16.2.23 MPI Leipzig

QUANTUM COHOMOLOGY & QUANTUM DIFFERENTIAL EQUATION

1. ~~...~~ QUANTUM COHOMOLOGY & CURVES IN $\mathbb{P}^2$
2. FROBENIUS MANIFOLDS AND QUANTUM DIFF. EQUATION
3. SOLUTIONS TO QUANTUM DIFF. EQUATION.

1.

~~From~~ prev:
X smooth Mfd$/\mathbb{C}$ (w/ no odd cohomology) $\{T_0, \dots, T_m\}$ basis of $H^*(X,\mathbb{C})$ $\{T_1,\dots,T_p\}$ basis of $H^2(X,\mathbb{C})$

s.th. $T_0 = [X_0] \in H^0(X,\mathbb{C})$

Let $\gamma = \sum_{i=0}^m y_i T_i$. From previous talk $\langle \gamma^n \rangle_{0,\beta} = \langle \underbrace{\gamma, \dots, \gamma}_{n \text{ times}} \rangle_{0,\beta} = \int_{[M_{0,\beta}]^{vir}} ev^*(\gamma,\dots,\gamma)$

<u>Def</u> The GW potential is

$$\Phi := \sum_{n=0}^{\infty} \sum_{\beta \in H_2^*(X,\mathbb{Z})} \langle \gamma^n \rangle_{0,\beta} \in \mathbb{C}[[y_0, y_1, \dots, y_m]]$$

if for a given β $\exists$ finitely many n : $\langle \gamma^n \rangle_{0,\beta} \neq 0$

<u>e.g.</u> X Fano (see dimension formula for $M_{0,n}(X,\beta)$)

<u>Def</u> The (big) quantum cohomology ring of X is

$H^*(X, \mathbb{C}[[y_0,\dots,y_m]])$ w/ product given by

$$T_i * T_j := \sum_{k=0}^m (\partial_i \partial_j \partial_k \Phi) T^k$$

where T^k are dual to T_k w.r.t. $\int_X - \cup -$

<u>i.e.</u>
- $\int_X T^i \cup T_k = \delta_{ik}$
- $T_i \cup T_j = \sum_k a_k T^k \xrightarrow{\int_X - \cup T_\ell} a_\ell$

so that $a_\ell = \int_X T_i \cup T_j \cup T_\ell$.

set $g_{ij} = \int_X T_i \cup T_j$ and $(g^{ij}) = (g_{ij})^{-1}$ so that $T^e = \sum_f g^{ef} T_f$

$*$ associative implies:

$\cancel{T_i * T_j}$

$$\boxed{T^e = \sum_f g^{ef} T_f}$$

WDVV equation

$$(T_i * T_j) * T_k =$$

$$= \left(\sum_e (\partial_i \partial_j \partial_e \Phi) T^e\right) * T_k$$

$$= \left(\sum_{e,f} (\partial_i \partial_j \partial_e \Phi) g^{ef} T_f\right) * T_k$$

$$= \sum_{e,f} (\partial_{ije} \Phi) g^{ef} \sum_c \partial_{fkc} T^c$$

$$= \sum_{e,f} (\partial_{ije} \Phi) g^{ef} \sum_c (\partial_{fkc} \Phi g^{cd}) T_d$$

$$T_i * (T_j * T_k) =$$

$$= T_i * \sum_e \partial_{jke} \Phi T^e$$

$$= T_i * \sum_{e,f} (\partial_{jke} \Phi) g^{ef} T_f$$

$$= \sum_{e,f} (\partial_{jke} \Phi) g^{ef} T_i * T_f$$

$$= \sum_{e,f} (\partial_{jke} \Phi) g^{ef} \sum_c (\partial_{ifc} \Phi) T^c$$

$$= \sum_{e,f} (\partial_{jke} \Phi) g^{ef} \sum_{c,d} \partial_{ifc} \Phi g^{cd} T_d$$

$k = c = 2$
$i = j = 1$

$\Gamma_{222} = \Gamma_{112}^2 - \Gamma_{111} \Gamma_{122}$

$$\sum_{e,f} \boxed{(\partial_{ije} \Phi) g^{ef} (\partial_{fkc} \Phi) = (\partial_{jke} \Phi) g^{ef} (\partial_{ifc} \Phi)}$$

Split the $\sum_\beta$ in Φ in

$$\Phi = \underbrace{\Phi_{\text{classical}}}_{\beta=0} + \underbrace{\Phi_{\text{quantum}}}_{\beta\neq 0}$$

$\underline{\Phi_{\text{classical}}}$:

$$\frac{1}{n!}\langle \gamma^n\rangle_{0,0} = \begin{cases} \frac{1}{3!}\int_X \gamma\cup\gamma\cup\gamma & \text{if } n=3 \\ 0 & \text{otherwise} \end{cases}$$

and one gets

$$\partial_{y_i}\partial_{y_j}\partial_{y_k}\Phi_{\text{classical}} = \int_X T_i\cup T_j\cup T_k$$

so that

think of

$$(y_0T_0 + y_1T_1 + y_2T_2)^{\cup 3} = \frac{3!}{2!\,1!}y_0^2y_1\,T_0^{\cup 2}\cup T_1 + \dots \frac{3!}{3!}y_0^3\,T_0^{\cup 3} + \frac{3!}{1!\,1!\,1!}T_0\cup T_1\cup T_2 + \dots$$

so the contribution to $T_i * T_j$ is

$$\sum_k\left(\int_X T_i\cup T_j\cup T_k\right)T^k = T_i\cup T_j$$

$\underline{\Phi_{\text{quantum}}}$:

by fundamental class axiom y_0T_0 doesn't appear

$$\sum_{n\geq 0}\frac{1}{n!}\langle\gamma^n\rangle_{0,\beta} =$$

divisor axiom

$$= \sum_{n_1,\dots,n_m\geq 0}\left(\prod_{i=1}^{p}\frac{1}{n_i!}\left(y_i\int_\beta T_i\right)^{n_i}\right)\left\langle T_{p+1}^{n_{p+1}},\dots,T_m^{n_m}\right\rangle_{0,\beta}\frac{y_{p+1}^{n_{p+1}}\cdots y_m^{n_m}}{n_{p+1}!\cdots n_m!}$$

$$= \prod_{i=1}^{p} e^{y_i\int_\beta T_i}\cdot\sum\langle\quad\rangle_{0,\beta}\cdot\ ''$$

$\underline{\text{For } X=\mathbb{P}^2}$: $\beta = d\cdot[\ell]$

$$\Phi_{\text{quantum}}(y_0,y_1,y_2) = \sum_{d>0}\langle T_2^{3d-1}\rangle_{0,d[\ell]}\,e^{dy_1}\cdot\frac{y_2^{3d-1}}{(3d-1)!}$$

$$2\left(n+(\dim\mathbb{P}^2-3)(1-0)+\overbrace{\int_{d[\ell]}c_1(T_{\mathbb{P}^2})}^{=3d}\right) = \overbrace{4-n}^{\deg T_2}$$

$$\Phi_{\text{classical}}(y_0,y_1,y_2) = \frac{y_0^2y_2}{2} + \frac{y_0y_1^2}{2}$$

⇝ COMPUTE

$T_0 * T_i = T_i$ $\qquad$ $T_1 * T_1 = T_2 + \Gamma_{111} T_1 + \Gamma_{112} T_0$

$T_1 * T_2 = \Gamma_{121} T_1 + \Gamma_{122} T_0$ $\qquad$ $T_2 * T_2 = \Gamma_{221} T_1 + \Gamma_{222} T_0$

where $\boxed{\Gamma_{ijk} = \partial_i \partial_j \partial_k \Phi}$

$(T_1 * T_1) * T_2 = \dots$

$T_1 * (T_1 * T_2) = \dots$

compare coeff of T_0 ⇝ $\Gamma_{222} = \Gamma_{112}^2 - \Gamma_{111}\Gamma_{122}$.

Compute $\Gamma_{111} = \sum_{d \geq 1} d^3 \langle T_2^{3d-1} \rangle_{0,d[\ell]} \cdot e^{dy_1} \frac{y_2^{3d-1}}{(3d-1)!}$

$\Gamma_{112} = \dots$

$\vdots$

⇝ Vafa-Intriligator formula

$$\langle T_2^{3d-1} \rangle_{0,d[\ell]} = \sum_{\substack{d_1, d_2 \geq 0 \\ d_1 + d_2 = d}} \langle T_2^{3d_1-1} \rangle_{0,d_1[\ell]} \cdot \langle T_2^{3d_2-1} \rangle_{0,d_2[\ell]} \cdot \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^2 d_2 \binom{3d-4}{3d_1-1} \right)$$

2. FROBENIUS MANIFOLDS & quantum diff equation

Def • Let M be a (germ of a) Mfd$/\mathbb{C}$

A pre-Frobenius structure is given by (∇, g, A) where

~~1? g is a $\mathbb{C}$-metric, i.e. $g: S^2(TM) \to \mathcal{O}_M$~~

1. $\nabla : T_M \to T_M \otimes \Omega^1$ is a flat connection (on T_M)

2. g is a metric, i.e. $g: S^2(TM) \to \mathcal{O}_M$ giving $TM \simeq T^*M$

compatible w/ ∇

$$d(g(X,Y))(-) = g(\nabla_X -, Y) + g(X, \nabla_Y -) \quad : T_M \to \mathcal{O}_M$$

3. $A : S^3(TM) \to \mathcal{O}_M$ is a symmetric tensor

s.th. A defines a product by

$$A(X,Y,Z) = g(X \circ Y, Z)$$

• M is Frobenius if

1. $\_\circ\_$ is associative
2. $\exists$ locally a potential Φ s.th. $A(X,Y,Z) = XYZ\,\Phi$

$\boxed{\text{Ex.}}$

X smooth$/\mathbb{C}$ (Fano) no odd-cohomology $\quad M := \operatorname{Spec} \mathbb{C}[[y_0, \dots, y_m]]$

g constant metric : $g(\partial_{y_i}, \partial_{y_j}) = \int_X T_i \cup T_j$

$A(\partial_i, \partial_j, \partial_k) = \partial_i \partial_j \partial_k \Phi$

X, Y vector fields

$\rightsquigarrow X \circ Y = X * Y$ □

Assume to take $T_1, \dots T_p \in H^2(X, \mathbb{R})$ and s.th.

$\int_\beta T_i > 0 \quad \forall \beta \in H_2(X, \mathbb{Z})$ effective

define $\quad K_i := e^{y_i} \quad \Rightarrow \quad \Phi \in \mathbb{C}[K_1, \dots K_p][[y_0, \dots, y_m]]$

$dK_i = e^{y_i} dy_i : \partial_{y_i} \mapsto \underbrace{e^{y_i} \cdot 1}_{= K_i}$ i.e. $\partial_{y_i} = K_i \partial_{K_i}$

$M = \operatorname{Spec} \mathbb{C}[\underline{K}^{\pm 1}][[\underline{y}]]$

$\overline{M} = \operatorname{Spec} \mathbb{C}[\underline{K}][[\underline{y}]]$

$\partial K_i \mapsto 1$

Def $(M,D,g,\cancel{A})$ pre-Frobenius, then

1. e is the identity vector field if $e\circ Y = Y \quad \forall Y$
2. E is an Euler vector field if ~~$\forall Y,Z$~~

$$\forall Y,Z \qquad E(g(Y,Z)) - g([E,Y],Z) - g(Y,[E,Z]) = D\cdot g(Y,Z)$$

for some $D\in\mathbb{C}$

$$[E, Y\circ Z] - [E,Y]\circ Z - Y\circ[E,Z] = d_0 Y\circ Z$$

for some $d_0\in\mathbb{C}$

M Frobenius, consider $M\times \mathbb{C}^*_\hbar \xrightarrow{pr} M$

and define $\hat{\nabla}$ on $p_1^* TM$

$$\hat{\nabla}_x(Y) = \nabla_x(Y) + \hbar^{-1} x\circ Y$$

for $x=\partial_x$, x coord on M

1st structure/Dubrovin connection

$$d_0\hat{\nabla}_{\hbar\partial_\hbar}(Y) = \hbar\partial_\hbar Y - \hbar^{-1} E\circ Y + Gr_E(Y)$$

$$Gr_E(Y) := [E,Y]$$

QUANTUM DIFF. EQUATION

$$\hat{\nabla}_{\partial_{y_i}}\circ s = 0 \qquad i=0,\dots,m$$

Q. Solutions?

$$\Leftrightarrow \hbar\,\partial_{y_i}(s_j) = -T_i * s_j$$

Notation:

$$\langle\langle \Psi^{d_1}_{a_1}\alpha_1,\dots,\Psi^{d_n}_{a_n}\alpha_n\rangle\rangle := \sum_{k=0}^{\infty}\sum_{\beta}\frac{1}{k!}\langle \Psi^{d_1}_{a_1}\alpha_1,\dots,\Psi^{d_n}\alpha_n;\gamma^k\rangle_{0,\beta}$$

$\Rightarrow$ • $\langle\langle 1\rangle\rangle = \Phi$

• $\partial_{y_i}\langle\langle \Psi^{d_1}\alpha_1,\dots \Psi^{d_n}\alpha_n\rangle\rangle = \langle\langle \_\_\_, T_i\rangle\rangle$

$$s_i := T_i - \sum_{j=0}^{m}\left\langle\left\langle \frac{T_i}{\hbar+\Psi}, T_j\right\rangle\right\rangle T^j \quad \text{where} \quad \frac{T_i}{\hbar+\Psi} = \sum_{k=0}^{\infty}(-1)^k\hbar^{-(k+1)}\Psi^k T_i$$

Q. 1. Do they converge?

2. $\nabla_{\hbar\partial_\hbar} s_i = 0$? NO, because $\nabla_{\hbar\partial_\hbar} s=0$ is singular in $\hbar=0$, $\hbar=\infty$

irregular — regular

Def $\tilde{M} = (\mathbb{C}^l, \mathcal{O}_{\tilde{M}})$ the a ringed space where ~~$y_1, \dots, y_l$ coordin~~ w/ coordinates $y_1, \dots, y_l$ and

$$\mathcal{O}_{\tilde{M}}(U) = \left\{ \sum_{i_0, i_1, \dots, i_m \geq 0} f_I \, y_0^{i_0} y_1^{i_1} \cdots y_m^{i_m} : f_I \in \mathcal{O}_{\mathbb{C}^l}(U) \text{ holomorphic} \right\}$$

Also consider

$$\mathcal{O}_{\tilde{M}}\{\hbar, \hbar^{-1}\} = \left\{ \quad - \quad : f_I \text{ is holomorphic on } \left\{ (y, \hbar) \in U \times \mathbb{C} : 0 < \hbar < \varepsilon(y) \text{ for some } \varepsilon : U \to \mathbb{R}_{>0} \right\} \right\}$$

$$S : H^*(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{M}}\{\hbar, \hbar^{-1}\} \longrightarrow H^*(X, \mathbb{C}) \otimes \mathcal{O}_{\tilde{M}}\{\hbar, \hbar^{-1}\}$$

$$\alpha \longmapsto \alpha - \sum_{j=0}^{m} \left\langle\!\left\langle \frac{\alpha}{\hbar + \psi}, T_j \right\rangle\!\right\rangle T^j$$