TU Chemnitz:

2008 Summer School on Applied Analysis

Four lectures on

Theory and numerical analysis of Volterra functional equations

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Lecture I:
Theory of Volterra functional equations

Classical and delay Volterra integral operators:

- $$(\mathcal{V}\mathbf{u})(\mathbf{t}) := \int_0^{\mathbf{t}} \mathbf{K_0}(\mathbf{t},\mathbf{s})\mathbf{u}(\mathbf{s})\,\mathbf{ds}, \quad \mathbf{t} \in \mathbf{I}$$

- $$(\mathcal{V}_\theta\mathbf{u})(\mathbf{t}) := \int_0^{\theta(\mathbf{t})} \mathbf{K_1}(\mathbf{t},\mathbf{s})\mathbf{u}(\mathbf{s})\,\mathbf{ds}, \quad \mathbf{t} \in \mathbf{I}$$

- $$(\mathcal{W}_\theta\mathbf{u})(\mathbf{t}) := \int_{\theta(\mathbf{t})}^{\mathbf{t}} \mathbf{K}(\mathbf{t},\mathbf{s})\mathbf{u}(\mathbf{s})\,\mathbf{ds}, \quad \mathbf{t} \in \mathbf{I}$$

Here, $\mathbf{t} \in \mathbf{I} := [\mathbf{0}, \mathbf{T}]$, and the *delay function* (or: *lag function*) θ has the form

$$\theta(\mathbf{t}) := \mathbf{t} - \tau(\mathbf{t}) \; .$$

We refer to τ as the *delay*.

- *Non-vanishing delay:* $\tau(\mathbf{t}) \geq \tau_0 > \mathbf{0}$ $(\mathbf{t} \in \mathbf{I})$

- *Vanishing delay:* $\tau(\mathbf{0}) = \mathbf{0},\; \tau(\mathbf{t}) > \mathbf{0}$ $(\mathbf{t} > \mathbf{0})$

Volterra functional equations

- Volterra functional integral equations (VFIEs):

$$\mathbf{u(t) = g(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \quad t \in I}$$

- Volterra functional integro-differential equations (VFIDEs):

$$\mathbf{u'(t) = a(t)u(t) + b(t)u(\theta(t)) + g(t)}$$

$$\mathbf{+(\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \quad t \in I}$$

$\hookrightarrow$ *Special case:* **Delay differential equation** (DDE):

$$\mathbf{u'(t) = a(t)u(t) + b(t)u(\theta(t)) + g(t), \quad t \in I}$$

- First-kind VFIE:

$$\mathbf{(\mathcal{W}_\theta u)(t) = g(t), \quad t \in I}$$

$\hookrightarrow$ $\mathbf{\theta(t) = qt}$ $\mathbf{(0 < q < 1)}$: **Volterra** (1897)

Vito Volterra (1860 - 1940)

DDEs; Effect of delay on solutions

Exercise

The solutions of the **ODE**

$$\mathbf{u'(t) = au(t), \quad t \geq 0; \quad \text{Re}(a) < 0,}$$

satisfies

$$\lim_{\mathbf{t}\to\infty} \mathbf{u(t) = 0}.$$

What is the **asymptotic behaviour**, as $\mathbf{t} \to \infty$, of the solutions to the **DDE**s

$$\mathbf{u'(t) = bu(t-\tau), \quad t \geq 0, \quad \text{Re}(b) < 0,}$$

with $\tau > \mathbf{0}$ and $\mathbf{u(t) = 1}$ if $\mathbf{t \leq 0}$, and

$$\mathbf{u'(t) = bu(qt), \quad t \geq 0, \quad \text{Re}(b) < 0,}$$

with $\mathbf{0 < q < 1}$ and $\mathbf{u(0) = u_0}$?

Illustration:

$$\mathbf{u'(t) = bu(qt), \quad u(0) = 1; \quad b < 0, \ 0 < q < 1.}$$

$\hookrightarrow$ The solution is given by

$$\mathbf{u(t) = \sum_{j=0}^{\infty} \frac{q^{j(j-1)/2}}{j!}(bt)^j, \quad t \geq 0} \ :$$

It is an **entire function** of **order zero**.
(*Comparison:* For $\mathbf{q = 1}$, the solution is an *entire function* of **order one**: $\mathbf{u(t)} = \exp(\mathbf{bt})$.)

<u>Example:</u> $\mathbf{b = -1, \ q = 0.95}$:

Properties of solutions of VFIEs and VFIDES

(Representation / regularity)

- Classical VIES:

$$\mathbf{u(t) = g(t) + \int_0^t K(t,s)u(s)\,ds, \quad t \in I := [0,T]}$$

<u>**Theorem**</u> (Volterra, 1896)
If $\mathbf{K \in C(D)}$ $(\mathbf{D := \{(t,s):\ 0 \le s \le t \le T\}})$, then for any $\mathbf{g \in C(I)}$ the VIE has a unique solution $\mathbf{u \in C(I)}$. This solution is given by

$$\mathbf{u(t) = g(t) + \int_0^t R(t,s)g(s)\,ds, \quad t \in I,}$$

where $\mathbf{R \in C(D)}$ denotes the **resolvent kernel** of $\mathbf{K}$:

$$\mathbf{R(t,s) := \sum_{j=1}^{\infty} K_j(t,s), \quad (t,s) \in D}$$

(*Neumann series* of $\mathbf{K}$). The *iterated kernels* $\mathbf{K_j}$ of $\mathbf{K}$ are defined by $\mathbf{K_1(t,s) := K(t,s)}$ and

$$\mathbf{K_{j+1}(t,s) := \int_s^t K(t,v)K_j(v,s)\,dv \quad (j \ge 1).}$$

Moreover,

$$\mathbf{K \in C^d(D)} \quad \text{and} \quad \mathbf{g \in C^d(I)} \quad \Rightarrow \quad \mathbf{u \in C^d(I).}$$

• VIES: with delay function $\theta(\mathbf{t}) = \mathbf{qt}$ $(\mathbf{0} < \mathbf{q} < \mathbf{1})$

$$\mathbf{u(t) = g(t) + \int_0^{qt} K(t,s)u(s)\,ds, \quad t \in I := [0,T]}$$

<u>**Theorem**</u> (Andreoli (1914); Chambers (1990))
If $\mathbf{K} \in \mathbf{C}(\mathbf{D}_\theta)$ $(\mathbf{D}_\theta := \{(\mathbf{t},\mathbf{s}):\ \mathbf{0} \le \mathbf{s} \le \theta(\mathbf{t})\ (\mathbf{t} \in \mathbf{I})\})$, then for any $\mathbf{g} \in \mathbf{C}(\mathbf{I})$ the VIE has a unique solution $\mathbf{u} \in \mathbf{C}(\mathbf{I})$. This solution is given by

$$\mathbf{u(t) = g(t) + \sum_{j=1}^{\infty} \int_0^{q^j t} K_j(t,s)g(s)\,ds, \quad t \in I\,.}$$

The *iterated kernels* $\mathbf{K_j}$ of $\mathbf{K}$ are defined by $\mathbf{K_1(t,s) := K(t,s)}$ and

$$\mathbf{K_{j+1}(t,s) := \int_{q^{-j}s}^{qt} K(t,v)K_j(v,s)\,dv \quad (j \ge 1).}$$

For $\mathbf{0 < q < 1}$ the kernel $\mathbf{K}$ <u>does not</u> have a *Neumann series* !

However, as for classical VIEs,

$$\mathbf{K \in C^d(D_\theta) \quad \text{and} \quad g \in C^d(I) \quad \Rightarrow \quad u \in C^d(I).}$$

- Classical VIDEs:

$$\mathbf{u}'(\mathbf{t}) = \mathbf{a}(\mathbf{t})\mathbf{u}(\mathbf{t}) + \mathbf{g}(\mathbf{t}) + \int_0^{\mathbf{t}} \mathbf{K}(\mathbf{t},\mathbf{s})\mathbf{u}(\mathbf{s})\,\mathbf{ds}, \quad \mathbf{t} \in \mathbf{I}$$

$\Rightarrow$ VIDE is equivalent to VIE

$$\mathbf{u}(\mathbf{t}) = \mathbf{g_0}(\mathbf{t}) + \int_0^{\mathbf{t}} \mathbf{H}(\mathbf{t},\mathbf{s})\mathbf{u}(\mathbf{s})\,\mathbf{ds},$$

with

$$\mathbf{g_0}(\mathbf{t}) := \mathbf{u_0} + \int_0^{\mathbf{t}} \mathbf{g}(\mathbf{s})\,\mathbf{ds}$$

and

$$\mathbf{H}(\mathbf{t},\mathbf{s}) := \mathbf{a}(\mathbf{s}) + \int_{\mathbf{s}}^{\mathbf{t}} \mathbf{K}(\mathbf{v},\mathbf{s})\,\mathbf{dv}\ .$$

Theorem: (Grossman & Miller (1970))
If $\mathbf{a} \in \mathbf{C}(\mathbf{I})$ and $\mathbf{K} \in \mathbf{C}(\mathbf{D})$, then for any $\mathbf{g} \in \mathbf{C}(\mathbf{I})$ and any $\mathbf{u_0} \in \mathbf{R}$ the VIDE has a unique solution $\mathbf{u} \in \mathbf{C}(\mathbf{I})$ satisfying $\mathbf{u}(\mathbf{0}) = \mathbf{u_0}$. This solution is given by

$$\mathbf{u}(\mathbf{t}) = \mathbf{r}(\mathbf{t},\mathbf{0})\mathbf{u_0} + \int_0^{\mathbf{t}} \mathbf{r}(\mathbf{t},\mathbf{s})\mathbf{g}(\mathbf{s})\,\mathbf{ds}, \quad \mathbf{t} \in \mathbf{I},$$

where the (differential) resolvent kernel $\mathbf{r}(\mathbf{t},\mathbf{s})$ depends on $\mathbf{a}$ and $\mathbf{K}$.

Moreover,

$\mathbf{a},\ \mathbf{g} \in \mathbf{C}^{\mathbf{d}}(\mathbf{I})$ and $\mathbf{K} \in \mathbf{C}^{\mathbf{d}}(\mathbf{D}) \ \Rightarrow\ \mathbf{u} \in \mathbf{C}^{\mathbf{d+1}}(\mathbf{I})$.

- VFIDEs with delay function $\theta(t) = qt \ (0 < q < 1)$

$$\mathbf{u}'(\mathbf{t}) = \mathbf{b}(\mathbf{t})\mathbf{u}(\theta(\mathbf{t})) + \mathbf{g}(\mathbf{t}) + \int_0^{\theta(\mathbf{t})} \mathbf{K}(\mathbf{t},\mathbf{s})\mathbf{u}(\mathbf{s})\,\mathbf{ds},$$

with $\mathbf{t} \in \mathbf{I} := [\mathbf{0},\mathbf{T}]$ and $\mathbf{u}(\mathbf{0}) = \mathbf{u_0}$.

Theorem: (Brunner & Hu (2007))
Assume that $\mathbf{b},\ \mathbf{g} \in \mathbf{C^d}(\mathbf{I})$ and $\mathbf{K} \in \mathbf{C^d}(\mathbf{D}_\theta)$. Then for each $\mathbf{q} \in (\mathbf{0},\mathbf{1})$ and given $\mathbf{u_0}$ the VFIDE has a unique solution $\mathbf{u} \in \mathbf{C^{d+1}}(\mathbf{I})$ that satisfies $\mathbf{u}(\mathbf{0}) = \mathbf{u_0}$.
This solution has the representation

$$\mathbf{u}(\mathbf{t}) = \left(\mathbf{1} + \sum_{\mathbf{j}=\mathbf{1}}^{\infty} \tilde{\mathbf{H}}_{\mathbf{j}}(\mathbf{t},\mathbf{s})\,\mathbf{ds}\right)\mathbf{u_0} + \int_0^{\mathbf{t}} \mathbf{g}(\mathbf{s})\,\mathbf{ds}$$

$$+ \sum_{\mathbf{j}=\mathbf{1}}^{\infty} \int_0^{\theta^{\mathbf{j}}(\mathbf{t})} \tilde{\mathbf{H}}_{\mathbf{j}}(\mathbf{t},\mathbf{s})\mathbf{g}(\mathbf{s})\,\mathbf{ds}, \quad \mathbf{t} \in \mathbf{I}.$$

where

$$\tilde{\mathbf{H}}_{\mathbf{j}}(\mathbf{t},\mathbf{s}) := \int_{\mathbf{s}}^{\theta^{\mathbf{j}}(\mathbf{t})} \mathbf{H}_{\mathbf{j}}(\mathbf{t},\mathbf{v})\,\mathbf{dv} \ \ (j \geq 1)\ .$$

Here, the $\mathbf{H}_{\mathbf{j}}(\mathbf{t},\mathbf{s})$ are the iterated kernels of

$$\mathbf{H_1}(\mathbf{t},\mathbf{s}) := \mathbf{b}(\theta^{-1}(\mathbf{s}))\theta'(\theta^{-1}(\mathbf{s})) + \int_{\theta^{-1}(\mathbf{s})}^{\mathbf{t}} \mathbf{K}(\mathbf{v},\mathbf{s})\,\mathbf{dv}\ .$$

- **Summary:**

If $\theta(t) = qt \ (0 < q < 1)$, or if θ is **nonlinear** and satisfies

(i) $\theta(0) = 0$, θ is strictly increasing on I, and

(ii) $\theta(t) \leq q_1 t$ for some $q_1 \in (0, 1)$, then

Smooth data $\Rightarrow$ Solution of VFE is (globally) smooth on $[0, T]$.

- **VFEs with non-vanishing delays**

Assume: the delay function $\theta(t) = t - \tau(t)$ satisfies:

(D1) $\tau(t) \geq \tau_0 > 0$ for $t \in I := [t_0, T]$

(D2) θ is *strictly increasing* on I;

(D3) $\tau \in C^d(I)$ for some $d \geq 0$.

Definition:

The points $\{\xi_\mu\}$ generated by

$$\theta(\xi_\mu) = \xi_{\mu-1}, \quad \mu \geq 1 \quad (\xi_0 := t_0),$$

are called the **primary discontinuity points** (or: *breaking points*) induced by the delay function θ. $\hookrightarrow$ By (D1): $\xi_\mu - \xi_{\mu-1} \geq \tau_0$ $(\mu \geq 1)$.

$\hookrightarrow$ Assume: T such that

$$T = \xi_{M+1} \quad \text{for some} \quad M \geq 1.$$

$\hookrightarrow$ **'Method of steps' :**

Solve VFE on $\mathbf{I}^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ $(\mu = \mathbf{0}, \ldots, \mathbf{M})$.

<u>Illustration:</u>

$$\mathbf{u}'(\mathbf{t}) = \mathbf{a}\mathbf{u}(\mathbf{t}) + \mathbf{b}\mathbf{u}(\theta(\mathbf{t})) + \int_{\mathbf{0}}^{\theta(\mathbf{t})} \mathbf{K}(\mathbf{t},\mathbf{s})\mathbf{u}(\mathbf{s})\,\mathbf{ds},$$

$\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$, with $\mathbf{u}(\mathbf{t}) = \phi(\mathbf{t})$ for $\mathbf{t} \leq \mathbf{0}$.

For $\underline{\mathbf{t} \in \mathbf{I}^{(\mathbf{0})}} := [\mathbf{0}, \xi_{\mathbf{1}}]$; $(\Rightarrow\ \theta(\mathbf{t}) \in \mathbf{I}^{(-\mathbf{1})} := [\theta(\mathbf{0}), \mathbf{0}])$:

$$\mathbf{u}'(\mathbf{t}) = \mathbf{a}\mathbf{u}(\mathbf{t}) + \mathbf{\Phi}_{\mathbf{0}}(\mathbf{t}).$$

where

$$\mathbf{\Phi}_{\mathbf{0}}(\mathbf{t}) := \mathbf{b}\phi(\theta(\mathbf{t})) + \int_{\mathbf{0}}^{\theta(\mathbf{t})} \mathbf{K}(\mathbf{t},\mathbf{s})\phi(\mathbf{s})\,\mathbf{ds}$$

is known.

For $\underline{\mathbf{t} \in \mathbf{I}^{(\mu)}} := [\xi_\mu, \xi_{\mu+\mathbf{1}}]$, $(\mu = \mathbf{1}, \ldots, \mathbf{M})$:

$$\mathbf{u}'(\mathbf{t}) = \mathbf{a}\mathbf{u}(\mathbf{t}) + \mathbf{\Phi}_\mu(\mathbf{t}).$$

with known

$$\mathbf{\Phi}_\mu(\mathbf{t}) := \mathbf{b}\mathbf{u}(\theta(\mathbf{t})) + \int_{\mathbf{0}}^{\theta(\mathbf{t})} \mathbf{K}(\mathbf{t},\mathbf{s})\mathbf{u}(\mathbf{s})\,\mathbf{ds}.$$

<u>Question:</u> *Regularity* of solution $\mathbf{u}(\mathbf{t})$ at ξ_μ ?

Representation of solutions:
Let $\theta(t) = t - \tau(t)$, $\tau(t) \geq \tau_0 > 0$. For continuous data, the solution of the **VFIDE**

$$u'(t) = a(t)u(t) + b(t)u(\theta(t)) + g(t)$$

$$+(\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \quad t \in I$$

(with $u(t) = \phi(t)$, $t \leq t_0$) on $I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ is given by

$$u(t) = r_1(r, \xi_\mu)u(\xi_\mu) + \int_{\xi_\mu}^{t} r_1(t,s)g(s)\,ds$$

$$+F_\mu(t) + \Phi_\mu(t), \quad t \in I^{(\mu)}.$$

Here,

$$F_\mu(t) := \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\xi_{\nu+1}} r_{\mu,\nu}(t,s)g(s)\,ds$$

$$\sum_{\nu=0}^{\mu-1} p_{\mu,\nu}(t)u(\xi_\nu) + G_\mu^{(1)}(t;\phi)$$

and

$$\Phi_\mu(t) := \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\theta^{\mu-\nu}(t)} r_{\mu,\nu}(t,s)g(s)\,ds + G_\mu^{(2)}(t;\phi).$$

Non-vanishing $\tau(t)$: **Regularity results**

- **VFIDEs** (and **DDEs**):

$$\mathbf{u}'(\mathbf{t}) = \mathbf{a}\mathbf{u}(\mathbf{t}) + \mathbf{b}\mathbf{u}(\theta(\mathbf{t})) + (\mathcal{V}\mathbf{u})(\mathbf{t}) + (\mathcal{V}_\theta \mathbf{u})(\mathbf{t}) :$$

Theorem: (*Smoothing* of solutions)
Assume that the given functions are arbitrarily *smooth* , and $\mathcal{V}_\theta \neq \mathbf{0}$. Then at $\mathbf{t} = \xi_\mu$,

$$\mathbf{u} \in \mathbf{C}^\mu \quad \textbf{but} \quad \mathbf{u} \notin \mathbf{C}^{\mu+1} \quad (\mu = 0, \ldots, M).$$

If $b \equiv 0$, then at $\mathbf{t} = \xi_\mu$ $(\mu = 0, \ldots, M)$,

$$\mathbf{u} \in \mathbf{C}^{\mathbf{2}\mu} \qquad ('super - smoothing').$$

Neutral VFIDE:

$$\mathbf{u}'(\mathbf{t}) = \mathbf{a}\mathbf{u}(\mathbf{t}) + \mathbf{b}\mathbf{u}(\theta(\mathbf{t})) + \underline{cu'(\theta(t))}$$

$$+(\mathcal{V}\mathbf{u})(\mathbf{t}) + (\mathcal{V}_\theta \mathbf{u})(\mathbf{t}) \ :$$

Theorem: (*Non-smoothing* of solutions)
If given functions are smooth and $c \neq 0$, then at $\mathbf{t} = \xi_\mu$ $(\mu = \mathbf{0}, \mathbf{1}, \ldots, \mathbf{M})$,

$$\mathbf{u} \in \mathbf{C}^\mathbf{0} \qquad \text{but} \qquad \mathbf{u} \notin \mathbf{C}^\mathbf{1} \ :$$

there is **no smoothing** at $\mathrm{t} = \xi_\mu$ as μ increases.

• **VFIEs** with non-vanishing delay: :

$$\mathbf{u(t) = g(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t)},$$

$\mathbf{t \in (0, T]}$, with $\mathbf{u(t) = \phi(t)}$ for $\mathbf{t \leq 0}$.

<u>**Theorem:**</u> (*Smoothing* of solutions)
For smooth data and $\mathbf{V_\theta \neq 0}$, the solution satisfies

$$\mathbf{u \in C^{\mu-1}} \qquad \text{but} \qquad \mathbf{u \notin C^{\mu}}$$

for $\mu = \underline{1}, \ldots, \mathbf{M}$. At $\mathbf{t = \xi_0 = 0}$ the solution is in general **discontinuous**; that is, $\mathbf{u}$ has a **finite jump** at $\mathbf{t = \xi_0}$ (except for specially chosen initial functions ϕ).

(**Note:**

$$\mathbf{u(0^-) = \phi(0)}. \quad \mathbf{u(0^+) = g(0) - \int_{\theta(0)}^{0} K_1(0,s)\phi(s)\, ds} \ )$$

<u>**Exercise:**</u> Regularity of solution of

$$\mathbf{u(t) = g(t)} + \underline{b(t)u(\theta(t))} + \mathbf{(\mathcal{W}_\theta u)(t)},$$

where

$$\mathbf{(\mathcal{W}_\theta u)(t) := \int_{\theta(t)}^{t} K(t,s)u(s)\, ds} \ ?$$

- **State-dependent delays**

Example:
Mathematical model of population whose *life span* depends on the size of the population (*crowding sffects*) (Bélair, 1990):

$$\mathbf{u}(\mathbf{t}) = \int_{\mathbf{t}-\tau(\mathbf{y}(\mathbf{t}))}^{\mathbf{t}} \mathbf{k}(\mathbf{t}-\mathbf{s})\mathbf{G}(\mathbf{u}(\mathbf{s}))\,\mathbf{ds}, \quad \mathbf{t} > \mathbf{0},$$

with $\mathbf{u}(\mathbf{t}) = \phi(\mathbf{t})$ for $\mathbf{t} \leq \mathbf{0}$.

$\hookrightarrow$ **Survey** of state-dependent DDEs:

F. Hartung, T. Krisztin, H.-O. Walther & J. Wu, Functional differential equations with state-dependent delays: theory and applications, in: *Handbook of Differential Equations: Ordinary Differential Equations*, Vol. 3 (A. Cañada *et al.*, eds.), pp. 435-545, Elsevier, 2006.

$\hookrightarrow$ The theory and **numerical analysis** of **state-dependent VFEs** and **VFIDEs** remain to be established !

Lecture I: **Basic references**

H. Brunner, *Lecture Notes*, 2008 Summer School on Applied Analysis, TU Chemnitz: Sections 1 and 2.

J.K. Hale & S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, 1993.

T. Kato & J.B. McLeod, The functional differential equation $y'(x) = ay(\lambda x) + by(x)$, *Bull. Amer. Math. Soc.* **77** (1971), 891-937.

A. Iserles, On the generalized pantograph functional differential equation, *Europ. J. Appl. Math.* **4** (1993), 1-33.

A. Bellen & M. Zennaro, *Numerical Methods for Delay Differential Equations*, Oxford University Press, 2003. (Chapters 1 and 2)

H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press, 2004. (Chapters 2-5)

Lecture II: Collocation in piecewise polynomial spaces

Mesh (or: *grid*) on $\mathbf{I} := [\mathbf{t_0}, \mathbf{T}]$:

$$\mathbf{I_h} := \{\mathbf{t_n} : \ \mathbf{t_0} < \mathbf{t_1} < \cdots < \mathbf{t_N} = \mathbf{T}\},$$

with

$$\mathbf{e_n} := (\mathbf{t_n}, \mathbf{t_{n+1}}], \quad \mathbf{h_n} := \mathbf{t_{n+1}} - \mathbf{t_n};$$

$\mathbf{h} :=$ max $\{\mathbf{h_n} : \ \mathbf{0} \le \mathbf{n} \le \mathbf{N} - \mathbf{1}\}$ is called the *mesh diameter*.

Definition: For given integers $\mathbf{r} \ge \mathbf{1}, \ -\mathbf{1} \le \mathbf{d} < \mathbf{r}$,

$$\mathbf{S_r^{(d)}(I_h)} := \{\mathbf{v} \in \mathbf{C^d(I)} : \ \mathbf{v}|_{\mathbf{e_n}} \in \pi_{\mathbf{r}} \ (\mathbf{0} \le \mathbf{n} \le \mathbf{N} - \mathbf{1})\}$$

denotes the space of **piecewise polynomials** (with respect to the given mesh $\mathbf{I_h}$) of **degree** $\mathbf{r}$; if $\mathbf{d} \ge \mathbf{0}$ these functions are *globally* in $\mathbf{C^d(I)}$.

$\hookrightarrow$ dim $\mathbf{S_r^{(d)}(I_h)} = \mathbf{N}(\mathbf{r} - \mathbf{d}) + (\mathbf{d} + \mathbf{1})$.

For $\underline{\mathbf{d} = -\mathbf{1}}$,

$$\mathbf{S_r^{(-1)}(I_h)} := \{\mathbf{v} : \ \mathbf{v}|_{\mathbf{e_n}} \in \pi_{\mathbf{r}} \ (\mathbf{0} \le \mathbf{n} \le \mathbf{N} - \mathbf{1})\}.$$

Illustration:

Approximation of the solution of the **ODE**

$$\mathbf{u'(t) = f(t, u(t)),\ \ t \in [0, T];\ \ u(0) = u_0},$$

by **collocation** in $\mathbf{S_m^{(0)}(I_h)}$ ($r = m$, $d = 0$).
Since dim $\mathbf{S_m^{(0)}(I_h) = Nm + 1}$, choose

$$\mathbf{X_h} :=$$

$$\mathbf{\{t_n + c_i h_n :\ 0 < c_1 < \cdots < c_m \le 1\ (0 \le n \le N-1)\}}$$

as **collocation points** ($\Rightarrow$ $|\mathbf{X_h}| = \mathbf{Nm}$),
$\hookrightarrow$ Find $\mathbf{u_h \in S_m^{(0)}(I_h)}$ satisfying the ODE on the *finite subset* $\mathbf{X_h}$ of $\mathbf{[0, T]}$:

$$\mathbf{u_h'(t) = f(t, u_h(t))} \quad \text{for all} \quad \underline{\mathbf{t \in X_h}},$$

with $\mathbf{u_h(0) = u_0}$.

Remark:

For kth-order ODEs ($\mathbf{k \ge 2}$),

$$\mathbf{u^{(k)}(t) = f(t, u(t), \ldots, u^{(k-1)}(t))},$$

choose the collocation space

$$\mathbf{S_{m+d}^{(d)}(I_h)} \quad \text{with} \quad \underline{\mathbf{d := k - 1}},$$

and the <u>same set</u> of collocation points $\mathbf{X_h}$ (since

$$\text{dim}\ \mathbf{S_{m+d}^{(d)}(I_h) = Nm + d + 1 = Nm + k}\ ).$$

Questions:

- Collocation for ODEs (and VEs) in **smoother** piecewise polynomial spaces:

$$\mathbf{S_r^{(d)}(I_h)} \quad \text{with} \quad \mathbf{d \geq 1} \;\; (d < r)\ ?$$

- **Computational form** of collocation equation ?

- **Global** order of convergence (on $\mathbf{I}$):

$$\|\mathbf{u} - \mathbf{u_h}\|_\infty \leq \mathbf{Ch^p} \;:\; \mathbf{p} = \,?$$

- **Local** order of convergence (on $\mathbf{I_h}$):

$$\max\{|\mathbf{u(t)} - \mathbf{u_h(t)}| : \mathbf{t} \in \mathbf{I_h}\} \leq \mathbf{Ch^{p^*}} \;:\; \mathbf{p^* > p}\ ?$$

$\hookrightarrow$ *Local superconvergence* on $\mathbf{I_h}$?

- Do the above optimal orders remain true for **VFEs** ?

• **Computational form of collocation equation**: Let

$$\mathbf{L_j(v)} := \prod_{k\neq j}^{m} \frac{\mathbf{v - c_k}}{\mathbf{c_j - c_k}}\,, \quad \mathbf{v} \in [\mathbf{0}, \mathbf{1}] \quad (\mathbf{j} = \mathbf{1}, \ldots, \mathbf{m})$$

denote the **Lagrange** canonical polynomials with respect to the *collocation parameters* $\{\mathbf{c_i}\}$. Setting $\mathbf{Y_{n,j}} := \mathbf{u'_h(t_n + c_j h_n)}$ and

$$\mathbf{u'_h(t_n + vh_n)} = \sum_{\mathbf{j=1}}^{\mathbf{m}} \mathbf{L_j(v) Y_{n,j}}, \quad \mathbf{v} \in (\mathbf{0}, \mathbf{1}],$$

we obtain the **local representation** of the collocation solution $\mathbf{u_h} \in \mathbf{S_m^{(0)}(I_h)}$ on the subinterval $[\mathbf{t_n}, \mathbf{t_{n+1}}]$:

$$\mathbf{u_h(t_n + vh_n)} = \mathbf{u_h(t_n)} + \mathbf{h_n} \sum_{\mathbf{j=1}}^{\mathbf{m}} \beta_{\mathbf{j}}(\mathbf{v}) \mathbf{Y_{n,j}},\ \mathbf{v} \in [\mathbf{0}, \mathbf{1}],$$

with

$$\beta_{\mathbf{j}}(\mathbf{v}) := \int_{\mathbf{0}}^{\mathbf{v}} \mathbf{L_j(s)}\, \mathbf{ds}.$$

$\hookrightarrow$ Computation of $\{\mathbf{Y_{n,j}}\}$ ($0 \leq n \leq N-1$):

$$\mathbf{Y_{n,i}} = \mathbf{f}\left(\mathbf{t_n} + \mathbf{c_i h_n}, \mathbf{y_n} + \mathbf{h_n} \sum_{\mathbf{j=1}}^{\mathbf{m}} \mathbf{a_{i,j} Y_{n,j}}\right) (i = 1, \ldots, m)$$

where $\mathbf{y_n} := \mathbf{u_h(t_n)}$ and $\mathbf{a_{i,j}} := \beta_{\mathbf{j}}(\mathbf{c_i})$.

$\hookrightarrow$ The pair of equations (for $\mathbf{0 \le n \le N-1}$):

$$\mathbf{u_h(t_n + vh_n) = u_h(t_n) + h_n \sum_{j=1}^{m} \beta_j(v) Y_{n,j},\ v \in [0,1]}$$

(**local representation** of the collocation solution $\mathbf{u_h \in S_m^{(0)}(I_h)}$ on the subinterval $\mathbf{[t_n, t_{n+1}]}$)
<u>*and*</u>

$$\mathbf{Y_{n,i} = f(t_n + c_i h_n, y_n + h \sum_{j=1}^{m} a_{i,j} Y_{n,j})}\ (i = 1, \dots, m)$$

(**collocation equations** for $\mathbf{t = t_n + c_i h_n}$)
represent an $\mathbf{m}$-stage **continuous implicit Runge-Kutta method** for solving the **ODE** initial-value problem

$$\mathbf{u'(t) = f(t, u(t)), \quad t \in [0,T]; \quad u(0) = u_0.}$$

For *arbitrary* $\mathbf{\{c_i\}}$ (and $\mathbf{u \in C^d(I)}$ with $\mathbf{d \ge m+1}$):

$$\mathbf{\|u^{(k)} - u_h^{(k)}\|_\infty \le Ch^m} \quad (k = 0, 1).$$

$\hookrightarrow$ **Question:**
Choice of collocation parameters $\mathbf{\{c_i\}}$?

Convergence results for **ODEs**: $\mathbf{u_h} \in \mathbf{S_m^{(0)}(I_h)}$.

- If $\mathbf{u} \in \mathbf{C^{m+1}(I)}$:

$\|\mathbf{u^{(k)}} - \mathbf{u_h^{(k)}}\|_\infty \leq \mathbf{Ch^m}$ $(\mathbf{k = 0, 1})$ for arbitrary $\{\mathbf{c_i}\}$.

Let

$$\mathbf{J}_\nu := \int_0^1 \mathbf{s}^\nu \prod_{\mathbf{i=1}}^{\mathbf{m}} (\mathbf{s} - \mathbf{c_i})\, \mathbf{ds} \quad (\nu \in \mathbb{N}).$$

- If $\mathbf{u} \in \mathbf{C^{m+2}(I)}$ and $\mathbf{J_0 = 0}$:

$$\|\mathbf{u} - \mathbf{u_h}\|_\infty \leq \mathbf{Ch^{m+1}}.$$

- Let $\mathbf{u} \in \mathbf{C^{m+\kappa+1}(I)}$ $(\kappa \leq \mathbf{m})$.

If $\mathbf{J}_\nu = \mathbf{0}$, $\nu = \mathbf{0}, \ldots, \kappa - \mathbf{1}$, and $\mathbf{J}_\kappa \neq \mathbf{0}$:

$$\max\{|\mathbf{u(t)} - \mathbf{u_h(t)}| : \underline{\mathbf{t} \in \mathbf{I_h}}\} \leq \mathbf{Ch^{m+\kappa}}.$$

$\underline{\kappa = \mathbf{m}}$: $\Rightarrow$ $\{\mathbf{c_i}\}$ are the **Gauss points¿**

$$\max\{|\mathbf{u(t)} - \mathbf{u_h(t)}| : \underline{\mathbf{t} \in \mathbf{I_h}}\} \leq \mathbf{Ch^{2m}}.$$

The underlying method is the $\mathbf{m}$-stage continuous implicit **Runge-Kutta-Gauss method**.

Why $\mathcal{O}(\mathbf{h^{2m}})$ -convergence on $\mathbf{I_h}$?

Illustration:

$$\mathbf{u'(t) = a(t)u(t) + g(t), \ \ t \in I; \ \ u(0) = u_0}$$

Collocation equation: $\mathbf{u_h \in S_m^{(-1)}(I_h)}$:

$$\mathbf{u'_h(t) = a(t)u_h(t) + g(t) - \delta_h(t), \ t \in I; \ u_h(0) = u_0,}$$

where the **defect function** $\delta_\mathbf{h}$ vanishes at the collocation points $\mathbf{t_n + c_i h_n}$:

$$\delta_\mathbf{h}(\mathbf{t}) = \mathbf{0} \quad \textbf{for all} \quad \mathbf{t \in X_h} \ .$$

$\Rightarrow$ **Collocation error** $\mathbf{e_h := u - u_h}$ satisfies

$$\mathbf{e'_h(t) = a(t)e_h(t) + \delta_h(t), \ \ t \in I; \ \ e_h(0) = 0.}$$

Thus, setting $\mathbf{r(t,s)} := \exp\left(\int_\mathbf{s}^\mathbf{t} \mathbf{a(v)\,dv}\right)$:

$$\mathbf{e_h(t) = \int_0^t r(t,s)\delta_h(s)\,ds, \ \ t \in I.}$$

$\hookrightarrow$ For $\underline{\mathbf{t = t_n \in I_h}}$:

$$\mathbf{e_h(t_n) = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 r(t_n, t_\ell + s h_\ell)\delta_h(t_\ell + s h_\ell)\,ds.}$$

$\hookrightarrow$ Recall that

$$\mathbf{\delta_h(t_\ell + c_i h_\ell) = 0} \quad \text{for } \mathbf{i = 1, \ldots, m; \ 0 \le \ell \le N - 1.}$$

$\hookrightarrow$ $\mathbf{m}$-point interpolatory quadrature formula:
Abscissas $\{\mathbf{d_j}\}$ with $\mathbf{0 \le d_1 < \cdots < d_m \le 1}$:

$$\int_0^1 \phi(\mathbf{t_n + sh_n})\,\mathbf{ds} = \sum_{\mathbf{j=1}}^{\mathbf{m}} \mathbf{w_j}\phi(\mathbf{t_n + d_jh_n}) + \mathbf{E_n}(\phi),$$

with quadrature weights

$$\mathbf{w_j} := \int_0^1 \mathbf{L_j(s)\,ds} \quad (\mathbf{j = 1, \ldots, m})$$

$\hookrightarrow$ **Quadrature error** $\mathbf{E_n}(\phi)$.

- For arbitrary abscissas $\{\mathbf{d_j}\}$ (and $\phi \in \mathbf{C^m}$):

$$|\mathbf{E_n}(\phi)| \le \mathbf{Q_m h_n^m} \,.$$

- If the $\{\mathbf{d_j}\}$ satisfy

$$\mathbf{J}_\nu := \int_0^1 \mathbf{s}^\nu \prod_{\mathbf{i=1}}^{\mathbf{m}} (\mathbf{s - d_j})\,\mathbf{ds} \quad (\nu = \mathbf{0}, \ldots, \kappa - \mathbf{1})$$

and $\mathbf{J}_\kappa \ne \mathbf{0}$ $(1 \le \kappa \le m)$, then

$$|\mathbf{E_n}(\phi)| \le \mathbf{Q_m h_n^{m+\kappa}},$$

provided that $\phi \in \mathbf{C^{m+\kappa}}$.

$\underline{\kappa = \mathbf{m}}$: The $\{\mathbf{d_j}\}$ are the **Gauss** (-Legendre) points (zeros of $\mathbf{P_m(2s-1)}$).

$\underline{\kappa = \mathbf{m-1}}$ and $\mathbf{d_m = 1}$:
The $\{\mathbf{d_j}\}$ are the **Radau II points**.

$\underline{\kappa = \mathbf{m-2}}$ and $\mathbf{d_1 = 0}$, $\mathbf{d_m = 1}$ $(m \ge 2)$:
The $\{\mathbf{d_j}\}$ are the **Lobatto points**.

The **collocation error** $\mathbf{e_h := u - u_h}$ satisfies

$$\mathbf{e_h(t) = \int_0^t r(t,s)\delta_h(s)\,ds, \quad t \in I.}$$

$\hookrightarrow$ For $\underline{\mathbf{t = t_n \in I_h}}$:

$$\mathbf{e_h(t_n) = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 r(t_n, t_\ell + sh_\ell)\delta_h(t_\ell + sh_\ell)\,ds.}$$

Set

$$\mathbf{\phi_n(t_\ell + sh_\ell) := r(t_n, t_\ell + sh_\ell)\delta_h(t_\ell + sh_\ell).}$$

Since $\underline{\mathbf{\delta_h(t) = 0}}$ for $\mathbf{t = t_\ell + c_j h_\ell \in X_h}$ $\Rightarrow$ choose the *collocation parameters* as *quadrature abscissas* ($\mathbf{d_j = c_j}$) :

$$\mathbf{e_h(t_n) = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 \phi_n(t_\ell + sh_\ell)\,ds = 0 + \sum_{\ell=0}^{n-1} h_\ell E_{n,\ell}}$$

for $\mathbf{n = 1, \ldots, N}$. This implies that

$$\mathbf{|e_h(t_n)| \le \sum_{\ell=0}^{n-1} h_\ell |E_{n,\ell}| \le Q_m h^{m+\kappa} \sum_{\ell=0}^{n-1} h_\ell \le C_m h^{m+\kappa}}\ ,$$

with $\mathbf{C_m := Q_m T}$. The **optimal order** (on the mesh $\mathbf{I_h}$) is attained when $\underline{\mathbf{\kappa = mi}}$ $\Leftrightarrow$ the $\mathbf{\{c_i\}}$ are the **Gauss points**.
However, we then only have

$$\max\{\mathbf{|u'(t) - u_h'(t)| :\ t \in I_h \setminus \{0\}\} \le C_m' h^m}\ !$$

<u>**ODE**s:</u> Collocation in **smoother** piecewise polynomial spaces ?

- $\mathbf{u_h} \in \mathbf{S_m^{(m-1)}(I_h)}$ $(d = m-1)$:

$\hookrightarrow$ $\mathbf{u_h}$ is **divergent** (as $\mathbf{h} \to \mathbf{0}$) whwn $\mathbf{m} \geq \mathbf{4}$!

(Loscalzo & Talbot, 1967)

- $\mathbf{u_h} \in \mathbf{S_4^{(2)}(I_h)}$, $\mathbf{0 < c_1 < c_2 = 1}$:

$\mathbf{u_h}$ is **divergent** if

$$\frac{\mathbf{1 - c_1}}{\mathbf{c_1}} > 1.$$

- $\mathbf{u_h} \in \mathbf{S_m^{(2)}(I_h)}$ $(m \geq 4)$:

$\mathbf{u_h}$ is **divergent** if the $\{\mathbf{c_i}\}$ are the **Radau II points**.

(Complete convergence / divergence analysis for ODEs: **Mülthei**, 1979)

<u>**Remark:**</u>

The *natural* (and *optimal*) piecewise polynomial spaces for (first-order) ODEs and VIDEs are the spaces $\mathbf{S_m^{(0)}(I_h)}$ with $\mathbf{m \geq 1}$

For VIEs the natural spaces are $\mathbf{S_{m-1}^{(-1)}(I_h)}$. .

Notes

1. Higher-order ODEs:

$$\mathbf{u^{(k)}(t) = f(t, u(t), \dots, u^{(k-1)}(t))} \quad \mathbf{(k \geq 2)}:$$

$\hookrightarrow$ Collocation in $\mathbf{S^{(d)}_{m+d}(I_h)}$ with $\underline{\mathbf{d := k-1}}$ and collocation points

$$\mathbf{X_h = \{t_n + c_i h_n : \; 0 < c_1 < \cdots < c_m \leq 1\,\}}$$

$\Rightarrow$ Continuous Runge-Kutta-**Nyström** methods.

2. The collocation solutions $\mathbf{u_h \in S^{(0)}_m(I_h)}$ for the **ODE**

$$\mathbf{u'(t) = au(t), \quad t \in I; \quad u_0 = u_0,}$$

and $\mathbf{v_h \in S^{(-1)}_{m-1}(I_h)}$ for the '**integrated ODE**'

$$\mathbf{u(t) = u_0 + \int_0^t au(s)\,ds, \quad t \in I}$$

(*Volterra integral equation*), using the same set $\mathbf{X_h}$ of collocation points, are *identical* **only if** $\underline{\mathbf{c_m = 1}}$. In particular:

$$\mathbf{u_h(t_n) = v_h(t_n) \quad (1 \leq n \leq N) \quad \Leftrightarrow \quad c_m = 1.}$$

$\hookrightarrow$ For the **Gauss** points: $\mathbf{u_h(t_n) \neq v_h(t_n)}$!

Observations:

• Assume that the solution of a given functional (differential or integral) equation admits a '*resolvent representation*' of the form

$$\mathbf{u(t) = r(t,0)u(0) + \int_0^t r(t,s)g(s)\,ds, \quad t \in I.}$$

or

$$\mathbf{u(t) = g(t) + \int_0^t R(t,s)g(s)\,ds, \quad t \in I.}$$

Then the collocation solution (or a closely related 'iterated collocation solution') in the 'natural' piecewise polynomial space for the given VFE has the <u>same</u> superconvergence orders as the one for ODEs.
This is **true** for classical *Volterra integral* and *integro-differential equations*, and for *delay differential* and *Volterra functional equations* with **non-vanishing delays** (but **not** for VFEs with **vanishing delays** like $\theta(\mathbf{t}) = \mathbf{qt},\ \mathbf{0 < q < 1}$).

• The attainable order of *superconvergence* is governed by the **regularity** of the solution $\mathbf{u}$ and the choice of the *collocation parameters*.

Volterra integro-differential equations:

$$\mathbf{u'(t) = a(t)u(t) + g(t) + \int_0^t K(t,s)u(s)\,ds, \;\; t \in I,}$$

with continuous $\mathbf{a}$, $\mathbf{g}$ and $\mathbf{K}$. For given initial value $\mathbf{u(0) = u_0}$ the (unique) solution $\mathbf{u \in C^1(I)}$ is given by

$$\mathbf{u(t) = r(t,0)u_0 + \int_0^t r(t,s)g(s)\,ds, \quad t \in I,}$$

where the (differential) **resolvent kernel** $\mathbf{r(t,s)}$ is defined by the *resolvent equation*

$$\mathbf{\frac{\partial r}{\partial s} = -r(t,s)a(s) - \int_s^t r(t,v)K(v,s)\,dv,}$$

$\mathbf{(0 \leq s \leq t \leq T)}$, with $\mathbf{r(s,s) = 1}$, $\mathbf{s \in I}$.

$\hookrightarrow$ Collocation in $\mathbf{S_m^{(0)}(I_h)}$: the **collocation error** $\mathbf{e_h := u - u_h}$ has the representation

$$\mathbf{e_h(t) = \int_0^t r(t,s)\delta_h(s)\,ds, \quad t \in I.}$$

Thus: for the **Gauss points** $\mathbf{\{c_i\}}$,

$$\max\{\mathbf{|e_h(t)| : \; t \in I_h\} \leq C_m h^{2m}},$$

as for *ODEs* !

Volterra integral equations:

$$\mathbf{u(t) = g(t) + \int_0^t K(t,s)u(s)\,ds, \quad t \in I.}\ :$$

For continuous $\mathbf{g}$ and $\mathbf{K}$ the (unique) solution $\mathbf{u \in C(I)}$ is given by

$$\mathbf{u(t) = g(t) + \int_0^t R(t,s)g(s)\,ds, \quad t \in I,}$$

where $\mathbf{R(t,s)}$ is the **resolvent kernel** of $\mathbf{K}$:

$$\mathbf{R(t,s) := \sum_{j=1}^{\infty} K_j(t,s)} \quad (\textit{Neumann series}),$$

with *iterated kernels* $\mathbf{K_1 := K}$ and

$$\mathbf{K_{j+1}(t,s: = \int_s^t K(t,v)K_j(v,s)\,dv \quad (j \geq 1).}$$

Collocation: $\mathbf{u_h \in S_{m-1}^{(-1)}(I_h)}$, and corresponding **iterated** collocation solution

$$\mathbf{u_h^{it}(t) := g(t) + \int_0^t K(t,s)u_h(s)\,ds, \quad t \in I}\ :$$

resulting errors $\mathbf{e_h := u - u_h}$ and $\mathbf{e_h^{it} := u - u_h^{it}}$ have the representations

$$\mathbf{e_h(t) = \delta_h(t) + \int_0^t R(t,s)\delta_h(s)\,ds, \quad t \in I}$$

and

$$\mathbf{e_h^{it}(t) = e_h(t) - \delta_h(t), \quad t \in I}\ :$$

$$\Rightarrow \quad \mathbf{e_h^{it}(t) = \int_0^t R(t,s)\delta_h(s)\,ds, \quad t \in I.}$$

Collocation at **Gauss points**:

$\hookrightarrow$ Iterated collocation error at the *mesh points* $\mathbf{t = t_n}$ $(1 \leq n \leq N)$ satisfies

$$\max\{|\mathbf{e_h^{it}(t)}| : \mathbf{t \in I_h \setminus \{0\}}\} \leq \mathbf{C_m h^{2m}}.$$

<u>But:</u>

$$\max\{|\mathbf{e_h(t)}| : \mathbf{t \in I_h \setminus \{0\}}\} \leq \mathbf{C_m h^m}$$

only !

<u>Note:</u>

If $\mathbf{c_m = 1}$, then

$$\mathbf{u_h^{it}(t_n) = u_h(t_n), \quad n = 1, \ldots, N,}$$

and thus $\mathbf{e_h^{it}(t_n) = e_h(t_n)}$. $\Rightarrow$

$$\max\{|\mathbf{e_h(t_n)}| : \mathbf{1 \leq n \leq N}\} \leq \mathbf{C_m h^{2m-1}}$$

if the collocation parameters $\{\mathbf{c_i}\}$ are the **Radau II points** ($\kappa = \mathbf{m} - 1$: zeros of $(\mathbf{P_m - P_{m-1}})(\mathbf{2s - 1})$).

Lecture II: **<u>Basic references</u>**

L.L. Schumaker, *Spline Functions: Basic Theory*, Wiley-Interscience, 1981.

H.N. Mülthei, Splineapproximationen von beliebigem Defekt zur numerischen Lösung von gewöhnlichen Differentialgleichungen I,II,III, *Numer. Math.*, 32 (1979), 147-157 *and* 343-358; *Numer. Math.*, 34 (1980), 143-154.

E. Hairer, Ch. Lubich & G. Wanner, *Geometric Numerical Integration* (2nd ed.), Springer-Verlag, 2006. (Section II.1)

H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press, 2004. (Chapters 1,2,3)

H. Brunner, On the divergence of collocation solutions in smooth piecewise polynomial spaces for Volterra integral equations, *BIT*, 44 (2004), 631-650.

Lecture III:
VFEs with non-vanishing delays

<u>Illustration:</u> Collocation for **DDE**

$$\mathbf{u}'(\mathbf{t}) = \mathbf{f}(\mathbf{t}, \mathbf{u}(\mathbf{t}), \mathbf{u}(\theta(\mathbf{t}))), \quad \mathbf{t} \in [\mathbf{0}, \mathbf{T}],$$

with $\theta(\mathbf{t}) = \mathbf{t} - \tau \ (\tau > \mathbf{0})$; $\mathbf{u}(\mathbf{t}) = \phi(\mathbf{t}),\ \mathbf{t} \leq \mathbf{0}$.
Primary discontinuity points: $\xi_\mu = \mu \cdot \tau \ (\mu \geq \mathbf{0})$
$\hookrightarrow$ *Assume:* $\mathbf{T} = \xi_{\mathbf{M}+\mathbf{1}}$ for some $\mathbf{M} \geq \mathbf{1}$, and let $\mathbf{I}^{(\mu)} := [\xi_\mu, \xi_{\mu+\mathbf{1}}]$ $(0 \leq \mu \leq M)$.
Collocation for DDE in $\mathbf{S}_{\mathbf{m}}^{(\mathbf{0})}(\mathbf{I}_{\mathbf{h}})$, with **constrained mesh** $\mathbf{I}_{\mathbf{h}}$,

$$\mathbf{I}_{\mathbf{h}} := \bigcup_{\mu=\mathbf{0}}^{\mathbf{M}} \mathbf{I}_{\mathbf{h}}^{(\mu)}$$

(containing the points $\{\xi_\mu\}$). $\mathbf{I}_{\mathbf{h}}$ is defined by the *local meshes*

$$\mathbf{I}_{\mathbf{h}}^{(\mu)} := \{\mathbf{t}_{\mathbf{n}}^{(\mu)} : \ \xi_\mu = \mathbf{t}_{\mathbf{0}}^{(\mu)} < \mathbf{t}_{\mathbf{1}}^{(\mu)} < \cdots < \mathbf{t}_{\mathbf{N}}^{(\mu)} = \xi_{\mu+\mathbf{1}}\}.$$

$\hookrightarrow$ **Local representation** of $\mathbf{u}_{\mathbf{h}}$ on $[\mathbf{t}_{\mathbf{n}}^{(\mu)}, \mathbf{t}_{\mathbf{n}+\mathbf{1}}^{(\mu)}]$:
for $\mathbf{t} = \mathbf{t}_{\mathbf{n}}^{(\mu)} + \mathbf{v}\mathbf{h}_{\mathbf{n}}^{(\mu)},\ \mathbf{v} \in [\mathbf{0}, \mathbf{1}]$; $\mathbf{h}_{\mathbf{n}}^{(\mu)} := \mathbf{t}_{\mathbf{n}+\mathbf{1}}^{(\mu)} - \mathbf{t}_{\mathbf{n}}^{(\mu)}$:

$$\mathbf{u}_{\mathbf{h}}(\mathbf{t}) = \mathbf{y}_{\mathbf{n}}^{(\mu)} + \mathbf{h}_{\mathbf{n}}^{(\mu)} \sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{m}} \beta_{\mathbf{j}}(\mathbf{v}) \mathbf{Y}_{\mathbf{n},\mathbf{j}}^{(\mu)},$$

with $\mathbf{y}_{\mathbf{n}}^{(\mu)} := \mathbf{u}_{\mathbf{h}}(\mathbf{t}_{\mathbf{n}}^{(\mu)})$, $\mathbf{Y}_{\mathbf{n},\mathbf{j}}^{(\mu)} := \mathbf{u}_{\mathbf{h}}'(\mathbf{t}_{\mathbf{n}}^{(\mu)} + \mathbf{c}_{\mathbf{j}}\mathbf{h}_{\mathbf{n}}^{(\mu)})$.

$\hookrightarrow$ Collocation points:

$$\mathbf{X_h} := \bigcup_{\mu=0}^{M} \mathbf{X_h^{(\mu)}},$$

with

$$\mathbf{X_h^{(\mu)}} := \{\mathbf{t_n^{(\mu)}} + \mathbf{c_i h_n^{(\mu)}} :\ \mathbf{i} = 1, \ldots, \mathbf{m};\ 0 \le \mathbf{n} \le \mathbf{N} - 1\}$$

and prescribed $0 < \mathbf{c_1} < \cdots < \mathbf{c_m} \le 1$.
For $\mu = 0, \ldots, M$: generate $\mathbf{u_h} \in \mathbf{S_m^{(0)}}(\mathbf{I_h})$ by

$$\mathbf{u_h'}(\mathbf{t}) = \mathbf{f}(\mathbf{t}, \mathbf{u_h}(\mathbf{t}), \mathbf{u_h}(\theta(\mathbf{t}))), \quad \mathbf{t} \in \mathbf{X_h^{(\mu)}},$$

with known $\mathbf{u_h}(\xi_\mu)$ and (when $\mu = 0$)

$$\mathbf{u}(\theta(\mathbf{t_n^{(0)}} + \mathbf{c_i h_n^{(0)}})) = \phi(\theta(\mathbf{t_n^{(0)}} + \mathbf{c_i h_n^{(0)}})).$$

$\hookrightarrow$ Choose θ-**invariant mesh**:

$$\theta(\mathbf{I_h^{(\mu)}}) = \mathbf{I_h^{(\mu-1)}} \quad \text{for} \quad \mu = 1, \ldots, M.$$

Note that here we have

$$\theta(\mathbf{t_n^{(\mu)}} + \mathbf{c_i h_n^{(\mu)}}) = \mathbf{t_n^{(\mu-1)}} + \mathbf{c_i h_n^{(\mu-1)}} \quad (\mu \ge 1),$$

since θ is **linear**.
$\hookrightarrow$ If θ is **nonlinear**:

$$\theta(\mathbf{t_n^{(\mu)}} + \mathbf{c_i h_n^{(\mu)}}) = \mathbf{t_n^{(\mu-1)}} + \tilde{\mathbf{c}}_\mathbf{i} \mathbf{h_n^{(\mu-1)}}$$

for some $\tilde{\mathbf{c}}_\mathbf{i} \in (0, 1]$.

$\hookrightarrow$ **Collocation solution** $\mathbf{u_h \in S_m^{(0)}(I_h)}$:

$$\mathbf{u_h' t = f(t, u_h(t), u_h(t-\tau)), \ \ t \in X_h,}$$

with $\mathbf{u_h(t) := \phi(t)}$ if $\mathbf{t \in [-\tau, 0]}$.

Using the *local representation* of $\mathbf{u_h}$ on $\mathbf{[t_n^{(\mu)}, t_{n+1}^{(\mu)}]}$,

$$\mathbf{u_h(t_n^{(\mu)} + v h_n^{(\mu)}) = y_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} \beta_j(v) Y_{n,j}^{(\mu)}, \ v \in [0,1]}$$

we obtain (setting $\mathbf{t_{n,i}^{(\mu)} := t_n^{(\mu)} + c_i h_n^{(\mu)}}$)

$$\mathbf{Y_{n,j}^{(\mu)} = f\left(t_{n,i}^{(\mu)}, y_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} a_{i,j} Y_{n,i}^{(\mu)}, \Phi_{n,i}^{(\mu)}\right),}$$

with

$$\mathbf{\Phi_{n,i}^{(\mu)} := u_h(\underbrace{t_n^{(\mu)} + c_i h_n^{(\mu)} - \tau}_{= \theta(t_n^{(\mu)} + c_i h_n^{(\mu)})}) \quad (i = 1, \dots, m).}$$

If the mesh $\mathbf{I_h}$ is θ-**invariant**:

$$\mathbf{u_h(t_{n,i}^{(\mu)} - \tau) = u_h(t_n^{(\mu-1)} + c_i h_n^{(\mu-1)}) = u_h(t_{n,i}^{(\mu-1)}),}$$

when θ is **linear**. For **nonlinear** θ we have

$$\mathbf{u_h(\theta(t_{n,i}^{(\mu)})) = u_h(t_n^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)}).}$$

$\Rightarrow$ $\mathbf{m}$-stage **continuous implicit Runge-Kutta method** for the **DDE**

$$\mathbf{u'(t) = f(t, u(t), u(t-\tau)), \ \ t \in I.}$$

Optimal convergence estimates

Assume that the delay function $\theta(\mathbf{t}) = \mathbf{t} - \tau(\mathbf{t})$ satisfies:

(D1) $\tau(\mathbf{t}) \geq \tau_{\mathbf{0}} > \mathbf{0}$ for $\mathbf{t} \in \mathbf{I} := [\mathbf{t_0}, \mathbf{T}]$

(D2) θ is *strictly increasing* on $\mathbf{I}$;

(D3) $\tau \in \mathbf{C^d}(\mathbf{I})$ for some $\mathbf{d} \geq \mathbf{0}$.

<u>**Theorem:**</u> (**Bellen (1984)**)

Suppose that the mesh $\mathbf{I_h}$ in $\mathbf{S_m^{(0)}}(\mathbf{I_h})$ is θ-**invariant**, and let the collocation parameters $\{\mathbf{c_i}\}$ satisfy

$$\mathbf{J}_\nu := \int_0^1 \mathbf{s}^\nu \prod_{\mathbf{i}=1}^{\mathbf{m}} (\mathbf{s} - \mathbf{c_i})\, \mathbf{ds} = \mathbf{0},$$

$\nu = \mathbf{0}, \ldots, \kappa - \mathbf{1}$, for some κ with $\mathbf{1} \leq \kappa \leq \mathbf{m}$.
If the given functions in the DDE (including θ) are sufficiently smooth, then:

(a) $\|\mathbf{u} - \mathbf{u_h}\|_\infty \leq \mathbf{C_m h^{m+1}}$.

(b) $\max\{|\mathbf{u}(\mathbf{t}) - \mathbf{u_h}(\mathbf{t})| : \mathbf{t} \in \mathbf{I_h}\} \leq \mathbf{C_m^* h^{m+\kappa}}$.

Here, $\mathbf{h} := \max_{(\mu)}\{\mathbf{h}^{(\mu)}\}$.

Summary: **Constrained** and θ-**invariant meshes**

- **Primary discontinuity points** (or: *breaking points*) $\{\xi_\mu\}$ induced by the delay function θ :

$$\theta(\xi_\mu) = \xi_{\mu-1} \quad (\mu \geq 1;\ \xi_0 := 0),$$

with $\xi_\mu - \xi_{\mu-1} \geq \tau_0 > 0$ for all $\mu \geq 1$.
$\hookrightarrow$ *Assume:* $T = \xi_{M+1}$ for some $M \geq 1$.

- **Definition:**
A mesh I_h on $I := [0, T]$ is called a **constrained mesh** if it contains the *primary discontinuity points* $\{\xi_\mu\}$ induced by θ; *i.e.*,

$$I_h := \bigcup_{\mu=0}^{M} I_h^{(\mu)}$$

is defined by the **local meshes**

$$I_h^{(\mu)} := \{t_n^{(\mu)} :\ \xi_\mu = t_0^{(\mu)} < t_1^{(\mu)} < \cdots < t_N^{(\mu)} = \xi_{\mu+1}\}.$$

- **Definition:**
A *constrained* mesh I_h is said to be θ-**invariant** if

$$\theta \ : \quad I_h^{(\mu)} \longrightarrow I_h^{(\mu-1)} \quad \text{for} \quad \mu = 1, \ldots, M;$$

that is, if

$$\theta(t_n^{(\mu)}) = t_n^{(\mu-1)} \quad (n = 0, 1, \ldots, N)$$

for $\mu = 1, \ldots, M$.

Superconvergence analysis: VFIDEs

Let $\theta(t) = t - \tau(t)$, $\tau(t) \geq \tau_0 > 0$. For $t \in [\xi_\mu, \xi_{\mu+1}]$ the collocation error $e_h := u - u_h$ associated with the collocation equation

$$u_h'(t) = a(t)u_h(t) + b(t)u(\theta(t)) + g(t)$$

$$+(\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u)_h(t) - \delta_h(t), \quad t \in I,$$

with $\delta_h(t) = 0$ for $t \in X_h$, has the representation

$$e_h(t) = r_1(t, \xi_\mu)e_h(\xi_\mu) + \int_{\xi_\mu}^{t} r_1(t,s)d_h(s)\,ds$$

$$+F_\mu(t) + \Phi_\mu(t), \quad t \in I^{(\mu)}.$$

Here,

$$F_\mu(t) := \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\xi_{\nu+1}} r_{\mu,\nu}(t,s)d_h(s)\,ds +$$

$$\sum_{\nu=0}^{\mu-1} p_{\mu,\nu}(t)e_h(\xi_\nu) + G_\mu^{(1)}(t;\phi) \quad \text{and}$$

$$\Phi_\mu(t) := \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\theta^{\mu-\nu}(t)} r_{\mu,\nu}(t,s)d_h(s)\,ds + G_\mu^{(2)}(t;\phi),$$

with

$$d_h(t) := \int_0^t \delta_h(s)\,ds \,.$$

Equation for **collocation error**:

$$\mathbf{e_h(t) = r_1(t,\xi_\mu)e_h(\xi_\mu) + \int_{\xi_\mu}^{t} r_1(t,s)d_h(s)\,ds}$$

$$\mathbf{+F_\mu(t) + \Phi_\mu(t), \quad t \in I^{(\mu)},}$$

with

$$\mathbf{F_\mu(t) := \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\xi_{\nu+1}} r_{\mu,\nu}(t,s)d_h(s)\,ds +}$$

$$\mathbf{\sum_{\nu=0}^{\mu-1} p_{\mu,\nu}(t)e_h(\xi_\nu) + G_\mu^{(1)}(t;\phi)} \quad \text{and}$$

$$\mathbf{\Phi_\mu(t) := \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\theta^{\mu-\nu}(t)} r_{\mu,\nu}(t,s)d_h(s)\,ds + G_\mu^{(2)}(t;\phi).}$$

$\hookrightarrow$ $\underline{\mathbf{t = t_n^{(\mu)}}}$:

If the mesh $\mathbf{I_h}$ is θ-**invariant**, then

$$\theta^{\mu-\nu}(\mathbf{t_n^{(\mu)}}) = \mathbf{t_n^{(\nu)}} \quad (\nu = \mathbf{0}, \ldots, \mu).$$

Hence, we can estimate the integrals by employing the techniques used for *non-delay* VIDEs (and ODEs).

An analogous error representation holds for VFIEs.

Superconvergence results for VFIDEs and VFIEs with **non-vanishing** delays:

Theorem: (**Bellen (1984); Brunner (2004)**)
Let the delay function $\theta(t) = t - \tau(t)$ satisfy
(D1) $\tau(t) \geq \tau_0 > 0$ for $t \in I := [t_0, T]$
(D2) θ is *strictly increasing* on I;
(D3) $\tau \in C^d(I)$, with sufficiently large d.
Then:
For sufficiently smooth data (including the initial function ϕ), the collocation solutions in $S_m^{(0)}(I_h)$ (for VFIDEs) or in $S_{m-1}^{(-1)}(I_h)$ (for VFIEs) possess the **same optimal orders** of **local superconvergence** on I_h as the ones for *classical* VIDEs and VIEs with similarly smooth data if, and only if, the underlying mesh I_h is θ-**invariant**.

For example, if the $\{c_i\}$ are the **Gauss points**:

$$\max\{|u(t) - u_h(t)| : t \in I_h\} \leq C_m^* h^{2m}$$

for **VFIDE**s, and

$$\max\{|u(t) - u_h^{it}(t)| : t \in I_h \setminus \{0\}\} \leq C_m^* h^{2m}$$

for **VFIE**s.

Remarks:

- **Fully discretised** collocation equations:

The integrals occurring in the collocation equations for VFIDEs and VFIEs,

$$\int_0^1 \mathbf{K}(\mathbf{t}_{\mathbf{n},\mathbf{i}}^{(\mu)}, \mathbf{t}_\ell^{(\mu)} + \mathbf{s}\mathbf{h}_\ell^{(\mu)})\beta_{\mathbf{j}}(\mathbf{s})\,\mathbf{ds}$$

and

$$\int_0^1 \mathbf{K}(\mathbf{t}_{\mathbf{n},\mathbf{i}}^{(\mu)}, \mathbf{t}_\ell^{(\mu)} + \mathbf{s}\mathbf{h}_\ell^{(\mu)})\mathbf{L}_{\mathbf{j}}(\mathbf{s})\,\mathbf{ds},$$

can in general <u>not</u> be found analytically and thus have to be *approximated* by appropriate *quadrature formulas*.

$\hookrightarrow$ Use $\mathbf{m}$-**point interpolatory quadrature** with **abscissas** given by the **collocation points**.

$\Rightarrow$ Order of quadrature error is (at least) equal to the local order of the *exact* collocation solution.

- *Non-monotonic delay functions:*

See monograph by **Bellen** & **Zennaro (2003)**; also: **Brunner** & **Maset (2008)**.

Lecture III: **Basic references**

H. Brunner, *Lecture Notes*, 2008 Summer School on Applied Analysis, TU Chemnitz: Section 3.

A. Bellen, One-step collocation for delay differential equations, *Computing* 10 (1984), 275-283.

A. Bellen & M. Zennaro, *Numerical Methods for Delay Differential Equations*, Oxford Universirty Press, 2003. (Chapters 5-7)

H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press, 2004. (Section 4.2)

N. Guglielmi & E. Hairer, Computing breaking points in implicit delay differential equations, *Adv. Comput. Math.*, (2008) (to appear).

Lecture IV:
VFEs with vanishing delays

Volterra functional equations (on $\mathbf{I} := [0, \mathbf{T}]$):

- $\mathbf{u}'(\mathbf{t}) = \mathbf{a}(\mathbf{t})\mathbf{u}(\mathbf{t}) + \mathbf{b}(\mathbf{t})\mathbf{u}(\theta(\mathbf{t})) + (\mathcal{V}_\theta \mathbf{u})(\mathbf{t})$,
- $\mathbf{u}(\mathbf{t}) = \mathbf{g}(\mathbf{t}) + (\mathcal{V}_\theta \mathbf{u})(\mathbf{t})$;
- $\mathbf{u}(\mathbf{t}) = \mathbf{g}(\mathbf{t}) + \mathbf{b}(\mathbf{t})\mathbf{u}(\theta(\mathbf{t})) + (\mathcal{V}_\theta \mathbf{u})(\mathbf{t})$;

Volterra integral operators $(\mathbf{C}(\mathbf{I}) \to \mathbf{C}(\mathbf{I}))$

$$(\mathcal{V}\mathbf{u})(\mathbf{t}) := \int_0^{\mathbf{t}} \mathbf{K}_0(\mathbf{t},\mathbf{s})\mathbf{u}(\mathbf{s})\,\mathbf{ds}$$
$$(\mathcal{V}_\theta\mathbf{u})(\mathbf{t}) := \int_0^{\theta(\mathbf{t})} \mathbf{K}_1(\mathbf{t},\mathbf{s})\mathbf{u}(\mathbf{s})\,\mathbf{ds}.$$

Also: $(\mathcal{W}_\theta\mathbf{u})(\mathbf{t}) := \int_{\theta(\mathbf{t})}^{\mathbf{t}} \mathbf{K}(\mathbf{t},\mathbf{s})\mathbf{u}(\mathbf{s})\,\mathbf{ds}$.

Assume that the delay function $\theta = \theta(\mathbf{t})$ satisfies:
(D1) $\theta(0) = 0$; $\theta(\mathbf{t}) \le \mathbf{q}_1\mathbf{t}$ for some $\mathbf{q}_1 \in (0,1)$;
(D2) θ is **strictly increasing** in $\mathbf{I}$;
(D3) $\theta \in \mathbf{C}^{\mathbf{d}}(\mathbf{I})$ for some $\mathbf{d} \ge 1$.

$\hookrightarrow$ **Pantograph equation:**

$$\mathbf{u}'(\mathbf{t}) = \mathbf{a}\mathbf{u}(\mathbf{t}) + \mathbf{b}\mathbf{u}(\mathbf{q}\mathbf{t}),\ \mathbf{t} \in \mathbf{I}\ :$$

$\theta(\mathbf{t}) = \mathbf{q}\mathbf{t} = \mathbf{t} - (1-\mathbf{q})\mathbf{t},\ 0 < \mathbf{q} < 1.$

Illustration: Pantograph DDE: $\theta(t) = qt$

$$u'(t) = au(t) + bu(qt), \quad t \in [0, T] \;\; (0 < q < 1),$$

with $u(0) = u_0$.
Collocation solution $u_h \in S_m^{(0)}(I_h)$, with **uniform** mesh I_h:

$$u_h'(t) = au_h(t) + bu_h(qt), \;\; t \in X_h; \;\; u_h(0) = u_0.$$

Define:

$$q^I := \lceil \frac{q}{1-q} c_1 \rceil, \qquad q^{II} := \lceil \frac{q}{1-q} c_m \rceil \, .$$

For the collocation points $t = t_n + c_i h \in e_n$, the **images** $q(t_n + c_i h)$ satisfy

- *Phase I:* $0 \leq n < q^I$

$q(t_n + c_i h) \in (t_n, t_{n+1})$ for **all** $i = 1, \ldots, m$.

- *Phase II:* $q^I \leq n < q^{II}$

$q(t_n + c_i h) \leq t_n$ for **some** $i < m$.

- *Phase III:* $q^{II} \leq n \leq N - 1$

$q(t_n + c_i h) \leq t_n$ for **all** $i = 1, \ldots, m$.

Continuous implicit Runge-Kutta method:

$\mathbf{u_h} \in \mathbf{S_m^{(0)}}(\mathbf{I_h})$, with **uniform** $\mathbf{I_h}$:

$$\mathbf{u_h}(\mathbf{t_n} + \mathbf{vh}) = \mathbf{u_h}(\mathbf{t_n}) + \mathbf{h} \sum_{\mathbf{j=1}}^{\mathbf{m}} \beta_{\mathbf{j}}(\mathbf{v}) \mathbf{Y_{n,j}}, ; \ \mathbf{v} \in [\mathbf{0}, \mathbf{1}].$$

Let $\mathbf{Y_n} := (\ \mathbf{Y_{n,1}}, \ldots, \mathbf{Y_{n.m}}\ )^{\mathbf{T}} \in \mathbb{R}^{\mathbf{m}}$.

- *Phase I:* $\mathbf{0} \leq \mathbf{n} < \mathbf{q^I}$

$\Rightarrow$ Linear algebraic systems for $\mathbf{Y_n}$:

$$[\mathcal{I}_{\mathbf{m}} - \mathbf{h}(\mathcal{A}_{\mathbf{n}} + \mathcal{B}_{\mathbf{n}}^{\mathbf{I}}(\mathbf{q}))]\mathbf{Y_n} = \mathbf{r_n^I}.$$

- *Phase II:* $\mathbf{q^I} \leq \mathbf{n} < \mathbf{q^{II}}$

$$[\mathcal{I}_{\mathbf{m}} - \mathbf{h}(\mathcal{A}_{\mathbf{n}} + \mathcal{B}_{\mathbf{n}}^{\mathbf{II}}(\mathbf{q}))]\mathbf{Y_n} = \mathbf{r_n^{II}} + \tilde{\mathcal{B}}_{\mathbf{n}}^{\mathbf{II}}(\mathbf{q})\mathbf{Y_{n-1}}.$$

- *Phase III:* $\mathbf{q^{II}} \leq \mathbf{n} \leq \mathbf{N} - \mathbf{1}$

$$[\mathcal{I}_{\mathbf{m}} - \mathbf{h}\mathcal{A}_{\mathbf{n}}]\mathbf{Y_n} = \mathbf{r_n^{III}} + \mathcal{B}_{\mathbf{n}}^{\mathbf{III}}(\mathbf{q})\mathbf{Y_{\tilde{n}}},$$

for some $\tilde{\mathbf{n}} < \mathbf{n}$. Here, $\mathcal{I}_{\mathbf{m}}$ denotes the *identity matrix* in $\mathbb{R}^{\mathbf{m}\times\mathbf{m}}$, and $\mathcal{A}_{\mathbf{n}}$ is the *Runge-Kutta matrix* corresponding to the **ODE part** in the pantograph DDE.

Collocation solutions for VFIDEs

$$\mathbf{u'(t) = a(t)u(t) + b(t)u(\theta(t))}$$

$$\mathbf{+(\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t)}, i \ \mathbf{t \in I},$$

with delay function θ satisfying

(D1) $\theta(\mathbf{0}) = \mathbf{0}$; $\theta(\mathbf{t}) \leq \mathbf{q_1 t}$ for some $\mathbf{q_1} \in (\mathbf{0}, \mathbf{1})$;

(D2) θ is **strictly increasing** in $\mathbf{I}$;

(D3) $\theta \in \mathbf{C^d(I)}$ for some $\mathbf{d \geq 0}$.

Let $\mathbf{I_h} := \{\mathbf{t_n}: \ \mathbf{0 = t_0 < t_1 < \cdots < t_N = T}\}$, with

$$\mathbf{e_n := (t_n, t_{n+1}]}, \quad \mathbf{h_n := t_{n+1} - t_n}, \quad \mathbf{h} := \max_{(\mathbf{n})}\{\mathbf{h_n}\}.$$

- **Collocation space:**

$\mathbf{S_m^{(0)} I_h}) := \{\mathbf{v \in C(I)}: \ \mathbf{v}|_{\mathbf{e_n}} \in \pi_{\mathbf{m}} \ (0 \leq n \leq N-1)\}$,

with $\Rightarrow$ dim $(\mathbf{S_m^{(0)}(I_h) = Nm + 1}$.

- **Collocation equation:** Find $\mathbf{u_h \in S_m^{(0)}(I_h)}$

so that for all $\underline{\mathbf{t \in X_h}}$,

$$\mathbf{u_h'(t) = a(t)u_h(t) + b(t)u_h(\theta(t))}$$

$$\mathbf{+(\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t)} \ ,$$

with $\mathbf{u_h(0) = u_0}$.

$\hookrightarrow$ **Structure** of algebraic equations for $\mathbf{Y_n}$ in the local representation of $\mathbf{u_h}$,

$$\mathbf{u_h(t) = u_h(t_n) + h \sum_{j=1}^{m} \beta_j(v) Y_{n.j}}, \quad \mathbf{v \in [0,1]} \ :$$

- <u>*Phase I:*</u> $\mathbf{0 \le n < q^I}$

$$[\mathcal{I}_\mathbf{m} - \mathbf{h}(\mathcal{A}_\mathbf{n} + \mathcal{B}^\mathbf{I}_\mathbf{n}(\mathbf{q})) - \mathbf{h^2}(\mathcal{C}_\mathbf{n} + \mathcal{C}^\mathbf{I}_\mathbf{n}(\mathbf{q}))]\mathbf{Y_n} = \mathbf{r^I_n}.$$

- <u>*Phase II:*</u> $\mathbf{q^I \le n < q^{II}}$

$$[\mathcal{I}_\mathbf{m} - \mathbf{h}(\mathcal{A}_\mathbf{n} + \mathcal{B}^\mathbf{II}_\mathbf{n}(\mathbf{q})) - \mathbf{h^2}(\mathcal{C}_\mathbf{n} + \mathcal{C}^\mathbf{II}_\mathbf{n}(\mathbf{q}))]\mathbf{Y_n}$$

$$= \mathbf{r^{II}_n} + \mathbf{h}[\tilde{\mathcal{B}}^\mathbf{II}_\mathbf{n}(\mathbf{q}) + \mathbf{h}\tilde{\mathcal{C}}^\mathbf{II}_\mathbf{n}(\mathbf{q})]\mathbf{Y_{n-1}}.$$

- <u>*Phase III:*</u> $\mathbf{q^{II} \le n \le N-1}$

$$[\mathcal{I}_\mathbf{m} - \mathbf{h}(\mathcal{A}_\mathbf{n} + \mathbf{h}\mathcal{C}_\mathbf{n})]\mathbf{Y_n}$$

$$= \mathbf{r^{III}_n} + \mathbf{h}[\tilde{\mathcal{B}}^\mathbf{III}_\mathbf{n}(\mathbf{q}) + \mathbf{h}\tilde{\mathcal{C}}^\mathbf{III}_\mathbf{n}(\mathbf{q})]\mathbf{Y_{\tilde{n}}},$$

for some $\mathbf{\tilde{n} < n}$.

Collocation solutions for VFIEs

$$\mathbf{u(t) = g(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \;\; t \in [0,T]} \;,$$

with

$$(\mathcal{V}\mathbf{u})(\mathbf{t}) := \int_0^{\mathbf{t}} \mathbf{K_0(t,s)u(s)\,ds}$$

and

$$(\mathcal{V}_\theta \mathbf{u})(\mathbf{t}) := \int_0^{\theta(\mathbf{t})} \mathbf{K_1(t,s)u(s)\,ds} \;.$$

Let $\mathbf{I_h := \{t_n :\; 0 = t_0 < t_1 < \cdots < t_N = T\}}$, with

$$\mathbf{e_n := (t_n, t_{n+1}], \quad h_n := t_{n+1} - t_n, \quad h := \max_{(n)}\{h_n\}}.$$

- **Collocation space:**

$$\mathbf{S_{m-1}^{(-1)}(I_h) := \{v :\; v|_{e_n} \in \pi_{m-1}} \;\; (0 \le n \le N)\} \;.$$

- **Collocation points:**

$$\mathbf{X_h := \{t_n + c_k h_n :} \;\; 0 \le n \le N-1\} \;,$$

with $\mathbf{0 \le c_1 < \cdots < c_m \le} 1$.

- **Collocation equation:** Find $\mathbf{u_h \in S_{m-1}^{(-1)}(I_h)}$ so that

$$\mathbf{u_h(t) = g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \;\; \underline{t \in X_h}}.$$

$\hookrightarrow$ **Iterated collocation solution:**

$$\mathbf{u_h^{it}(t) := g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \;\; t \in I}.$$

Note that $\mathbf{u_h^{it}(t) = u_h(t)}$ for all $\underline{\mathbf{t \in X_h}}$.

VFEs with **vanishing delays:**
Global (super-) convergence on uniform I_h:

- $u_h \in S_{m-1}^{(-1)}(I_h)$ for the **VFIE**

$$u(t) = g(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t),\ t \in I:$$

Theorem: (**B. & Hu (2005)**)
(i) For **general** $\{c_k\}$:

$$\|u - u_h\|_\infty \le C_m h^m .$$

(ii) If the $\{c_k\}$ are the m **Gauss points** in $(0, 1)$:

$$\|u - u_h^{it}\|_\infty \le \tilde{C}_m h^{m+1} .$$

- $u_h \in S_m^{(0)}(I_h)$ for the **VFIDE**

$$u'(t) = au(t) + bu(\theta(t)) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t),\ t \in I:$$

Theorem: (**B. & Hu (2007)**)
(i) For **general** $\{c_k\}$:

$$\|u - u_h\|_\infty \le C_m h^m .$$

(ii) For the **Gauss points** $\{c_k\}$:

$$\|u - u_h\|_\infty \le \tilde{C}_m h^{m+1} .$$

VIDEs with **vanishing delays:**

Local superconvergence on uniform $\mathbf{I_h}$

Collocation solution $\mathbf{u_h} \in \mathbf{S^{(d)}_{m+d}(I_h)}$ ($d := k-1$), with **uniform mesh** $\mathbf{I_h}$, for

$$\mathbf{u^{(k)}(t) = a(t)u(t) + b(t)u(\theta(t))}$$

$$\mathbf{+(\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t) \;\; t \in I := [0, T],}$$

with delay function $\theta(\mathbf{t}) = \mathbf{qt} \; (\mathbf{0} < \mathbf{q} < \mathbf{1})$.

Theorem: (**B. & Hu (2007)** ($k = 1$); **B. (2008)**)
If the $\{\mathbf{c_j}\}$ are the **Gauss points** :

$$\max_{\mathbf{t} \in \mathbf{I_h}} |\mathbf{u^{(j)}(t) - u_h^{(j)}(t)}| \leq \mathbf{C^*_m(q)} \begin{cases} \mathbf{h^{2m}} & \text{if } \mathbf{m = 1, 2} \\ \mathbf{h^{m+2}} & \text{if } \mathbf{m > 2,} \end{cases}$$

for $\mathbf{j = 0, \ldots, k-1}$ and **all** $\mathbf{q \in (0, 1)}$.

Special case: **Pantograph DDE**:

$$\mathbf{u'(t) = a(t)u(t) + b(t)u(qt)} \;\; (0 < q < 1) \; .$$

The **proofs** of the *optimal local superconvergence results* for VFIDEs and VFIEs are based on the **representations of the solutions** $\mathbf{e_h := u - u_h}$ of the **error equations**.

VFIEs with **vanishing delays:**

Local superconvergence on uniform I_h

Collocation solution $\mathbf{u_h} \in \mathbf{S^{(-1)}_{m-1}(I_h)}$ and corresponding **iterated** collocation solution $\mathbf{u^{it}_h}$ for

$$\mathbf{u(t) = g(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t)}, \quad t \in [0, T] \, ,$$

with $\underline{\theta(\mathbf{t}) = \mathbf{qt}}$ $(\mathbf{0 < q < 1})$

↪ **Observation:** (**B. (1997)**)

$$\mathbf{u(t) = u_0 + \int_0^{qt} (b/q) u(s) \, ds}, \quad t \geq 0, \quad \underline{\mathbf{0 < q < 1}} \; :$$

If collocation is at **Gauss points**, then $\mathbf{u^{it}_h(h)}$ is **not** the $(\mathbf{m}, \mathbf{m})$-**Padé approximant** to $\mathbf{u(h)}$:

$$|\mathbf{u(h) - u_h(h)}| = \mathcal{O}(\mathbf{h^{p^*}}) \quad \textit{with} \quad \underline{\mathbf{p^* < 2m + 1}} \, .$$

<u>**Theorem:**</u> (**B. & Hu (2005)**)
If the $\{\mathbf{c_k}\}$ are the **Gauss points** and $\mathbf{m \geq 2}$:

$$\max_{\mathbf{t \in I_h}} |\mathbf{u(t) - u^{it}_h(t)}| \leq \mathbf{C^*_m(q)} \begin{cases} \mathbf{h^{m+2}} & \Leftrightarrow \underline{\mathbf{q = 1/2}} \\ & \textit{and } \mathbf{m} \text{ \underline{even}}, \\ \mathbf{h^{m+1}} & \text{otherwise.} \end{cases}$$

Comparison: For $\mathbf{q = 1}$ (classical VIE):

$$\max\{|\mathbf{u(t) - u^{it}_h(t)}| : \ \mathbf{t \in I_h \setminus \{0\}}\} \leq \mathbf{C^*_m h^{2m}}.$$

Open Problem:

Superconvergence analysis of *iterated collocation solution* $\mathbf{u_h^{it}}$ corresponding to $\mathbf{u_h} \in \mathbf{S_{m-1}^{(-1)}(I_h)}$ (on **uniform** mesh $\mathbf{I_h}$) for the **VFIE**s

$$\mathbf{u(t) = g(t) + \underline{b(t)u(\theta(t))} + (\mathcal{V}_\theta u)(t)}, \;\; t \in [0,T] \, ,$$

and

$$\mathbf{u(t) = g(t) + \underline{b(t)u(\theta(t))} + (\mathcal{W}_\theta u)(t)}, \;\; t \in [0,T] \, ,$$

with

$$\mathbf{(\mathcal{W}_\theta u)(t) := \int_{\theta(t)}^{t} K(t,s)u(s)\, ds},$$

and $\theta(\mathbf{t}) = \mathbf{qt}$ $(0 < q < 1)$?

$\hookrightarrow$ **Ill-posed problem** (Denisov & Lorenzi (1997))

Special case:

$$\mathbf{u(t) = g(t) + b(t)u(\theta(t))}, \;\; t \in [0,T]$$

(**Liu (1995)**: $\mathrm{m} = 1$).

Representation of collocation errors: VFIDEs

The **collocation error** $\mathbf{e_h} := \mathbf{u} - \mathbf{u_h}$ for

$$\mathbf{u^{(k)}(t) = a(t)u(t) + b(t)u(\theta(t))}$$

$$+(\mathcal{V}\mathbf{u})(\mathbf{t}) + (\mathcal{V}_\theta \mathbf{u})(\mathbf{t}) \ \ (k \geq 1) ,$$

with vanishing delay function $\theta(\mathbf{t})$ (*e.g.* $\theta(\mathbf{t}) = \mathbf{qt}$) satisfies the VFIDE

$$\mathbf{e_h^{(k)}(t) = a(t)e_h(t) + b(t)e_h(\theta(t)) + \delta_h(t)}$$

$$+(\mathcal{V}\mathbf{e_h})(\mathbf{t}) + (\mathcal{V}_\theta \mathbf{e_h})(\mathbf{t}), \ \ t \in [0, T],$$

with $\mathbf{e_h^{(j)}(0) = 0}$, $j = 0, \ldots, k-1$. The **defect function** $\delta_\mathbf{h}(\mathbf{t})$ is piecewise smooth and **vanishes on** $\mathbf{X_h}$.

For $\mathbf{a(t) \equiv 0}$, $\mathcal{V} = \mathbf{0}$ the solution of the error equation is given by

$$\mathbf{e_h(t) = d_h(t) + \sum_{j=1}^{\infty} \int_0^{\theta^j(t)} H_{k,j}(t,s) d_h(s)\, ds}, \ \ t \in [0, T],$$

where the kernels $\mathbf{H_{k,j}}$ are smooth and

$$\mathbf{d_h(t) := \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} \delta_h(s)\, ds} .$$

For $\mathbf{t = t_n}$ (uniform mesh), $\theta^\mathbf{j}(\mathbf{t_n}) = \mathbf{t_{q_{n,j}}} + \gamma_\mathbf{n,j}\mathbf{h}$, where

$$\mathbf{q_{n,j}} := \lfloor \theta^\mathbf{j}(\mathbf{t_n})/\mathbf{h} \rfloor \in \mathbb{N}, \quad \gamma_\mathbf{n,j} := \theta^\mathbf{j}(\mathbf{t_n})/\mathbf{h} - \mathbf{q_{n,j}} \in [\mathbf{0}, \mathbf{1}) .$$

For $\underline{\theta(\mathbf{t}) = \mathbf{qt}}$, $\mathbf{t} = \mathbf{t_n} = \mathbf{nh}$ $(1 \leq n \leq N)$:

$$\mathbf{e_h(t_n) = d_h(t_n) + \sum_{j=1}^{\infty} \int_0^{q^j t_n} H_{k,j}(t_n, s) d_h(s)\, ds} \ ,$$

with

$$\mathbf{d_h(t)} := \int_0^{\mathbf{t}} \frac{\mathbf{(t-s)^{k-1}}}{\mathbf{(k-1)!}} \delta_{\mathbf{h}}(\mathbf{s})\, \mathbf{ds} \quad \text{if} \quad \mathbf{k \geq 1},$$

and

$$\mathbf{d_h(t)} := \delta_{\mathbf{h}}(\mathbf{t}) \qquad \text{if} \quad \mathbf{k = 0}.$$

(Recall that $\underline{\delta_{\mathbf{h}}(\mathbf{t}) = \mathbf{0}}$ for $\underline{\mathbf{t} \in \mathbf{X_h}}$.)

Since $\theta^{\mathbf{j}}(\mathbf{t_n}) = \mathbf{t_{q_{n,j}}} + \gamma_{\mathbf{n,j}}\mathbf{h}$, we have

$$\int_0^{\mathbf{q^j t_n}} \mathbf{H_{k,j}(t_n, s) d_h(s)\, ds} = \int_0^{\mathbf{t_{q_{n,j}}}} \mathbf{H_{k,j}(t_n, s) d_h(s)\, ds}$$

$$+\mathbf{h} \int_0^{\gamma_{\mathbf{n,j}}} \mathbf{H(t_n, t_{q_{n,j}} + sh) d_h(t_{q_{n,j}} + sh)\, ds} \ .$$

(*etc.*)

Collocation on (quasi-) geometric meshes

I. Non-vanishing delay techniques:
On $[0, t_0]$ (with suitably *small* $t_0 = t_0(q; N) > 0$), assume given *initial approximation* to $u(t)$.
$\hookrightarrow$ Choose **geometric macro-mesh** on $[t_0, T]$ given by

$$\{\xi_\mu := q^{\kappa-\mu} T : 0 \leq \mu \leq \kappa\}, \quad \kappa = \kappa(q; N),$$

with appropriate κ such that $\xi_0 := t_0 \to 0$ as $N \to \infty$. $\hookrightarrow$ **Local** (uniform) **meshes:**

$$I_h^{(\mu)} := \{t_n^{(\mu)} : \ \xi_\mu = t_0^{(\mu)} < t_1^{(\mu)} < \cdots < t_N^{(\mu)} = \xi_{\mu+1}\}.$$

$\Rightarrow$ Collocation solution $u_h \in S_m^{(0)}(I_h)$ (at **Gauss points**, and on the *global* θ-**invariant** *mesh* $I_h := \bigcup_{\mu=0}^{\kappa-1} I_h^{(\mu)}$) for the VFIDE

$$u'(t) = a(t)u(t) + b(t)u(\theta(t)) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t) \ :$$

$$\Rightarrow \quad \max_{t \in I_h} |u(t) - u_h(t)| \leq C_m^*(q) N^{-2m} .$$

(**Bellen (2002)** (*DDEs*), **Bellen, B., Maset** & **Torelli (2006)** (*VFIDEs*))

II. Vanishing delay techniques:
(**Brunner, Hu** & **Lin (2001)**, **B.** & **Hu (2007)**)
Global **geometric** *mesh* on $[0, T]$:

$$\mathbf{I_h} := \{\mathbf{t_n} = \mathbf{t_n^{(N)}} := \mathbf{d^{N-n}T} : 0 \leq n \leq N\},$$

with suitably chosen $\mathbf{d} = \mathbf{d}(\mathbf{q}; \mathbf{m}, \mathbf{N}) \in (0, 1)$.
Collocation in $\mathbf{S_m^{(0)}(I_h)}$ for **VFIDE**

$$\mathbf{u'(t) = a(t)u(t) + b(t)u(\theta(t)) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t)},$$

using the **Gauss points**, yields

$$\max_{\mathbf{t} \in \mathbf{I_h}} |\mathbf{u(t) - u_h(t)}| \leq \mathbf{C_m^*(q) N^{-(2m-\varepsilon_N)}} \,,$$

where $\varepsilon_{\mathbf{N}} \to \mathbf{0}$, as $\mathbf{N} \to \infty$.

Question:

Numerical comparison of collocation solutions on **quasi-geometric meshes** (approach of *Bellen et al.*) and on **geometric meshes** (approach of *Brunner & Hu*) ?

Remark:

The *variable stepsize code* **RADAR5** (**Guglielmi** & **Hairer (2001, 2005)**), when applied to **pantograph-type** DDEs, appears to generate meshes $\mathbf{I_h}$ with stepsizes $\{\mathbf{h_n}\}$ that show **exponential-like growth** (**Guglielmi (2006)**).

Multiple vanishing delays:

The *attainable order of local superconvergence* at $\mathbf{t = t_1 = h}$ for the **double pantograph equation**,

$$\mathbf{u'(t) = au(t) + b_1u(q_1t) + b_2u(q_2t)}, \;\; t \in [0, T],$$

where $\mathbf{0 < q_1 < q_2 < 1}$, is discussed in **Zhao, Xu** & **Qiao (2005)**; see also **Qiu, Mitsui** & **Kuang (1999)** and **Liu** & **Li (2004)**.

- **Optimal superconvergence** of $\mathbf{u_h \in S_m^{(0)}(I_h)}$ on **uniform meshes** $\mathbf{I_h}$ for the *multiple delay* **VFIDE**

$$\mathbf{u'(t) = a(t)u(t) + \sum_{j=1}^{r} b_j(t)u(\theta_j(t))}$$

$$+ \sum_{j=1}^{r} (\mathcal{V}_{\theta_j}\mathbf{u})(\mathbf{t}), \;\; t \in [0, T],$$

where $\theta_{\mathbf{j}}(\mathbf{t}) = \mathbf{q_j t}, \;\; \mathbf{0 < q_1 < \cdots < q_r < 1}$.

Theorem: (**B. (2008)**)

Collocation at **Gauss points** leads to

$$\max\{|\mathbf{u(t) - u_h(t)}| : \mathbf{t \in I_h}\} \leq \mathbf{C_m^*(q)h^{m+2}}$$

for any $\mathbf{q} := (\mathbf{q_1, \ldots, q_r})$ $(\mathbf{r \geq 2})$ and all $\mathbf{m \geq 2}$.

• Optimal orders of **superconvergence** of $\mathbf{u_h} \in \mathbf{S_{m-1}^{(-1)}(I_h)}$ and corresponding $\mathbf{u_h^{it}}$, on *uniform meshes*, for **VFIEs** with multiple vanishing delays,

$$\mathbf{u(t) = g(t) + \sum_{j=1}^{r} (\mathcal{V}_{\theta_j} u)(t), \quad t \in [0, T],}$$

where $\theta_{\mathbf{j}}\mathbf{(t) = q_j t,\ 0 < q_1 < \cdots < q_r < 1}$:

<u>**Theorem:**</u> (**B., 2008**)
Local superconvergence for $\mathbf{u_h}$ or $\mathbf{u^{it}}$ with $\mathbf{p^* = m + 2}$ $(m \geq 2)$ is **not possible**. If the $\{\mathbf{c_i}\}$ are the **Gauss points**, then the optimal local order of convergence on **uniform meshes** $\mathbf{I_h}$ is described by

$$\max\{|\mathbf{u(t) - u_h^{it}(t)}|\ \mathbf{t \in I_h \setminus \{0\}}\} \leq \mathbf{C_m^*(q) h^{m+1}}$$

for all $\mathbf{q := (\, q_1, \ldots, q_r\,)}$. It coincides with the optimal *global* order of superconvergence of $\mathbf{u_h^{it}}$ on $\mathbf{I}$.

‘Integral-algebraic’ VFEs
(VFEs with *non-local comstraints*)

Illustration:

$$\mathbf{u'(t) = F(t, u(t), u(\theta(t)), w(t), w(\theta(t))), \ \ t \in [0, T]},$$

$$\mathbf{0 = g(t) + \int_{\theta(t)}^{t} k(t-s) G(s, u(s), w(s))\, ds},$$

with delay function $\theta(\mathbf{t})$ satisfying $\theta(\mathbf{0}) = \mathbf{0}$ (etc.).
(Collocation for delay DAEs with non-vanishing delays and *local* (algebraic) constraints was studied by **Hauber (1997)**.)

$\hookrightarrow$ **Open problem:** For $\theta(\mathbf{t}) = \mathbf{qt}$ $(0 < q < 1)$, the convergence analysis ($\mathbf{h} \to \mathbf{0}$) of collocation solutions $\mathbf{u_h} \in \mathbf{S_{m-1}^{(-1)}(I_h)}$ (with uniform $\mathbf{I_h}$) for the **first-kind VFIE** ($\hookrightarrow$ **Volterra (1897) !**)

$$\mathbf{0 = g(t) + \int_{qt}^{t} K(t, s) u(s)\, ds, \ \ t \in [0, T]},$$

where $\mathbf{g(0) = 0}$, $\mathbf{g \in C^1(I)}$; $|\mathbf{K(t,t)} \geq \kappa_0 > \mathbf{0}$ and $\mathbf{K \in C^1(D_\theta)}$, is *open*.

- $\mathbf{q = 0}$:

$$\mathbf{\|u - u_h\|_\infty \longrightarrow 0} \qquad \Leftrightarrow \qquad \prod_{i=1}^{m} \frac{\mathbf{1 - c_i}}{\mathbf{c_i}} \leq \mathbf{1}\ .$$

Lecture IV: **Basic references**

H. Brunner, *Lecture Notes*, 2008 Summer School on Applied Analysis, TU Chemnitz: Sections 4 and 5

A. Iserles, Numerical analysis of delay differential equations with variable delays, *Ann. Numer. Math.* **1** (1994), 133-152.

A. Bellen & M. Zennaro, *Numerical Methods for Delay Differential Equations*, Oxford University Press, 2003. (Section 6.4)

Y.K. Liu, Numerical investigation of the pantograph equation, *Appl. Numer. Math.* **24** (1997), 516-528.

H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press, 2004. (Chapter 5)

A. Bellen *et al.*, Superconvergence in collocation methods on quasi-graded meshes for functional differential equations with vanishing delays, *BIT* **46** (2006), 229-247.

V. Concluding Remarks:
Current and future research work

- **DDEs and VFEs with non-monotonic (vanishing) delay functions**

↪ *Illustration:*

$$\theta(t) = q_1 t + (q_2 - q_1) t \sin^2(\omega t) \quad t \geq 0,$$

with $0 < q_1 < q_2 < 1, \ \omega \geq 1.$

Brunner & **Maset** (2008); **B.** & **Guglielmi** (2008)

- **VFEs with weakly singular kernels**

VFIDEs and VFIEs corresponding to delay integral operators of the form

$$(\mathcal{V}_{\theta,\alpha} u)(t) := \int_0^{\theta(t)} (t-s)^{-\alpha} K_1(t,s) u(s)\, ds$$

and

$$(\mathcal{W}_{\theta,\alpha} u)(t) := \int_{\theta(t)}^{t} (t-s)^{-\alpha} K(t,s) u(s)\, ds,$$

with $0 < \alpha < 1$.

↪ $\theta(t) = t - \tau(t), \ \tau(t) \geq \tau_0 > 0$:

Brunner, *Appl. Numer. Math.* 57 (2007), 533-548.

↪ $\theta(t) = qt \ (0 < q < 1)$:

Current work with **Q.-Y. Hu** (*collocation*) and **D. Schötzau** (*discontinuous Galerkin method*).

• **Analysis of asymptotic stability (and contractivity)** of collocation solutions on **uniform meshes** for **VFIDE**s with **vanishing delays** (e.g. for $\theta(t) = qt \ (0 < q < 1)$?

The solutions of the pantograph DDE

$$u'(t) = au(t) + bu(qt), \quad t \geq 0,$$

satisfy $\lim_{t\to\infty} u(t) = 0$ if

$$\text{Re}(a) < 0 \quad \text{and} \quad |b| < |a|. \qquad (1)$$

↪ <u>**Open Problem 1:**</u>
For which $\{c_i\}$ does the collocation solution $u_h \in S_m^{(0)}(I_h)$, with **uniform** I_h , satisfy

$$\lim_{t\to\infty} u_h(t) = 0 \ ?$$

<u>*Special case:*</u> $m = 1,\ q = 1/2$: $c_1 \in [1/2, 1]$
(**Buhmann, Nørsett** & **Iserles (1994); Liu, Wang** & **Hu (2005)**).

↪ <u>**Open Problem 2:**</u>
Assume (1). For which (continuous) k_0 and k_1 are the solutions of the VFIDE

$$u'(t) = au(t) + bu(qt) + \int_0^t k_0(t-s)u(s)\,ds$$

$$+ \int_0^{qt} k_1(t-s)u(s)\,ds, \ \ t \geq 0,$$

asymptotically stable ?

- **DEs and VFEs with advanced arguments**

Illustration:

$$\mathbf{u'(t) = au(t) + bu(qt)}, \;\; t \geq 0, \;\; \underline{\mathbf{q > 1}} \; :$$

$\hookrightarrow$ *Ill-posed problem* !
$\hookrightarrow$ Numerical analysis remains open.

Application:
Modelling of cell growth: steady-state distribution of population of cells that grow and divide (each mother cell divides into $q > 1$ daughter cells of same size).

See, e.g., **Hall** & **Wake (1989,1990+), Wall (2007)**; *also:* **Marshall, van Brunt** & **Wake (2004)** and references.

- **Design of VFE software**

$\hookrightarrow$ Extension of **RADAR5** to **VFIDE**s (and VFIEs) ?

(See `www.unige.ch/~hairer` for details of **RADAR5**.)

• Collocation for VFEs with **state-dependent delays**

Illustration:

Population growth with 'crowding effects' (Bélair (1991)):

$$\mathbf{u(t) = \int_{t-\tau(u(t))}^{t} P(t-s)G(u(s))\,ds, \quad t > 0,}$$

$\hookrightarrow$ Attainable order of (super-) convergence of iterated collocation solution corresponding to collocation solution $\mathbf{u_h \in S_{m-1}^{(-1)}(I_h)}$?

Current work with Stefano **Maset** (Trieste)

• **Partial VFIE**s

Illustration:

Time-stepping for (semi-discretised) system corresponding to the partial VFIDE

$$\mathbf{u_t - \Delta u = \int_0^t k(t-s)G(u(s,\cdot), u(\theta(s),\cdot))\,ds,}$$

with $\mathbf{x \in \Omega \subset \mathbb{R}^d}$ $\mathbf{(d = 1, 2)}$, $\mathbf{u(t, 0) = u_0(x)}$ (plus homogeneous BCs), $\theta(\mathbf{0}) = \mathbf{0}$, and

$$\mathbf{G(u, w) = au^p + bw^r, \quad p > 1, \;\; r > 1.}$$

($\hookrightarrow$ **J. Wu**, *Theory and Applications of Partial Functional Differential Equations*, Springer-Verlag, 1996)