Semi-infinite variations of Hodge structures and Frobenius manifolds

aim: describe some (complex) differential geometry structure (and how to construct it) on the parameter space of (big) quantum cohomology / univ. deform. space of LG-potential ($\hat{=}$ moduli space of 2D top field theories)

Def (Dubrovin): A Frobenius manifold is a tuple $(M, \circ, e, E, g)$ where M is a complex manifold ($\dim(M) > 0$), $g \in \mathcal{T}_M^{*\otimes 2} =: \mathcal{T}_M^{(2,0)}$ is a complex metric, i.e. a bilinear symmetric non-degenerate form. Write $\nabla : \mathcal{T}_M \times \mathcal{T}_M \to \mathcal{T}_M$ for its Levi-Civita connection (defined as in the Riemannian case). $\circ \in \mathcal{T}_M^{(2,1)}$ is a symmetric & associative multiplication on $\mathcal{T}_M$

with global unit $e \in \Gamma(M, \mathcal{T}_M)$, $E \in \Gamma(M, \mathcal{T}_M)$ is another global vector field. These data are subject to the following conditions:

1.) g is $\circ$-invariant: $g(X \circ Y, Z) = g(X, Y \circ Z)$

2.) Let $\nabla \circ$ be the covariant derivative of $\circ$, we have $\nabla \circ \in \mathcal{T}_M^{(3,1)}: (X, Y, Z) \mapsto \nabla_X(Y \circ Z) - (\nabla_X Y) \circ Z - Y \circ (\nabla_X Z)$

Then $\nabla \circ$ should be symmetric in X, Y, Z

3.) ∇ is flat (curvature $R^\nabla \in \mathcal{T}_M^{(3,1)}$ defined by $R^\nabla_{X,Y} Z := \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y](Z)$ is zero)

4.) e is flat: $\nabla e = 0$

5.) E satisfies $Lie_E(\circ) = d \circ$ & $Lie_E(g) = D\, g$ for some constants $d, D \in \mathbb{C}$.

(here Lie_E is a tensor derivation defined by $Lie_E X = [E, X]$, $Lie_E(f) = E(f)$; recall that it is def. by $(Lie_E T)(Y) := Lie_E(T(Y)) - T(Lie_E Y)$)

Remarks: 1.) flatness of ∇ implies: locally, $\exists$ coordinates $x_1, \ldots, x_n$ s.t. $\nabla_{\partial_{x_i}}(\partial_{x_j}) = 0 \Leftrightarrow g(\partial_{x_i}, \partial_{x_j}) = \text{const.}$

2.) can show $(\nabla\circ) \in \mathcal{T}^{(3,1)}_M$ symmetric $\Leftrightarrow$ $(\nabla A) \in \mathcal{T}^{(4,0)}$ symm.

where $A(X,Y,Z) := g(X, Y\circ Z)$

3.) ∇ flat & (∇A) symmetric $\Leftrightarrow$ locally, $\exists\, \Phi \in \mathcal{O}_M$ s.t.

$A(X,Y,Z) = (XYZ)\Phi$ Let $t_0, \ldots, t_m$ be flat coordinates

$$\Rightarrow (\nabla_{\partial_{t_i}} A)(\partial_{t_j}, \partial_{t_k}, \partial_{t_l}) = \partial_{t_i}\, g(\partial_{t_j}, \partial_{t_k} \circ \partial_{t_l})$$

symmetric in all 4 arguments: e.g.

$$\partial_{t_i}\, g(\partial_{t_j}, \partial_{t_k} \circ \partial_{t_l}) = \partial_{t_j}\, g(\partial_{t_i}, \partial_{t_k} \circ \partial_{t_l})$$

for fixed k, l, put $g^{(i)} := g(\partial_{t_i}, \partial_{t_k} \circ \partial_{t_l})$

$$\Rightarrow \partial_{t_j} g^{(i)} = \partial_{t_i} g^{(j)} \Rightarrow g^{(i)} = \partial_{t_i} h \quad \text{etc} \ldots\ldots$$

Φ is called (local) potential of $(M, g, \ldots)$

Fact: $\circ$ associative $\Leftrightarrow$ Φ satisfies WDVV

$$\partial_i \partial_j \partial_e \Phi \cdot g^{ef} \partial_f \partial_k \partial_l \Phi = \partial_j \partial_k \partial_e F \cdot g^{ef} \cdot \partial_f \partial_i \partial_l F$$

<u>main example</u>: big quantum cohomology of smooth, proj, toric Fano X

We will ignore all convergence questions here and hence construct M as follows: assume $H^*(X,\mathbb{C}) = H^{2*}(X,\mathbb{C})$ and chose basis $T_0, \ldots, T_m$ of $H^*(X,\mathbb{R})$ s.t.:

1.) T_0 generates $H^0(X,\mathbb{R})$

2.) $T_1, \ldots, T_r$ generate $H^2(X,\mathbb{R})$, T_i are nef classes on X

3.) $-K_X \in \sum_{i=1}^{r} \mathbb{R}_{\geq 0} T_i$

4.) $T_{r+j} \in H^{>2}(X,\mathbb{R})$ $\forall j > 0$

write $t_0, \ldots, t_m$ for corresponding coordinates on $H^*(X,\mathbb{C})$

put $\mathcal{K} := H^2(X,\mathbb{C}) / 2\pi i H^2(X,\mathbb{Z})$ then $\mathcal{O}_{\mathcal{K}} = \mathbb{C}[\kappa_1^{\pm}, \ldots, \kappa_r^{\pm}]$

with $\kappa_i := e^{t_i}$, $i = 1, \ldots, r$ $\quad \overline{\mathcal{K}} \simeq \mathbb{C}^r = \operatorname{Spec} \mathbb{C}[\kappa_1, \ldots, \kappa_r]$

let $\mathcal{U} \subset H^0(X,\mathbb{C}) \times \overline{\mathcal{K}} \times H^{>2}(X,\mathbb{C})$ be (analytic) open neighborhood of 0, then $M := \mathcal{U} \cap (H^0 \times \mathcal{K} \times H^{>2})$

Put $g(\partial_{t_i}, \partial_{t_j}) := \int_X T_i \cup T_j \rightsquigarrow \partial_{t_i}$ flat, i.e.

$t_0, \dots, t_m$ are flat coordinates (notice $\partial_{t_i} = \kappa_i \partial_{\kappa_i}$)

Moreover, define $A(\partial_{t_i}, \partial_{t_j}, \partial_{t_k}) := (\partial_{t_i} \partial_{t_j} \partial_{t_k} \Phi)(\underline{t})$ (GW-pot. $\downarrow$ Φ)

$$= \sum_{n,\beta} \frac{1}{n!} \langle T_i, T_j, \underbrace{\underline{t}, \dots, \underline{t}}_{n\text{-times}}, T_k \rangle_{0,n+3,\beta}$$, then $\partial_{t_i} \circ \partial_{t_j}$

$$\partial_{t_i} \underset{\underline{t}}{*} \partial_{t_j} = \sum_{n,\beta,k,l} \frac{1}{n!} \langle T_i, T_j, \underline{t}, T_k \rangle_{0,n+3,\beta}\, g^{kl} T_l, \quad (g^{kl}) = (g_{kl})^{-1}$$

By fundamental class axiom: $e := \partial_{t_0}$ is unit

Put $$E := \sum_{i=0}^{m} \Big(1 - \tfrac{1}{2}\underbrace{\deg(T_i)}_{\in 2\mathbb{Z}}\Big) t_i \partial_{t_i} + \sum_{j=1}^{r} c_j \cdot \partial_{t_j}$$

where $c_1(T_X) = \sum_{i=1}^{r} c_i \cdot T_i$. Then $[E, \partial_{t_k}] = \left(1 - \frac{\deg T_k}{2}\right) \partial_{t_k}$

$$\Rightarrow (\mathrm{Lie}_E\, g)(\partial_{t_i}, \partial_{t_j}) = \left(2 - \frac{\overbrace{\deg T_i + \deg T_j}^{\dim X}}{2}\right) = \underbrace{(2 - \dim X)}_{D}\, g(\partial_{t_i}, \partial_{t_j})$$

similarly (more complicated): $\mathrm{Lie}_E \circ = 1 \cdot \circ$ follows by homogeneity argument

<u>Correspondence</u>: We consider holomorphic VB $\mathcal{E} \to \mathbb{P}^1 \times M$ which are fibrewise trivial, i.e. $E \simeq \pi^* \pi_* E$, with $\pi : \mathbb{P}^1 \times M \to M$, together with a <u>flat</u> meromorphic connection

$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{\mathbb{P}^1 \times M}(* \{0,\infty\} \times M)$ s.t.

$(E_{|\mathbb{C} \times M}, \nabla)$ has pole of Poincaré rk $= 1$ along $\{0\} \times M$

i.e. $z \nabla_X (\mathcal{E}) \subset \mathcal{E} \;\; \forall X \in \mathcal{T}_{\mathbb{C} \times M}(\log \{0\} \times M)$

$(E_{|(\mathbb{P}^1 \setminus 0) \times M}, \nabla)$ has log. pole (Poincaré rk $= 0$) along $\{\infty\} \times M$

in part. $(\mathcal{E}_{|\mathbb{C}^* \times M}, \nabla)$ is flat bundle corresponding to local system $\mathcal{L}$.

Let $P : \mathcal{L} \otimes \iota^* \mathcal{L} \to \mathbb{C}_{\mathbb{C}^*}$ be a symm., bilin. pairing

$\iota(z, \underline{t}) = (-z, \underline{t})$ s.t. $P(\mathcal{E}, \mathcal{E}) \subset \mathcal{O}_{\mathbb{P}^1 \times M}$ (i.e. no pole at 0 or ∞)

Construction theorem for Frobenius mfd's: Let $(\mathcal{E}, \nabla, P)$ as above be given. Take $x \in M$, assume that $\exists\, x \in U \subset M$, $\xi \in \Gamma(U, \pi_* \mathcal{E})$ s.t. ← primitive form (K. Saito)

$$\phi_\xi : \mathcal{T}_U \xrightarrow{\cong} [\mathcal{E}/z\mathcal{E}] \quad \& \quad \xi \text{ eigenvector of}$$
$$X \longmapsto -[z\nabla_X \xi] \qquad [z\nabla_{\partial_z}] \in \mathcal{E}nd(\mathcal{E}/z^{-1}\mathcal{E})$$

In part., $\operatorname{rk} \mathcal{E} = \dim M$.

Theorem (Dubrovin, Hertling-Manin): $(\mathcal{E}, \nabla, P)$ as above $\nabla^2 = 0$, $\nabla P = 0$, $\exists$ primitive form on $U \rightsquigarrow (U, \circ, E, e, g)$ Frob-mf.

Idea of proof: Since $\mathcal{E} \simeq \pi^* \pi_* \mathcal{E}$, we have canonical isomorphisms $K := \mathcal{E}_{|\{0\}\times M} \simeq \pi_* \mathcal{E} \simeq \mathcal{E}_{|\{\infty\}\times M}$

$(\mathcal{E}_{|\mathbb{C}\times M}, \nabla)$ has pole of $\operatorname{rk} = 1 \Rightarrow$ class of $z \cdot \nabla$ on $\mathcal{E}_{|\{0\}\times M}$ is Higgs field $(C := [z\nabla] : K \to K \otimes \Omega^1_M\ \mathcal{O}_M\text{-lin.})$

$(\mathcal{E}_{|(\mathbb{P}^1\setminus\{0\})\times M}, \nabla)$ has log pole $\Rightarrow$ class of ∇ on $\mathcal{E}_{|\{\infty\}\times M}$ is connection ∇^r on K, called residue connection

Write $V \in \mathrm{End}(K)$ for residue endomorphism along $\{\infty\} \times M$ (i.e. class of $[-z\nabla_{\partial_z}]$ on $\mathcal{E}_{|\{\infty\}\times M}$)

Write $\mathcal{U} \in \mathrm{End}(K)$ for class of $[z^2\nabla_z]$ on $\mathcal{E}_{|\{0\}\times M}$

Then $\forall s \in \Gamma(M, \pi_* \mathcal{E}) \simeq \mathcal{E}_{|\{0\}\times M} \simeq \mathcal{E}_{|\{\infty\}\times M}$, ∇ is

$$\nabla s = \left(\frac{1}{z^2}\mathcal{U}(s) + \frac{1}{z}V(s)\right)dz + \frac{1}{z}C(s) + \nabla^r(s) \in$$

$$\mathcal{E} \otimes \Omega^1_{\mathbb{P}^1\times M}(* \{0,\infty\} \times M)$$

Moreover, $P_{|\mathcal{E}_{|\{0\}\times M}} =: g$ gives symm, non-deg. pairing $g: K \times K \to \mathcal{O}_M$

Now use ϕ_ξ to pull-back $C, \nabla^r, \mathcal{U}, V, g$ to $\mathcal{T}_M$. Notice $\phi_\xi^* C : \mathcal{T}_M \times \mathcal{T}_M \to \mathcal{T}_M$ is Higgs field on $\mathcal{T}_M \iff$ multiplication

Concretely: $X \circ Y := \phi_\xi^{-1}(C_X C_Y \xi)$ (notice $[C_X, C_Y] = 0$)

Moreover, put $e := \phi_\xi^{-1}(\xi)$, $E := \phi_\xi^{-1}(\mathcal{U}(\xi))$.

generalization: assume only that ξ is such that

1.) Φ_ξ injective

2.) $[\pi_* \mathcal{E}]_{|\mathcal{U}}$ generated over $\mathcal{O}_\mathcal{U}$ by ξ and iterations of $C_X, X \in T\mathcal{U}$ and $\mathcal{U}$

3.) ξ is eigenvector of V

} still get Frob. struct. on $\mathcal{U}$

How to construct $(\mathcal{E}, \nabla, P)$?

<u>Def</u>: A bundle $\mathcal{F} \to \mathbb{C} \times M$, with flat connection $\nabla: \mathcal{F} \to \mathcal{F} \otimes \frac{1}{z} \Omega^1_M (\log \{0\} \times M)$ and flat pairing $P: \mathcal{F} \otimes \iota^* \mathcal{F} \to \mathcal{O}_{\mathbb{C}\times M}$, which is non-deg. (i.e. $[P] = \mathcal{F}/_{z\mathcal{F}} \otimes \mathcal{F}/_{z\mathcal{F}} \to \mathcal{O}_M$ is non-deg) is called $\frac{\infty}{2}$-VHS, TEP-structure, (part of) ncHodge-str

<u>clear</u>: $(M, \circ, \ldots)$ Frob $\rightsquigarrow (\mathcal{E}, \nabla, P)$ as above $\rightsquigarrow \mathcal{F} := \mathcal{E}_{|\mathbb{C}\times M}$

much less clear: given $(\mathcal{F}, \nabla, P) \overset{?}{\rightsquigarrow} (\mathcal{E}, \nabla, P)$

task: extend $\mathcal{F}$ to (fibrewise) trivial $\mathbb{P}^1$-bdl $\mathcal{E}$ s.t. ∇ aquires log. pole along $\{\infty\} \times M$

<u>example:</u> assume $M = \{pt\} \rightsquigarrow F$ VB on $\mathbb{C}$

$\nabla : F \to F \otimes \frac{1}{z^2} \Omega^1_{\mathbb{C}}$. Let $\underline{e} = (e_0, \dots, e_m)$ be basis

$$\rightsquigarrow \nabla \underline{e} = \underline{e} \cdot \left(A_0 \frac{1}{z^2} + A_1 \frac{1}{z} + \dots \right) dz$$

<u>aim:</u> construct extension $\mathcal{E} \to \mathbb{P}^1$ s.t.:

1.) $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}^{m+1}$

2.) ∇ has log. pole on $\mathcal{E}$ at ∞

both cond. are easy individually, but not together:

Strategy: when defining $\mathcal{E} := \bigoplus_{i=0}^{m} \mathcal{O}_{\mathbb{P}^1} \cdot e_i$, $\tau := z^{-1}$

$$d\tau = -\frac{1}{z^2} dz \iff dz = -\frac{1}{\tau^2} d\tau$$

$$\nabla \underline{e} = \underline{e} \cdot \Big(-A_0 - \underbrace{A_1 \cdot \frac{1}{\tau} - A_2 \frac{1}{\tau^2} - \dots}_{=0 \text{ if } \nabla \text{ has log pole}} \Big) d\tau$$

<u>hence:</u> find basis $(v_0, \dots, v_m)$ of F s.t.

$$\nabla \underline{v} = \underline{v} \cdot \left(A_0 \frac{1}{z^2} + A_1 \frac{1}{z} \right) dz.$$

back to general situation: let

$$\left(\mathcal{F} \to \mathbb{C}\times M,\ \nabla: \mathcal{F} \to \mathcal{F}\otimes z^{-1}\cdot\Omega^1_{\mathbb{C}\times M}(\log\{0\}\times M),\ P: \mathcal{F}\otimes \iota^*\mathcal{F} \to \mathcal{O}_M\right)$$

be $\infty/2$-VHS, then need to find basis

$\underline{v} = (v_0, \dots, v_m)$ of $\mathcal{F}$ s.t.

$$\nabla \underline{v} = \underline{v}\cdot\left[\left(\mathcal{U}\frac{1}{z^2} + \mathcal{V}\frac{1}{z}\right)dz + \frac{1}{z}C + \nabla^r\right]$$

---

rk on notation in Gross: $\frac{\infty}{2}$-VHS are defined only as $(\mathcal{F}, \nabla: \mathcal{F} \to z^{-1}\Omega^1_M, P)$ together with grading operator $Gr: \mathcal{F} \to \mathcal{F}$ and $E \in \mathcal{T}_M$ with

- $Gr(fs) = (z\partial_z + E)(f)s + f\,Gr(s)$
- $[Gr, \nabla_Y] = \nabla_{[E,Y]}$
- $(z\partial_z + E)\,P(s_1, s_2) = P(Gr(s_1), s_2) + P(s_1, Gr(s_2)) + D\cdot P(s_1, s_2)$ for $D \in \mathbb{C}$

This is contained in our definition

when adding the requirement:

$\exists E \in \hat{\mathcal{T}}_M$ s.t. classes $[\nabla_{z\partial_z}] = [-\nabla_E]$ in $\mathcal{F}/_{z\mathcal{F}}$

$\mathcal{G}r := \nabla_{z\partial_z + E} : \mathcal{F} \to \mathcal{F}$ satisfies above conditions

restriction of such $\mathcal{F}$ to int. curve of $\mathcal{F}$ is "rescaling"

Rk. on "moving subspace realization": assume that M is simply connected, then consider

$$\mathcal{H} = \left\{ s \in \Gamma(M, \mathcal{F}(*\{0\} \times M)) \;\middle|\; \nabla_X s = 0 \;\forall X \in \hat{\mathcal{T}}_M \right\}$$

this is a free $\mathcal{O}_{\mathbb{C}}(*0)$-module.

Notice $s_1, s_2 \in \mathcal{H} \Rightarrow P(s_1, s_2) \in \mathcal{O}_{\mathbb{C}\times M}(*\{0 \times M\})$

and $P(s_1,s_2)(z) = P(s_1(z), s_2(-z)) = P(s_2(z), s_1(-z))$,

hence $\Omega(s_1, s_2) := \mathrm{Res}_{z=0}\, P(s_1,s_2)(z)\, dz$

is anti-symmetric (symplectic form on $\mathcal{H}$ since constant)

Let $x \in M$, then $F_{|\mathbb{C}^\times\{x\}} \leftarrow \mathcal{H}$

$$s \longmapsto \tilde{s}$$

where $\tilde{s}$ is unique flat section s.t. $\tilde{s}(x) = s$. The subspace $F_{|\mathbb{C}^\times\{x\}}$ corresponds to point in affine Grassmannian $LGl(N,\mathbb{C}) / L^+Gl(N,\mathbb{C})$, where: $N := m+1$

$LGl(N,\mathbb{C}) = \mathcal{C}^\infty(S^1_\varepsilon, gl(N,\mathbb{C}))$

$L^+Gl(N,\mathbb{C}) = \{\gamma \in LGL(N,\mathbb{C}) \mid \gamma$ extends to $Hol(D_\varepsilon, Gl(N,\mathbb{C}))\}$

By fixing $\mathcal{O}_{\mathbb{C}}(x_0)$-basis $s_0, \dots, s_m$ of $\mathcal{H}$, a basis $e_0, \dots, e_m$ of $F_{|\mathbb{C}^\times\{x\}}$ over $\mathcal{O}_{\mathbb{C}}$ is given by matrix $S \in LGl(N,\mathbb{C})$, but $F_{|\mathbb{C}^\times\{x\}}$ only depends on class of S in $LGl(N,\mathbb{C})/L^+Gl(N,\mathbb{C})$

"opposite subspace": suppose that we are given an $\mathcal{O}_{\mathbb{P}^1,0}$-submodule $\mathcal{H}_-$ of $\mathcal{H}$ s.t. $\mathcal{H}_- \oplus F_{|\mathbb{C}\times\{x\}} \simeq \mathcal{H}$. Then (exercise):

$$F_{|\mathbb{C}\times\{x\}} \cap z\cdot\mathcal{H}_- \simeq F_{|\mathbb{C}\times\{x\}} / z\cdot F_{|\mathbb{C}\times\{x\}}$$

<u>Th (Dubrovin, Hertling-Manin)</u>: $\exists$ open neighb. $x \in U \subset M$ s.t. $\exists\ \mathcal{E} \to \mathbb{P}^1 \times U$ extension of F, $\mathcal{E} = \pi^*\pi_*\mathcal{E}$, log. pole along $\{\infty\}\times U$ and $[\pi_*\mathcal{E}]_x = F_{|\mathbb{C}\times\{x\}} \cap z\mathcal{H}_-$

If $\mathcal{H}_-$ is Ω-isotropic $\Rightarrow$ P z-const. on $\pi_*\mathcal{E}$

$\frac{\infty}{2}$-VHS from quantum cohomology (in principle, this is already defined via QCH $\rightsquigarrow$ Frob-mf $\rightsquigarrow$ $(\mathcal{E},\nabla,P)$ $\rightsquigarrow$ $F := \mathcal{E}_{|\mathbb{C}\times M}$): $M = H^*(X,\mathbb{C})$, coord. $t_0, \ldots, t_m$

let ∇^T (flat) conn. on $\mathcal{T}_M \simeq H^*(X,\mathbb{C}) \otimes \mathcal{O}_M$ for which $\partial_{t_0}, \ldots, \partial_{t_m}$ is flat.

$$\mathcal{F} = \mathcal{O}_{\mathbb{C}\times M} \otimes H^*(X,\mathbb{C}) = \pi^* \mathcal{T}_M \quad , \pi : \mathbb{C}\times M \to M$$

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes z^{-1} \Omega^1_{\mathbb{C}\times X} (\log \{0\}\times M) \text{ def. by}$$

$$\nabla_X s := \nabla^\tau_X s + \frac{1}{z} X * s$$ ← quantum product

$$\nabla_{z\partial_z} s := \frac{1}{z} E * s + V(s) \text{ , where } V \in End(\mathcal{F})$$

$$\text{def. by } V(T_i) := \frac{\deg T_i}{2} - 1$$

$$P(s_1, s_2)(z) = \int_X s_1(z) \cup s_2(-z)$$

} here: $\mathcal{E} := \mathcal{O}_{\mathbb{P}^1 \times M} \otimes H^*(X,\mathbb{C})$, ∇ satisfies all requirements

recall from Luca's talk: "horizontal flat" sections

$$s_i(\gamma, z^{-1}) := T_i - \sum_{j=0}^{m} \sum_{\substack{n\geq 0, \\ \beta \in \mathrm{Eff}_X}} \frac{1}{n!} \left\langle \frac{T_i}{z+\psi}, T_j, \gamma \right\rangle_{0,\beta} \cdot T^j$$

$i = 0, \dots, m$

$$\frac{T_i}{z+\psi} = \sum_{\nu \geq 0} (-1)^\nu \psi^\nu \cdot T_i \cdot z^{-(\nu+1)}$$

ψ inserted at k-th spot means ψ_k

$$S := (s_0 | \dots | s_m)$$

Since $\nabla_X s_i = 0 \;\; \forall X \in \mathcal{T}_M$, the moving subspace realization is given by $J := S^{-1} \in LGL(N, \mathbb{C})$

Prop: $\mathbb{J}(s) = e^{\sum_{k=0}^{r}(t_k T_k)/z}\Big[s +$

$$\sum_{\beta, n_{r+1}, \dots, n_m} \sum_{i=0}^{m} \left\langle s, T_{r+1}^{n_{r+1}}, \dots, T_m^{n_m}, \frac{T_i}{z-\psi} \right\rangle_{0,\beta} T^i \cdot \prod_{k=1}^{r} e^{t_k \int_\beta T_k} \cdot \frac{t_{r+1}^{n_{r+1}} \cdots t_m^{n_m}}{n_{r+1}! \cdots n_m!} \Big]$$

Def. Givental's J-function is the $H^*(X,\mathbb{C})$-valued section/series $J(z^{-1}) := \mathbb{J} \cdot T_0$.

notice: from $\nabla_{\partial_{t_j}} s_i = 0$ we get $z\partial_{t_i} J =$

$$\mathbb{J}(\nabla_{z\partial_{t_i}} T_0) = \mathbb{J}(T_0 * T_i) = \mathbb{J}(T_i)$$

$\leadsto \mathbb{J} = \mathfrak{Z}$ is primitive form

case $X = \mathbb{P}^2$: $H^*(\mathbb{P}^2, \mathbb{C}) = \mathbb{C}\cdot T_0 \oplus \mathbb{C}\cdot T_1 \oplus \mathbb{C}\cdot T_2$ $(r=1)$

$\beta = d[H] \in H_2(X, \mathbb{Z})$

$\Rightarrow s \in H^0(\mathbb{P}^2, \mathbb{C})$ and $\langle T_0, T_2^{n_2}, \psi^\nu T_i \rangle_{0,\beta} \overset{\text{fund. class ax}}{=} \langle T_2^{n_2}, \psi^{\nu-1} T_i \rangle_{0,n_2+1,\beta}$

$= 0$ unless $n_2 + \underbrace{3d}_{\int_\beta c_1(T_X)} = 2n_2 + i + \nu - 1 \Leftrightarrow n_2 = 3d - i - \nu + 1$

if $\beta \neq 0$

If $\beta = 0$, then $\langle T_0, T_2^{n_2}, \psi^\nu T_i \rangle_{0, n_2+2, 0} = 0$

unless $n_2 + 2 - 1 = n_2 + 1 = 2n_2 + i + \nu \iff$

$n_2 = 1 - i - \nu$

Rk. $\overline{\mathcal{M}}_{0,2}(X, 0) \simeq \overline{\mathcal{M}}_{0,2} \times X$ empty

since 2-pointed curve is unstable

$\implies n_2 > 0 \implies$ only $i = \nu = 0$ give contribution

$$\implies \langle T_0, T_2, T_0 \rangle_{0,3,0} = \int_X T_0 \cup T_2 \cup T_0 = 1$$

hence:

$$J_{\mathbb{P}^2} = J_{\mathbb{P}^2}(T_0) = e^{(t_0 T_0 + t_1 T_1)/z} \cup$$

$$\left[T_0 + \sum_{i=0}^{2} \left(\frac{t_2}{z} \delta_{2,i} + \sum_{\substack{d \geq 1 \\ \nu \geq 0}} \left\langle T_2^{3d+i-2-\nu}, \psi^\nu T_{2-i} \right\rangle_{0,d} z^{-(\nu+2)} \cdot e^{d t_1} \cdot \frac{t_2^{3d+i-2-\nu}}{(3d+i-2-\nu)!} \right) \right]$$