{R}^2)$ if

$$A := \sup_{x_1\in\mathbb{R}} \sup_{x_2\in\mathbb{R}} \sup_{h_1\in\mathbb{R}} \sup_{h_2\in\mathbb{R}} |h_1|^{-r_1}\, |h_2|^{-r_2}\, |\Delta_{h_1}^{m_1}(\Delta_{h_2}^{m_2} f(x_1, x_2))| < \infty$$

as well as

$$B := \sup_{x_1\in\mathbb{R}} \sup_{x_2\in\mathbb{R}} \sup_{h_1\in\mathbb{R}} |h_1|^{-r_1}\, |\Delta_{h_1}^{m_1} f(x_1, x_2))| < \infty$$

and

$$C := \sup_{x_1\in\mathbb{R}} \sup_{x_2\in\mathbb{R}} \sup_{h_1\in\mathbb{R}} |h_2|^{-r_2}\, |\Delta_{h_2}^{m_2} f(x_1, x_2)| < \infty\,.$$

The norm is obtained by taking

$$\|\, f\, |S^{r_1,r_2}_{\infty,\infty}B(\mathbb{R}^2)\| := \|\, f\, |C(\mathbb{R}^2)\| + A + B + C$$

One can prove that $\mathring{S}^{r_1,r_2}_{\infty,\infty}B(\mathbb{R}^d)$ coincides with the closure of $C_0^\infty(\mathbb{R}^d)$ with the respect to the norm of $S^{r_1,r_2}_{\infty,\infty}B(\mathbb{R}^d)$.

Similar characterizations can be given for $p < \infty$. We omit details and refer to [1, 33, 44].

6 The Smolyak algorithm and tensor products of Besov spaces

After these lenghty preparations its time to bring these two objects together.

6.1 The Smolyak algorithm for partial sums of wavelet expansions

Given a wavelet system generated by φ and ψ we consider the associated projections

$$P_N f := \sum_{j=0}^{N} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k}\rangle\, \psi_{j,k}\,, \qquad N \in \mathbb{N}_0\,.$$

Now we apply the Smolyak algorithm to this sequence, i.e. we put $L^i_j = P_j$, $i = 1, \ldots , d$, and obtain (similar to the periodic case)

$$A(m, d, L)f = \sum_{j_1+\ldots+j_d\le m} \sum_{k\in\mathbb{Z}^d} \langle f, \psi_{\bar{j},\bar{k}}\rangle \, \psi_{\bar{j},\bar{k}} \, , \qquad m \in \mathbb{N}_0 \, .$$

A first rough estimate

For simplicity we choose $r_1 = r_2 = \ldots = r_d = s$. We need a further preparation. We already know from Lemma 2 that $\| I - P_N \,|\mathcal{L}(b^s_p, b^0_p)\| \asymp N^{-s}$, $N \in \mathbb{N}_0$.

Lemma 11 *Let* $1 < p < \infty$. *Then*

$$\| I - P_N \,|\mathcal{L}(B^s_{p,p}(\mathbb{R}), L_p(\mathbb{R}))\| \asymp N^{-s} \, , \qquad N \in \mathbb{N}_0 \, .$$

Proof We give a sketch of the proof only. We shall use

$$B^0_{p,\min(p,2)}(\mathbb{R}) \hookrightarrow L_p(\mathbb{R}) \hookrightarrow B^0_{p,\max(p,2)}(\mathbb{R}) \, ,$$

see [40, 2.3.2]. Furthermore, Theorem 2 extends to $s = 0$, see [41]. Hence, if $p \le 2$ we obtain

$$\begin{aligned} \| I - P_N \,|\mathcal{L}(B^s_{p,p}(\mathbb{R}), L_p(\mathbb{R}))\| &\le \| I - P_N \,|\mathcal{L}(B^s_{p,p}(\mathbb{R}), B^0_{p,p}(\mathbb{R}))\| \\ &\asymp \| I - P_N \,|\mathcal{L}(b^s_p, b^0_p(\mathbb{R}))\| \\ &\asymp N^{-s} \, . \end{aligned}$$

For $p > 2$ one has to use the counterpart of Theorem 2 for the spaces $B^s_{p,q}(\mathbb{R})$ with $q \ne p$, see e.g. [6, 41, 48]. Then, as in Lemma 2 one can prove

$$\| I - P_N \,|\mathcal{L}(b^s_{p,q_1}, b^0_{p,q_0}(\mathbb{R}))\| \asymp N^{-s}$$

where $b^s_{p,q}$ is the image of $B^s_{p,q}(\mathbb{R})$ under mapping J and q_0 and q_1 are arbitrary. We omit details. From this the claim follows. ■

Let

$$f = \sum_{j_1,\ldots,j_d\le M} \sum_{k\in\mathbb{Z}^d} \langle f, \psi_{\bar{j},\bar{k}}\rangle \, \psi_{\bar{j},\bar{k}} \, .$$

Functions of this type are dense in $S^{s,\ldots,s}_{p,p}B(\mathbb{R}^d)$. Moreover, we have

$$A(d\,M, d, L)f = f \, .$$

Using the tensor product techniques and Lemma 11 we obtain

$$\begin{aligned}
\| f \; - \; & A(m,d,L)f|L_p(\mathbb{R}^d)\| = \| A(d\,M,d,L)f - A(m,d,L)f|L_p(\mathbb{R}^d)\| \\
= \; & \| \sum_{m<j_1+\ldots+j_d\le d\,M} \Delta_{j_1}(L^1) \otimes \ldots \otimes \Delta_{j_d}(L^d)\, f\, |L_p(\mathbb{R}^d)\| \\
\le \; & \sum_{m<j_1+\ldots+j_d\le d\,M} \| \Delta_{j_1}(L^1) \otimes \ldots \otimes \Delta_{j_d}(L^d)\, f\, |L_p(\mathbb{R}^d)\| \\
\le \; & \sum_{m<j_1+\ldots+j_d\le d\,M} \Big(\prod_{i=1}^{d} \| \Delta_{j_i} \,|\mathcal{L}(B^s_{p,p}(\mathbb{R}), L_p(\mathbb{R}))\| \Big) \| f\, |S^{s,\ldots,s}_{p,p}(\mathbb{R}^d)\| \\
\le \; & c_1 \sum_{m<j_1+\ldots+j_d\le d\,M} 2^{-(j_1+\ldots+j_d)s} \| f\, |S^{s,\ldots,s}_{p,p}(\mathbb{R}^d)\| \\
\le \; & c_2\, m^{d-1}\, 2^{-ms} \| f\, |S^{s,\ldots,s}_{p,p}(\mathbb{R}^d)\| \, .
\end{aligned}$$

Since c_2 is independent of M we can extend this estimate to all of $S^{s,\ldots,s}_{p,p}B(\mathbb{R}^d)$ (modification in case $p=\infty$). However, this estimate is not optimal.
We shall need a further assumption. Let $\mathcal{X}$ be the characteristic function of the interval $(0,1)$. We put

$$\mathcal{X}_{j,k}(t) = 2^{j/2}\, \mathcal{X}(2^j t - k)\, , \qquad t \in \mathbb{R}\, , \quad k \in \mathbb{Z}\, , \quad j \in \mathbb{N}_0\, .$$

Further we define

$$\mathcal{X}_{\bar{j},\bar{k}} = \mathcal{X}_{j_1,k_1} \otimes \ldots \otimes \mathcal{X}_{j_d,k_d}\, , \quad \bar{j} = (j_1, \ldots, j_d) \in \mathbb{N}_0^d, \quad \bar{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d.$$

Let $1<p<\infty$. We say that the system $(\psi_{\bar{j},\bar{k}})_{\bar{j},\bar{k}}$ has the Littlewood-Paley property if

$$\begin{aligned}
\Big\| \sum_{\bar{j}\in\mathbb{N}_0^d} \sum_{\bar{k}\in\mathbb{Z}^d} a_{\bar{j},\bar{k}}\, \psi_{\bar{j},\bar{k}} \Big| L_p(\mathbb{R}^d) \Big\| \quad & \asymp \quad \Big\| \Big(\sum_{\bar{j}\in\mathbb{N}_0^d} \sum_{\bar{k}\in\mathbb{Z}^d} |a_{\bar{j},\bar{k}}|^2\, \mathcal{X}^2_{\bar{j},\bar{k}} \Big)^{1/2} \Big| L_p(\mathbb{R}^d) \Big\| \\
& \asymp \quad \Big\| \Big(\sum_{\bar{j}\in\mathbb{N}_0^d} \sum_{\bar{k}\in\mathbb{Z}^d} |a_{\bar{j},\bar{k}}|^2\, (\psi_{\bar{j},\bar{k}})^2 \Big)^{1/2} \Big| L_p(\mathbb{R}^d) \Big\| \qquad (40)
\end{aligned}$$

holds for all finite sequences $(a_{\bar{j},\bar{k}})_{\bar{j},\bar{k}}$. The spline systems $(\psi^m_{\bar{j},\bar{k}})_{\bar{j},\bar{k}}$ and the systems generated by the Daubechies wavelets have the Littlewood-Paley property, see [11].

Theorem 7 *Let $0<p<\infty$ and*

$$\max\Big(0, \frac{1}{p} - 1\Big) < s\, .$$

For $1<p<\infty$ we suppose in addition that $(\psi_{\bar{j},\bar{k}})_{\bar{j},\bar{k}}$ has the Littlewood-Paley property. Then there exists a constant c s.t.

$$\| I - A(m,d,L)\, |\mathcal{L}(S^{s,\ldots,s}_{p,p}(\mathbb{R}^d), L_p(\mathbb{R}^d))\| \le c \begin{cases} 2^{-sm} & \text{if} \quad 0<p\le 2\, , \\ m^{(d-1)(\frac{1}{2}-\frac{1}{p})}\, 2^{-sm} & \text{if} \quad 2<p<\infty\, . \end{cases}$$

holds for all $m \in \mathbb{N}$.

Proof We only deal with the case $p \le 2$ here. For the general case we refer to [35]. Observe

$$S^{0,\dots 0}_{p,p}B(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d)\,, \qquad p \le 2\,,$$

see [46]. As shown above it will be sufficient to prove

$$\| \, I - A(m,d,L) \, | \mathcal{L}(s^{s,\dots,s}_p b, s^{0,\dots 0}_p b \| \le c\, 2^{-sm}\,.$$

But

$$\begin{aligned}
\| \sum_{|j|_1 > m} \sum_{k \in \mathbb{Z}^d} a_{\bar{j},\bar{k}}\, e_{\bar{j},\bar{k}} \, | s^{0,\dots 0}_p b\|^p &= \sum_{|j|_1>m} 2^{|\bar{j}|_1 (\frac{1}{2}-\frac{1}{p})p} \sum_{k\in\mathbb{Z}^d} |a_{\bar{j},\bar{k}}|^p \\
&\le 2^{-smp} \sum_{|j|_1>m} 2^{|\bar{j}|_1 (s+\frac{1}{2}-\frac{1}{p})p} \sum_{k\in\mathbb{Z}^d} |a_{\bar{j},\bar{k}}|^p \\
&\le 2^{-smp}\, \| \, a \, | s^{s,\dots,s}_p b\|^p\,.
\end{aligned}$$

This proves the claim. ■

In case of the spline system one can prove even more, see [35]. Let L^n refer to the sequence of partial sum operators with respect to the spline system of order n.

Theorem 8 *Let* $d > 1$*.*
(i) *Let* $0 < p < \infty$ *and* $\max(0, \frac{1}{p} - 1) < r < n - 1 + \min(1, 1/p)$*. Then we have*

$$\| \, I - A(m,d,L^n)\, | \mathcal{L}(S^r_{p,p}B(\mathbb{R}^d), L_p(\mathbb{R}^d))\| \asymp \begin{cases} 2^{-rm} & \text{if} \quad 0 < p \le 2\,, \\ & \\ m^{(d-1)(\frac{1}{2}-\frac{1}{p})}\, 2^{-rm} & \text{if} \quad 2 < p < \infty, \end{cases}$$

$m \in \mathbb{N}$.
(ii) *Let* $p = \infty$ *and* $0 < r < n - 1$*. Then we have*

$$\| \, I - A(m,d,L^n)\, | \mathcal{L}(S^r_{\infty,\infty}B(\mathbb{R}^d), L_\infty(\mathbb{R}^d))\| \asymp m^{d-1}\, 2^{-rm}\,, \qquad m \in \mathbb{N}\,.$$

6.2 Best approximation from sparse grid ansatz spaces by splines

Next we want to define the error of best approximation of a function $f \in L_p(\mathbb{R}^d)$, $0 < p \le \infty$, by splines of degree less than m related to the hyperbolic cross. For this purpose it will be convenient to introduce some further notation first. Let

$$V^n_j := \operatorname{span} \Big\{ \mathcal{N}_n(2^j \cdot - k) : \quad k \in \mathbb{Z} \Big\}\,, \qquad j \in \mathbb{N}_0\,,$$

and

$$\mathcal{V}^n_m := \operatorname{span} \Big\{ \mathcal{N}_n(2^{j_1} \cdot - k_1) \otimes \ldots \otimes \mathcal{N}_n(2^{j_d} \cdot - k_d) : \quad \bar{j} \in \mathbb{N}^d_0\,, \quad |\bar{j}|_1 = m\,, \quad \bar{k} \in \mathbb{Z}^d \Big\}\,,$$

$m \in \mathbb{N}_0$. Sometimes spaces of this type are called *sparse grid ansatz spaces*. Since "span" contains finite sums only we have $\mathcal{V}_m^n \subset L_p(\mathbb{R}^d)$. We put

$$E_m^n(f, L_p(\mathbb{R}^d)) := \inf \Big\{ \| f - g \,|L_p(\mathbb{R}^d)\| : \quad g \in \mathcal{V}_m^n \Big\} , \qquad m \in \mathbb{N}_0 \, .$$

Some comments are necessary. The classes $\mathcal{V}_m^n$ are nested, i.e. $\mathcal{V}_m^n \subset \mathcal{V}_{m+1}^n$, since

$$\mathcal{N}_n(t) = 2^{-n+1} \sum_{k=0}^{n} \binom{n}{k} \mathcal{N}_n(2t-k) \, , \qquad t \in \mathbb{R} \, .$$

Furthermore, the spaces $\mathcal{V}_m^n$ do not contain our basis functions $\psi_{\bar{j},\bar{k}}^n$. However, it becomes obvious from (10) that $\psi_{\bar{j},\bar{k}}^n$ belongs to the closure of $\mathcal{V}_m^n$, $|\bar{j}|_1 = m$, in $L_p(\mathbb{R}^d)$. Alternatively to the quantity $E_m^n(f, L_p(\mathbb{R}^d))$ one could consider the following

$$\begin{aligned} \widetilde{E}_m^n(f, L_p(\mathbb{R}^d)) \quad := \quad \inf \Big\{ & \| f - g \,|L_p(\mathbb{R}^d)\| : \quad g \in L_p(\mathbb{R}^d), \qquad \exists (a_{\bar{j},\bar{k}})_{\bar{j},\bar{k}} \quad \text{s.t.} \\ & a_{\bar{j},\bar{k}} = 0 \quad \text{if} \quad |\bar{j}|_1 > m \qquad \text{and} \qquad g \stackrel{L_p}{=} \sum_{\bar{j} \in \mathbb{N}_0^d} \sum_{k \in \mathbb{Z}^d} a_{\bar{j},\bar{k}} \, \psi_{\bar{j},\bar{k}}^n \Big\} , \end{aligned}$$

a concept which is related to the definition of $A(m, d, L^n)$. Fortunately we have

$$E_m^n(f, L_p(\mathbb{R}^d)) = \widetilde{E}_m^n(f, L_p(\mathbb{R}^d)) \, .$$

This can be seen by using (iv) in our list of properties of the ψ_n given above. To have a compact formulation we shall use the following quantity:

$$\mathcal{E}_m^n(F)_p := \sup_{\|f\|_F \le 1} E_m^n(f, L_p(\mathbb{R}^d))$$

where $F \hookrightarrow L_p$ denotes an arbitrary quasi-Banach space.

Theorem 9 *Let $d > 1$ and let $n \in \mathbb{N}$.*
(i) *Let $1 \le p < \infty$ and $0 < r < n - 1 + 1/p$. Then it holds*

$$\mathcal{E}_m^n(S_{p,p}^r B(\mathbb{R}^d))_p \asymp \begin{cases} 2^{-rm} & \text{if} \quad 1 \le p \le 2 \, , \\ m^{(d-1)(\frac{1}{2} - \frac{1}{p})} \, 2^{-rm} & \text{if} \quad 2 < p < \infty \, , \end{cases} \tag{41}$$

$m \in \mathbb{N}$.
(ii) *Let $0 < p < 1$ and $\frac{1}{p} - 1 < r < n$. Then there exists a constant c such that*

$$\mathcal{E}_m^n(S_{p,p}^r B(\mathbb{R}^d))_p \le c \, 2^{-rm} \tag{42}$$

holds for all $m \in \mathbb{N}$.
(iii) *Let $p = \infty$ and $0 < r < n - 1$. Then there exists a constant c such that*

$$\mathcal{E}_m^n(S_{\infty,\infty}^r B(\mathbb{R}^d))_\infty \le c \, m^{d-1} \, 2^{-rm} \, , \qquad m \in \mathbb{N} \, .$$

Remark 18 (i) DeVore, Konyagin, Temlyakov [11] have dealt with similar problems for Sobolev-type spaces.
(ii) In case $1 < p < \infty$ we refer to Kamont [16] for corresponding estimates in terms of a certain modulus of smoothness for functions defined on $[0, 1]^d$.
(iii) All estimates stated in Theorem 9 have counterparts in the classical periodic context of best approximation by polynomials with frequencies taken from a hyperbolic cross. We refer to Dinh Dung [12], Galeev [15], Romanyuk [29], Temlyakov [38], [34] and [45]. There are also unpublished notes of Bazarkhanov devoted to this topic. However, the Littlewood-Paley theory for the spline system Ψ^m differs from the Littlewood-Paley theory of the trigonometric system. So, at least partly, our test functions are not the same as used in the quoted literature.

At least for the most interesting case $p = 2$ there is a partial inverse of the inequality (41). In fact we have the following equivalence.

Corollary 3 *Let $0 < r < n - 1/2$. A function $f \in L_2(\mathbb{R}^d)$ belongs to $S^r_{2,2}B(\mathbb{R}^d)$ if and only if the sequence $(2^{rm}\, E^n_m(f, L_2(\mathbb{R}^d)))_m$ belongs to ℓ_2. Moreover, we have*

$$\| \, f \, |S^r_{2,2}B(\mathbb{R}^d)\| \quad \asymp \quad \| \, f \, |L_2(\mathbb{R}^d)\| + \Big(\sum_{m=0}^{\infty} \Big(2^{rm}\, E^n_m(f, L_2(\mathbb{R}^d)) \Big)^2 \Big)^{1/2} .$$

Remark 19 For $m = 1$ we refer to Oswald [28]. Let us further mention that there is not much hope to generalize Corollary 3 to $p \neq 2$. In a slightly different setting (approximation by entire analytic functions with frequencies in the hyperbolic cross) it has been shown in [32] that the approximation spaces $A^r_{p,q}(\mathbb{R}^d)$ characterized by a condition

$$\| \, f \, |L_p(\mathbb{R}^d)\| + \Big(\sum_{m=0}^{\infty} \Big(2^{rm}\, \mathcal{E}_m(f, L_p(\mathbb{R}^d)) \Big)^q \Big)^{1/q} < \infty$$

do not belong to the scales of Besov-Lizorkin-Triebel spaces except $p = q = 2$. Here $\mathcal{E}_m(f, L_p(\mathbb{R}^d))$ has to be understood in a different context of approximation by entire analytic functions with frequencies in the hyperbolic cross. For the approximation spaces related to approximation from hyperbolic crosses we refer to DeVore, Konyagin, Temlyakov [11].

6.3 The Smolyak algorithm in the classical periodic situations

We return to the two examples from Subsection 4.3.

Theorem 10 *(see [34]) Let $1 < p < \infty$ and $r > 0$. Then*

$$\begin{aligned} & \| \, I - A(m,d,S) \, | S^r_{p,p} B(\mathbb{T}^d) \to L_p(\mathbb{T}^d) \| \\ & \qquad \asymp \quad \begin{cases} 2^{-mr} & \text{if} \quad 1 < p \le 2 \,, \\ m^{(d-1)(\frac{1}{2}-\frac{1}{p})} \, 2^{-mr} & \text{if} \quad 2 < p < \infty \,, \end{cases} \end{aligned}$$

$m \in \mathbb{N}_0$.

Theorem 11 *(see [34], [45]) Let $1 < p < \infty$ and $r > 1/p$. Then there is a constant $c > 0$ such that*

$$\| \, I - A(m,d,D) \, | S^r_{p,p} B(\mathbb{T}^d) \to L_p(\mathbb{T}^d) \| \, \le c \, m^{(d-1)(1-1/p)} \, 2^{-mr} \,, \qquad m \in \mathbb{N}_0 \,.$$

Now we turn to the efficiency of these Smolyak algorithms. Both operators $A(m,d,S)$ and $A(m,d,D)$ use approximately $K := m^{d-1} \, 2^m$ information about the function which is to be approximated. This implies

$$\begin{aligned} & \| \, I - A(m,d,S) \, | S^r_{p,p} B(\mathbb{T}^d) \to L_p(\mathbb{T}^d) \| \\ & \qquad \asymp \quad \begin{cases} K^{-r} \, (\log K)^{(d-1)r} & \text{if} \quad 1 < p \le 2 \,, \\ K^{-r} \, (\log K)^{(d-1)(r+\frac{1}{2}-\frac{1}{p})} & \text{if} \quad 2 < p < \infty \,, \end{cases} \end{aligned}$$

as well as

$$\| \, I - A(m,d,D) \, | S^r_{p,p} B(\mathbb{T}^d) \to L_p(\mathbb{T}^d) \| \, \le c \, K^{-r} \, (\log K)^{(d-1)(r+1-1/p)} \,.$$

The operator $A(m,d,S)$ is optimal in the sense of linear widths.

Proposition 6 *([38, Thm. 3.4.4]) Let $r > 0$. Then*

$$\lambda_K(S^r_{2,2} B(\mathbb{T}^d), L_2(\mathbb{T}^d)) \asymp K^{-r} \, (\log K)^{(d-1)r} \,, \qquad K \in \mathbb{N} \,.$$

Remark 20 Proposition 6 has a certain history. We mention only earlier work of Babenko, Mityagin and Galeev, see [38] for details.

6.4 Optimal Recovery of Functions from Besov Spaces of Dominating Mixed Smoothness

Let

$$\Psi_K(f,\xi)(x) := \sum_{j=1}^{K} f(\xi^j) \, \psi_j(x)$$

denote a general sampling operator for a class F of continuous, periodic functions defined on $\mathbb{T}^d$, where

$$\xi := \Big\{\xi^1, ..., \xi^K\Big\}, \qquad \xi^i \in \mathbb{T}^d, \quad i = 1, 2, \ldots, K,$$

is a fixed set of sampling points and $\psi_j : \mathbb{T}^d \to \mathbb{C}$, $j = 1, \ldots, K$, are fixed, continuous, periodic functions. Then the quantity

$$\rho_K(F, L_p(\mathbb{T}^d)) := \inf_{\xi} \inf_{\psi_1,\ldots,\psi_K} \sup_{\|f|F\|\le 1} \| f - \Psi_K(f,\xi) \,|L_p(\mathbb{T}^d)\|$$

measures the optimal rate of approximate recovery of the functions taken from F. We are interested in the case, when $F = S^r_{p,p}B(\mathbb{T}^d)$, $1 \le p \le \infty$, $r > 1/p$.

Theorem 12 *(see [34], [45]) Let $1 < p < \infty$ and $r > 1/p$. Then there exist positive constants c_1 and c_2 such that for all $K \in \mathbb{N}$*

$$\begin{aligned} c_1\, K^{-r} (\log K)^{(d-1)r}\, \eta(K,d,p) &\le \rho_K(S^r_{p,p}B(\mathbb{T}^d), L_p(\mathbb{T}^d)) \\ &\le c_2\, K^{-r}\, (\log K)^{(d-1)(r+1-1/p)}\,, \end{aligned}$$

where

$$\eta(K,d,p) := \begin{cases} (\log K)^{(d-1)(\frac{1}{2}-\frac{1}{p})} & \text{if} \quad 2 \le p < \infty\,, \\ 1 & \text{if} \quad 1 < p < 2\,. \end{cases} \tag{43}$$

Remark 21 The Smolyak algorithm uses samples of a very specific structure. Theorem 12 tells us that allowing arbitrary sets of sampling points of the same cardinality we can not do much better. The difference is at most $(\log K)^{(d-1)/2}$.

6.5 A final remark to the approximation of eigenfunctions

The classes $S_2^{1,\cdots 1}W(\mathbb{R}^d) = S_{2,2}^{1,\cdots,1}B(\mathbb{R}^d)$ allow a characterization in terms of the Fourier transform, see [33, Chapt. 2]. In fact

$$\| f\|^* := \Big(\int_{\mathbb{R}^d} \prod_{i=1}^d (1+|\xi_i|^2)\, |\mathcal{F}f(\xi)|^2\, d\xi\Big)^{1/2}$$

is an equivalent norm. If we compare this with the extra regularity of the eigenfunctions of the Hamilton operator then we are close. Switching to the tensor product

$$W^1_2(\mathbb{R}^3) \otimes_{\alpha_2} W^1_2(\mathbb{R}^3) \otimes_{\alpha_2} \ldots \otimes_{\alpha_2} W^1_2(\mathbb{R}^3)$$

then this space can be characterized by using the norm

$$\| f\|^{**} := \Big(\int_{\mathbb{R}^{3N}} \prod_{i=1}^{N} (1+|\xi_i|^2)\, |\mathcal{F}f(\xi)|^2\, d\xi \Big)^{1/2}, \qquad \xi_i := (\xi_i^1, \xi_i^2, \xi_i^3)\,.$$

A further modification by considering

$$W_2^2(\mathbb{R}^3) \otimes_{\alpha_2} W_2^1(\mathbb{R}^3) \otimes_{\alpha_2} \ \ldots \otimes_{\alpha_2} W_2^1(\mathbb{R}^3)$$

leads to the norm

$$\| f\|^{***} := \Big(\int_{\mathbb{R}^{3N}} (1+|\xi_1|^2)^2 \prod_{i=2}^{N} (1+|\xi_i|^2)\, |\mathcal{F}f(\xi)|^2\, d\xi \Big)^{1/2}.$$

For eigenfunctions $u \in H^1(\mathbb{R}^{3N})$ of physical relevance the latter norm is finite. Because of their exponential decay it will be enough to approximate them on some cube $Q := [-M, M]^d$. Daubechies wavelets have a compact support. If we apply the Smolyak algorithm with respect to the partial sum operator of the Daubechies expansion then inside Q we need to consider only those summands $\langle u, \psi_{\bar{j},\bar{k}} \rangle\, \psi_{\bar{j},\bar{k}}$, where

$$Q \cap \mathrm{supp}\ \psi_{\bar{j},\bar{k}} \neq \emptyset\,.$$

This leads to

$$\sum_{j_1+\ldots+j_N \le m} \ \sum_{\substack{|(k_i^1,k_i^2,k_i^3)| \le 2^{j_i} \widetilde{M} \\ i=1,\ldots,N}} \langle u, \psi_{\bar{j},\bar{k}} \rangle\, \psi_{\bar{j},\bar{k}}$$

for some $\widetilde{M} > M$ but independent of m. The number of information we use is at most $K \asymp m^{3N-1}\, 2^m$. By using the same principles as above we obtain the estimate

$$\| I - A(m,d,L)\, |W_2^1(Q) \otimes_{\alpha_2} \ \ldots \otimes_{\alpha_2} W_2^1(Q) \to L_2(Q)\| \asymp \ K^{-1}\, (\log K)^{(3N-1)}\,. \qquad (44)$$

The influence of the dimension is only of logarithmic order.
Now we turn back to our examples from the very beginning. By ignoring the constants behind $\asymp$ we find with $K = 10^{50}$

$$\frac{(\log K)^{29}}{K} \asymp 0.186\, (\log 10)^{29}\,.$$

We had not fixed the basis of the logarithm before. Changing the basis of the logarithm means we change the constant $C(3N)$ in (44). As long as we do not have estimates of $C(d)$ we can not give precise bounds for the error !

6.6 Open problems

There are many, of course. Here are two of some importance.

- Calculate $C(d)$ in Theorems 7-12
- Estimates of $C(d)$ from below and above
- Search for specific regularity properties of solutions and introduce new classes of functions (they should be smaller than the usual Sobolev or Besov spaces). One possible approach uses the philosophy that not all variables are equally important. Problems of such type occur in the mathematics of finance. This leads to weighted spaces. We refer to the monographs of Novak and Wozniakowski [26] for a very recent overview.

In a particular part of mathematics, namely the *Information based complexity*, there is even some interest to switch from *highdimensional* to $d \to \infty$. Then the asymptotic behaviour of $C(d)$ if d tends to infinity is of interest. We omit details and refer to [26].

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