Quadriken – Klassifikation unter Kongruenz im $\mathbb{R}^n$

Sei Q eine Quadrik in $\mathbb{R}^n$. Dann gibt es eine Kongruenzabbildung f, natürliche Zahlen $\tilde{\pi}, \tilde{\nu}$ und reelle Zahlen $\beta_i > 0, 1 \leq i \leq \tilde{\pi} + \tilde{\nu}$, so dass die Quadrik $f(Q)$ durch eine der folgenden Gleichungen beschrieben wird:

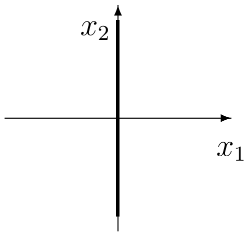
$$\sum_{i=1}^{\tilde{\pi}} \frac{x_i^2}{\beta_i^2} - \sum_{i=\tilde{\pi}+1}^{\tilde{\pi}+\tilde{\nu}} \frac{x_i^2}{\beta_i^2} = 0, \qquad \tilde{\pi} \geq \tilde{\nu}, \tag{1}$$

$$\sum_{i=1}^{\tilde{\pi}} \frac{x_i^2}{\beta_i^2} - \sum_{i=\tilde{\pi}+1}^{\tilde{\pi}+\tilde{\nu}} \frac{x_i^2}{\beta_i^2} = 1, \tag{2}$$

$$\sum_{i=1}^{\tilde{\pi}} \frac{x_i^2}{\beta_i^2} - \sum_{i=\tilde{\pi}+1}^{\tilde{\pi}+\tilde{\nu}} \frac{x_i^2}{\beta_i^2} = x_{\tilde{\pi}+\tilde{\nu}+1}, \qquad \tilde{\pi} + \tilde{\nu} < n. \tag{3}$$

Quadriken in $\mathbb{R}^2$

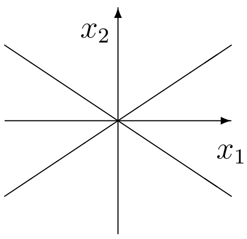
(1) $\nu = 0, \quad \pi = 1, \qquad x_1^2 = 0$



Gerade

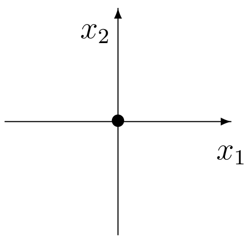
$\nu = 1, \quad \pi = 1, \qquad \frac{x_1^2}{\beta_1^2} - x_2^2 = 0$

$x_1 = \pm\beta_1 x_2$



zwei sich schneidende Geraden

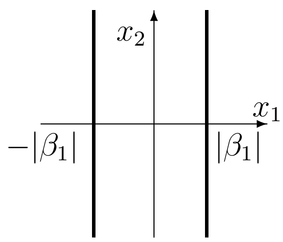
$\nu = 0, \quad \pi = 2, \qquad \frac{x_1^2}{\beta_1^2} + x_2^2 = 0$



Punkt

(2) $\nu = 1, \quad \pi = 0, \qquad \frac{-x_1^2}{\beta_1^2} = 1$ $\qquad \emptyset$

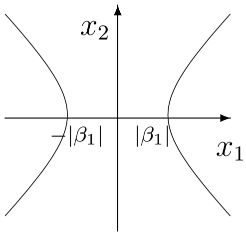
$\nu = 0, \quad \pi = 1, \qquad \frac{x_1^2}{\beta_1^2} = 1$



zwei parallele Geraden

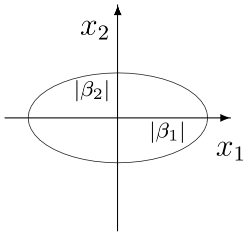
$\nu = 2, \quad \pi = 0, \qquad \frac{-x_1^2}{\beta_1^2} - \frac{x_2^2}{\beta_2^2} = 1$ $\qquad \emptyset$

$\nu = 1, \quad \pi = 1, \qquad \dfrac{x_1^2}{\beta_1^2} - \dfrac{x_2^2}{\beta_2^2} = 1$



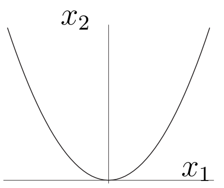
Hyperbel

$\nu = 0, \quad \pi = 2, \qquad \dfrac{x_1^2}{\beta_1^2} + \dfrac{x_2^2}{\beta_2^2} = 1$



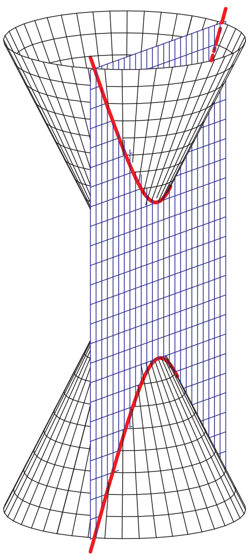
Ellipse

(3) $\nu = 0, \quad \pi = 1, \qquad \dfrac{x_1^2}{\beta_1^2} = x_2$

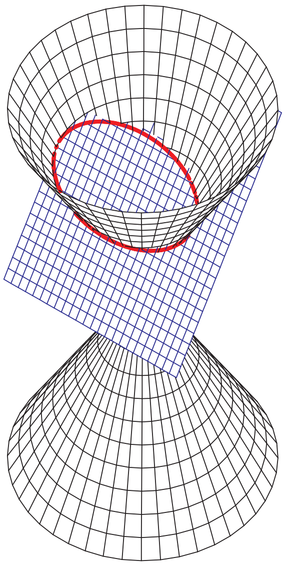


Parabel

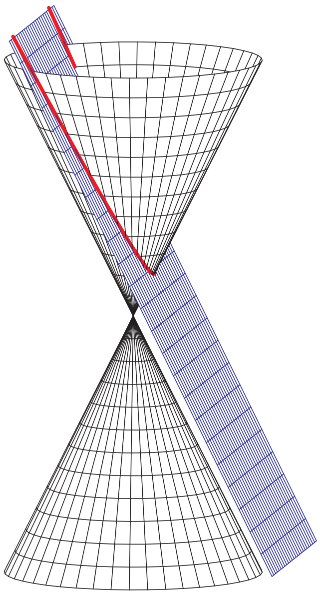
Quadriken in $\mathbb{R}^2$ sind Schnitte von Ebenen mit einem doppelten Kreiskegel, sie werden daher auch Kegelschnitte genannt.



Hyperbel



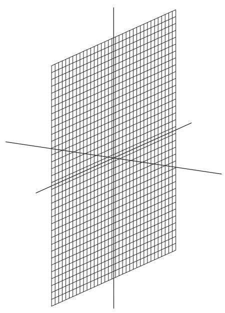
Ellipse



Parabel

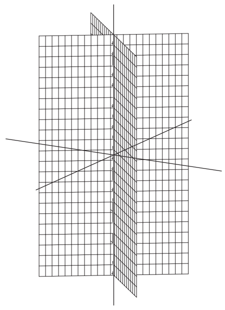
Quadriken in $\mathbb{R}^3$

(1) $\nu = 0, \quad \pi = 1, \qquad x_1^2 = 0$

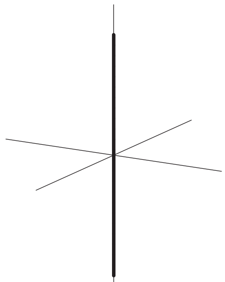


eine Ebene

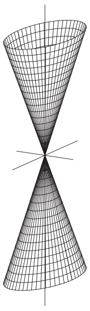
$\nu = 1, \quad \pi = 1, \qquad \dfrac{x_1^2}{\beta_1^2} - x_2^2 = 0$ — zwei sich schneidende Ebenen



$\nu = 0, \quad \pi = 2, \qquad \dfrac{x_1^2}{\beta_1^2} + x_2^2 = 0$ — eine Gerade



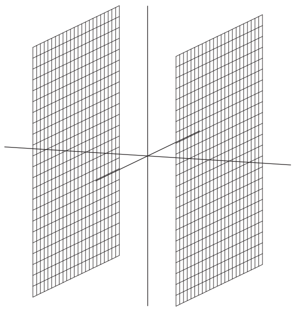
$\nu = 1, \quad \pi = 2, \qquad \dfrac{x_1^2}{\beta_1^2} + \dfrac{x_2^2}{\beta_2^2} - x_3^2 = 0$ — Ellipsenkegel



$\nu = 0, \quad \pi = 3, \qquad \dfrac{x_1^2}{\beta_1^2} + \dfrac{x_2^2}{\beta_2^2} + x_3^2 = 0$ — Punkt

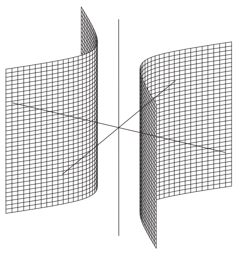
(2) $\nu = 1, \quad \pi = 0, \qquad \dfrac{-x_1^2}{\beta_1^2} = 1$ — $\emptyset$

$\nu = 0, \quad \pi = 1, \qquad \dfrac{x_1^2}{\beta_1^2} = 1$ — zwei parallele Ebenen

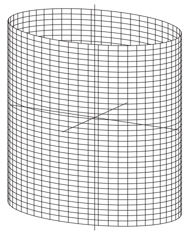


$\nu = 2, \quad \pi = 0, \qquad \dfrac{-x_1^2}{\beta_1^2} - \dfrac{x_2^2}{\beta_2^2} = 1$ — $\emptyset$

$\nu = 1, \quad \pi = 1, \qquad \dfrac{x_1^2}{\beta_1^2} - \dfrac{x_2^2}{\beta_2^2} = 1$ — Hyperbelzylinder



$\nu = 0, \quad \pi = 2, \qquad \frac{x_1^2}{\beta_1^2} + \frac{x_2^2}{\beta_1^2} = 1$

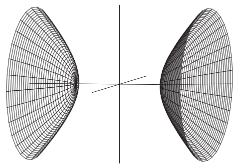


Ellipsenzylinder

$\nu = 3, \quad \pi = 0, \qquad \frac{-x_1^2}{\beta_1^2} - \frac{x_2^2}{\beta_2^2} - \frac{x_3^2}{\beta_3^2} = 1$

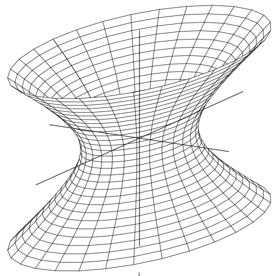
$\emptyset$

$\nu = 2, \quad \pi = 1, \qquad \frac{x_1^2}{\beta_1^2} - \frac{x_2^2}{\beta_2^2} - \frac{x_3^2}{\beta_3^2} = 1$



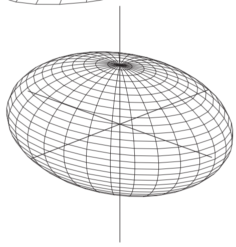
zweischaliges Hyperboloid

$\nu = 1, \quad \pi = 2, \qquad \frac{x_1^2}{\beta_1^2} + \frac{x_2^2}{\beta_2^2} - \frac{x_3^2}{\beta_3^2} = 1$



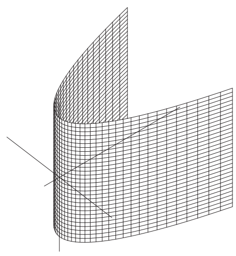
einschaliges Hyperboloid

$\nu = 0, \quad \pi = 3, \qquad \frac{x_1^2}{\beta_1^2} + \frac{x_2^2}{\beta_2^2} + \frac{x_3^2}{\beta_3^2} = 1$



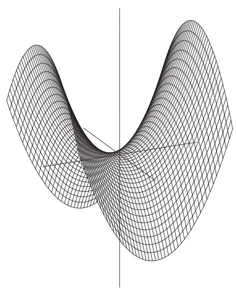
Ellipsoid

(3) $\nu = 0, \quad \pi = 1, \qquad \frac{x_1^2}{\beta_1^2} = x_2$



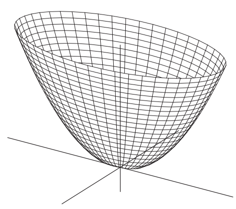
parabolischer Zylinder

$\nu = 1, \quad \pi = 1, \qquad \frac{x_1^2}{\beta_1^2} - \frac{x_2^2}{\beta_2^2} = x_3$



hyperbolisches Paraboloid

$\nu = 0, \quad \pi = 2, \qquad \frac{x_1^2}{\beta_1^2} + \frac{x_2^2}{\beta_2^2} = x_3$



elliptisches Paraboloid