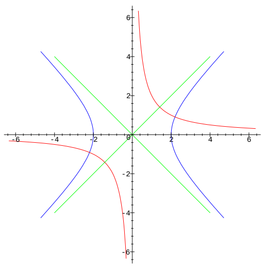
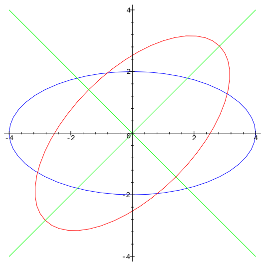
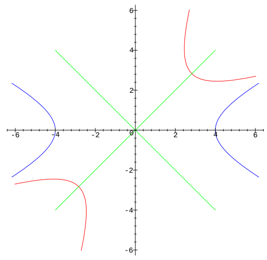
Lösungen zu Übung 4

1.

a) 

b) 

c) 

$$\frac{\tilde{x}^2}{4} - \frac{\tilde{y}^2}{4} = 1 \qquad \frac{\tilde{x}^2}{16} + \frac{\tilde{y}^2}{4} = 1 \qquad \frac{\tilde{x}^2}{16} - \frac{\tilde{y}^2}{4} = 1$$

2. Für $\tau_0 = \dfrac{\pi}{4}$: $\varkappa(\tau_0) = |p''(\tau_0)| = \ldots = \frac{1}{9}\sqrt{6 \cdot 26} \approx 1.3878$.

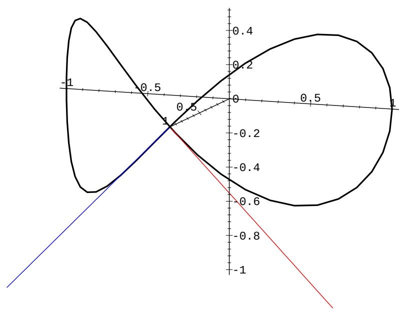
$$p = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \dot{p} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 1 \\ 0 \end{pmatrix}, \ddot{p} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ -2 \end{pmatrix}, \; |\dot{p}|^2 = \frac{3}{2}, \; |\ddot{p}|^2 = \frac{9}{2}, \; (\dot{p} \cdot \ddot{p}) = \frac{1}{2}.$$

3. $\tau_1 = \dfrac{\pi}{2}$, $\tau_2 = \dfrac{3\pi}{2}$

$$t_1 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\cos \sphericalangle(t_1, t_2) == 0$$

$$\Longrightarrow \quad \sphericalangle(t_1, t_2) = \frac{\pi}{2} \quad (= 90^\circ)$$



4. $\varkappa = |p''| = \dfrac{1}{5} \;\Rightarrow\; \varrho = 5$

$$M = p(\tau_0) + \varrho(\tau_0) \cdot \vec{n}(\tau_0) = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} + 5 \cdot \frac{1}{5} \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

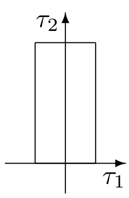
Bogenlänge: $s(\tau) = 5\tau \;\Rightarrow\; s\left(\frac{\pi}{2}\right) = \frac{5}{2}\pi \approx 7.854$

5. $p(\tau_1, \tau_2) = \begin{pmatrix} r\cos\tau_1 \cos\tau_2 \\ r\cos\tau_1 \sin\tau_2 \\ r\sin\tau_1 \end{pmatrix}$,

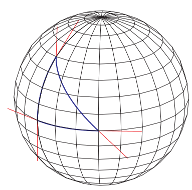
a) Tangentenrichtungen: $t_{12} = \begin{pmatrix} -r\sin(\tau) \\ r\cos(\tau) \\ 0 \end{pmatrix}$, $t_{13} = \begin{pmatrix} -r\sin(\tau) \\ 0 \\ r\cos(\tau) \end{pmatrix}$,

$$t_{23} = \begin{pmatrix} r\sin(\tau_2 - \tau_1) \\ -r\cos(\tau_2 - \tau_1) \\ r\cos(\tau_1) \end{pmatrix}, \text{ wobei } \tau_2 - \tau_1 = \tfrac{\pi}{4} - 2\tau, \quad \tau_1 = \tau.$$

$$\sphericalangle(P_1) = 90^\circ, \quad \sphericalangle(P_2) = 45^\circ, \quad \sphericalangle(P_3) \approx 35{,}26^\circ.$$



$\Longrightarrow$



b) Kürzester Weg zwischen P_2 und P_3: Kurve muss in einer Ebene mit dem Mittelpunkt O und P_2, P_3 liegen, also in der Ebene mit dem Normalenvektor $\overrightarrow{OP_2} \times \overrightarrow{OP_3}$, also $0 = x - y - z = r(\cos\tau_1 \cos\tau_2 - \cos\tau_1 \sin\tau_2 - \sin\tau_1)$.
Mit $\tau_2 = \tau$ und $\tau_1 = \arctan(\cos\tau - \sin\tau)$ folgt

$$p(\tau) = p(\tau_1(\tau), \tau_2(\tau)) = \frac{r}{\sqrt{2 - \sin 2\tau}} \begin{pmatrix} \cos\tau \\ \sin\tau \\ \cos\tau - \sin\tau \end{pmatrix}$$

6. Flächennormalenvektor:

$$f(l, \psi) = \frac{1}{\sqrt{R^2 \sin^2\psi + r^2 \cos^2\psi}} \begin{pmatrix} r\cos l \cos\psi \\ r \sin l \cos\psi \\ R\sin\psi \end{pmatrix} \approx \begin{pmatrix} 0.63073 \cos l \\ 0.63073 \sin l \\ 0.77600 \end{pmatrix}$$

Breitenkreis: $b = 50.825°,\ -\pi \le l \le \pi$

$$k(l) = \begin{pmatrix} 4030.896 \cos l \\ 4030.896 \sin l \\ 4926.675 \end{pmatrix}, \quad \varrho = R\cos\psi = 4030.896, \quad \varkappa = \frac{1}{\varrho} \approx 2.48 \cdot 10^{-4}.$$

Normalkrümmung:

$$\varkappa_n = \varkappa \cdot \cos\sphericalangle(f, n_k) \approx 1.565 \cdot 10^{-4},$$

entspricht einem Krümmungskreisradius $\varrho = 6390.4$ km.

Mittels Fundamentalformen:
Flächenkurve

$$l = \tau_1 = \tau,\ b = \tau_2 = \psi = const \quad \Rightarrow \quad dl = d\tau_1 = d\tau,\ db = d\tau_2 = 0, \quad \text{deshalb :}$$

$$\varkappa_n = \frac{\sum\limits_{i,j} h_{ij} d\tau_i d\tau_j}{\sum\limits_{i,j} g_{ij} d\tau_i d\tau_j} = \frac{h_{11}}{g_{11}} = \frac{\langle f, p_{ll}\rangle}{\langle p_l, p_l\rangle}$$

$$h_{11} = \frac{Rr\cos^2\psi}{\sqrt{R^2\sin^2\psi + r^2\cos^2\psi}}, \quad g_{11} = R^2\cos^2\psi$$

$$\varkappa_n = \frac{r}{R\sqrt{R^2\sin^2\psi + r^2\cos^2\psi}} \approx 1.565 \cdot 10^{-4}$$