TALK 3: PARAMETRIZED TROPICAL CURVES

READING GROUP "MIRROR SYMMETRY AND TROPICAL GEOMETRY"

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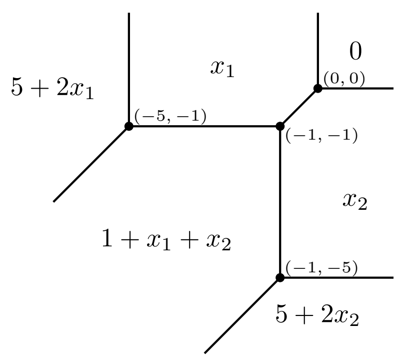
Abstract. One of the goals of this reading group is understanding (in the case of genus-0 and on a 2-dimensional toric variety) Mikhalkin's result that the number $N^{0,\mathrm{trop}}_{\Sigma,\Delta}$ of genus-0, degree-Δ tropical curves in a toric variety X_Σ passing through $|\Delta|-1$ general points equals the number $N^{0,\mathrm{hol}}_{\Sigma,\Delta}$ of genus-0, degree-Δ holomorphic curves in a toric variety passing through $|\Delta|-1$ general points.

The ultimate goal of this talk is to define $N^{0,\mathrm{trop}}_{\Sigma,\Delta}$ when $\dim(X_\Sigma)=2$.

1. Parametrized and marked tropical curves

Recall from the second talk that tropical *hypersurfaces* inside a two-dimensional space appear as piecewise linear graphs, for example:



(1.1)

Here, we will define *tropical curves* as one-dimensional piecewise linear subvarieties, even inside higher dimensional spaces.

Taking away the ambient two-dimensional space in the examples of the second talk, we find that in an "abstract sense", tropical hypersurfaces appear as graphs:

Definition 1.1. A *graph* $\overline{\Gamma}$ is a tuple $(\overline{\Gamma}^{[0]},\overline{\Gamma}^{[1]})$ where $\overline{\Gamma}^{[0]}$ is a finite set of elements called the *vertices* and $\overline{\Gamma}^{[1]}$ a set of *unordered* pairs of elements of $\overline{\Gamma}^{[0]}$ called the *edges*; in particular, we do not allow "loops" (an edge consisting of the same vertex twice). We say an edge E is *connected* to a vertex V if the vertex is one of the two elements defining the edge, i.e. $V\in E$; the pair (V,E) is called a *flag*. The *valency* of a vertex is the number of edges connected to it. A *leaf* is a flag (V,E) consisting of a vertex V of valency 1 and its unique connecting edge E; we denote by $\overline{\Gamma}^{[0]}_\infty\subset\overline{\Gamma}^{[0]}$ the set of all vertices in leaves and by $\overline{\Gamma}^{[1]}_\infty\subset\overline{\Gamma}^{[1]}$ the set of all edges in leaves.

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We will consider the vertices in $\overline{\Gamma}^{[0]}_\infty$ to "lie at infinity" and hence the edges in $\overline{\Gamma}^{[1]}_\infty$ to be "unbounded". In other words, we associate to $\overline{\Gamma}$ the topological space

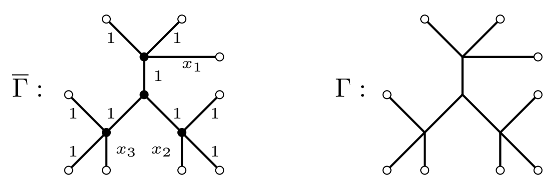
$$\Gamma = \left(\bigcup_{E \in \overline{\Gamma}^{[1]}} \{(1-t)V_1 + tV_2 \mid t \in [0,1]\} \right) \setminus \left(\bigcup_{V \in \overline{\Gamma}^{[0]}_\infty} \{V\} \right)$$

with the topology induced by the euclidean topology on $[0,1] \subset \mathbb{R}$. We will abuse notation by writing $E \in \Gamma^{[1]}$ for the topological "line segment" $E = \{(1-t)V_1 + tV_2 | \ t \in [0,1]\}$ where $E = \{V_1, V_2\} \in \overline{\Gamma}^{[1]}$, and we call $E \in \Gamma^{[1]}$ an "edge" of Γ. (In particular, $E \in \Gamma^{[1]}_\infty$ denotes the "unbounded line segment" corresponding to an unbounded edge $E \in \overline{\Gamma}^{[1]}_\infty$.) Similarly, we call the points $V \in \Gamma$ for $V \in \overline{\Gamma}^{[1]} \setminus \overline{\Gamma}^{[1]}_\infty$ the "vertices" of Γ, denoted by $\Gamma^{[1]}$.

Definition 1.2. A *weighted graph* $(\overline{\Gamma}, w)$ is a graph $\overline{\Gamma}$ with a *weight function* $w : \overline{\Gamma}^{[1]} \to \mathbb{Z}_{\geq 0}$ assigning a non-negative integer to each edge. A *marking* on a graph $\overline{\Gamma}$ is an embedding $\{x_i\} = \{x_1, \ldots, x_k\} \hookrightarrow \overline{\Gamma}^{[1]}_\infty : x_i \mapsto E_{x_i}$. A *marked graph* $(\overline{\Gamma}, w, \{x_i\})$ is a weighted graph $(\overline{\Gamma}, w)$ with a marking $\{x_i\} \hookrightarrow \overline{\Gamma}^{[1]}_\infty$ such that $w(E) = 0$ if and only if $E = E_{x_i}$ for some i.

Remark 1.3. The standard convention in tropical geometry is "$w(E) = 0$ if $E = E_{x_i}$", and *not* "if and only if".

Example 1.4. An example of a marked graph and its associated topological space related to the tropical hypersurface in (1.1) is:



(1.2)

Where $\overline{\Gamma}^{[0]}$ consists of all nodes, $\overline{\Gamma}^{[0]}_\infty$ consists of the white nodes, $\overline{\Gamma}^{[1]}$ consists of all edges, $\overline{\Gamma}^{[1]}_\infty$ consists of edges connecting one black and one white node, and where each edge E is labeled with either its nonzero weight $w(E)$ or with x_i when it is marked. For Γ, each line segment has the topology of $[0,1]$, except those with a circle at the end, which denote line segments with the topology $[0,1\rangle$ with the circle denoting the non-compact end. $\Diamond$

Now, we return to the conventions of the second talk: Let $M = \mathbb{Z}^n$ with $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$, and set $N = \mathrm{Hom}_\mathbb{Z}(M, \mathbb{Z})$.

Definition 1.5. A *parametrized marked tropical curve* $(\overline{\Gamma}, w, \{x_i\}, h)$ is the triple of a marked graph $(\overline{\Gamma}, w, \{x_i\})$ together with a continuous function $h : \Gamma \to M_\mathbb{R}$ satisfying:

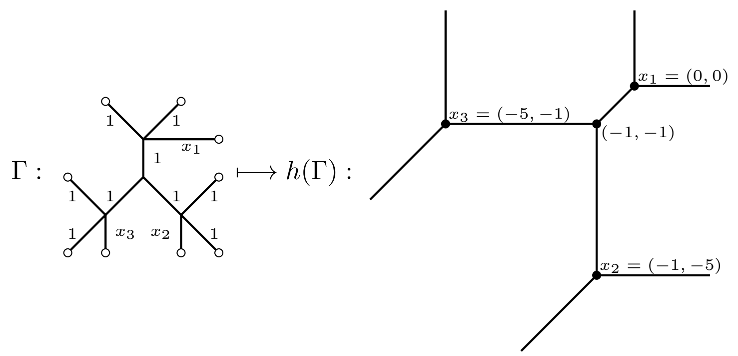
- **(i)** If $E \in \overline{\Gamma}^{[1]}$ has weight $w(E) = 0$, then $h|_E$ is constant (i.e. a point); else $h|_E$ is a proper embedding of $E \subset \Gamma$ into a line of *rational* slope in $M_\mathbb{R}$.
- **(ii)** The *balancing condition* at every $V \in \Gamma^{[0]}$: Let $E_1, \ldots, E_\ell \in \Gamma^{[1]}$ be all the edges connected to V and denote by $m_i \in M$ a primitive tangent vector to $h(E_i)$ pointing away from $h(V)$, then
$$\sum_{i=1}^{\ell} w(E_i)\, m_i = 0.$$

Since $h(E_{x_i})$ is a point in $M_\mathbb{R}$, we will simply write $h(x_i)$. It is also common to write $h : (\Gamma, x_1, \ldots, x_k) \to M_\mathbb{R}$ for a parametrized marked tropical curve. We call $b_1(\Gamma)$ the genus of h.

Definition 1.6. Two parametrized marked tropical curves $h : (\Gamma, \{x_i\}) \to M_{\mathbb{R}}$ and $h' : (\Gamma', \{x_i'\}) \to M_{\mathbb{R}}$ are called *equivalent* if there is a homeomorphism $\varphi : \Gamma \to \Gamma'$ with $\varphi(E_{x_i}) = E_{x_i'}$ and $h = h' \circ \varphi$. An equivalence class of parametrized marked tropical curves is called a *marked tropical curve.*

Remark 1.7. The balancing condition implies that the edges connecting a vertex of valency 2 are mapped opposite to each other (i.e. $m_2 = -m_1$) and that they have the same weight, so there is an equivalent parametrized marked tropical curve whose underlying graph has the vertex removed and the edges replaced with a single edge connecting the opposite vertices. Hence, from now on we assume the graphs underlying the parametrized marked tropical curves have *no* vertices of valency 2.

Example 1.8. Continuing the above examples, a marked tropical curve can be drawn as:



(1.3)

A parametrized marked tropical curve is obtained by choosing (continuously) the image of every point of Γ (a "parametrization" of the edges of $h(\Gamma)$"). Note that the images of the vertices *are* fixed by the marked tropical curve. Also note that the balancing condition is satisfied at all vertices. ◊

2. Marked tropical curves in a toric variety

Let X_Σ be the toric variety defined by the fan Σ, and denote by $\Sigma^{[1]}$ the set of one-dimensional cones ("rays") in Σ. We write T_Σ for the free abelian group generated by the rays: namely, $T_\Sigma = \bigoplus_{\rho \in \Sigma^{[1]}} \mathbb{Z} t_\rho$ for t_ρ a formal generator corresponding to the ray ρ. As $\rho \subset M_{\mathbb{R}}$ is contained in a line of rational slope, there exists a primitive $m_\rho \in M$ such that $\rho = \mathbb{R}_{\geq 0} m_\rho$. Hence we obtain a map $T_\Sigma \to M$ by sending $t_\rho \mapsto m_\rho$.

Definition 2.1. A marked tropical curve h *lies in* X_Σ if for every *unmarked* unbounded edge $E \in \Gamma_\infty^{[1]}$ its image $h(E)$ is a translate of some $\rho \in \Sigma^{[1]}$. In other words, if m_E is the primitive tangent vector to $h(E)$ pointing to the vertex "at infinity", then $m_E = m_\rho$ for some $\rho \in \Sigma^{[1]}$.

For a given ray $\rho \in \Sigma^{[1]}$, denote by

$$d_\rho = \sum\nolimits_{E \in \Gamma_\infty^{[1]}:\ m_E = m_\rho} w(E)$$

the number of (unmarked, else $w(E) = 0$) unbounded edges $E \in \Gamma_\infty^{[1]}$ with $h(E)$ a translate of ρ counted with weight. The *degree* $\Delta(h) \in T_\Sigma$ of h is defined as

$$\Delta(h) = \sum\nolimits_{\rho \in \Sigma^{[1]}} d_\rho t_\rho.$$

Moreover, we set $|\Delta| = \sum_{\rho \in \Sigma^{[1]}} d_\rho$.

Summing over the balancing conditions on the vertices of a marked tropical curve gives:

Lemma 2.2 ([Gross], Lemma 1.13). $r(\Delta(h)) = 0$, where $r(\Delta(h)) = \sum_{\rho\in\Sigma^{[1]}} d_\rho m_\rho \in M$.

Example 2.3. The marked tropical curve in (1.3) lies in $\mathbb{P}^2$: Recall from the first talk that $\mathbb{P}^2$ is obtained from the complete fan with rays in the directions $(1,0)$, $(0,1)$ and $(1,1)$, and note that the unbounded edges of the marked tropical curve point in the same directions. ◊

3. Marked tropical curves as tropical hypersurfaces in dimension two

Let h be a marked tropical curve, then we can see $h(\Gamma)$ as a one-dimensional polyhedral complex. On the other hand, for a tropical polynomial $f \in \mathbb{R}^{\text{trop}}[x_1,\dots,x_n]$ its tropical hypersurface $V(f)$ (the locus of non-linearity of f) is a polyhedral complex as well. However, $V(f)$ comes with a natural weight map assigning to each of its codimension-one polyhedra a weight. Hence, we would like to turn $h(\Gamma)$ into a *weighted* polyhedral complex as well.

We already have a weight map on $\overline{\Gamma}$, the marked weighted graph underlying $h : (\Gamma, \{x_i\}) \to M_\mathbb{R}$. The image $h(\Gamma)$ is a piecewise linear space itself, as h is continuous, so the idea to define a weight on each of the linear segments is simply to add up all the weights of the edges of Γ mapped to that given linear segment by h.

To define this more carefully, proceed as follows: let $\widetilde{E}$ be an "edge" (linear segment) of $h(\Gamma) \subset M_\mathbb{R}$, and take a point $m \in \widetilde{E}$ that is not a "vertex" (nonlinear point) of $h(\Gamma)$ nor the image of a vertex of Γ, then

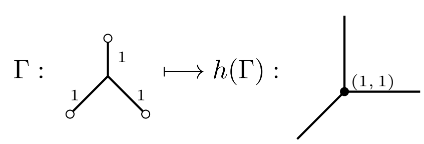
$$w(\widetilde{E}) = \sum_{E\in\Gamma^{[1]}:\ m\in h(E)} w(E).$$

The balancing condition on h implies that $w(\widetilde{E})$ does not depend on $m \in \widetilde{E}$, and moreover that $h(\Gamma)$ satisfies the balancing condition.

Now, restrict to the case $\dim(M_\mathbb{R}) = 2$, i.e. $\dim(X_\Sigma) = 2$, then marked tropical curves and tropical hypersurfaces are both weighted one-dimensional polyhedral complexes. It turns out that:

Proposition 3.1 ([Gross], Proposition 1.15). For every marked tropical curve $h : \Gamma \to M_\mathbb{R}$ with $\dim(M_\mathbb{R}) = 2$ there exists a tropical polynomial $f \in \mathbb{R}^{\text{trop}}[x_1, x_2]$ such that $h(\Gamma) = V(f)$ as weighted one-dimensional polyhedral complexes.

Proof (sketch). Let us start with an easy example. Consider the marked tropical curve given by



Clearly, $h(\Gamma)$ defines a polyhedral decomposition: the closures of the complements of $h(\Gamma)$ are the two-dimensional cells (so here the top right quarter, the lower third and the left third of the plane), the edges of $h(\Gamma)$ are one-dimensional cells and the vertices the zero-dimensional cells.

To define the a tropical function, pick one of the components, say the top-right. Take the function 0 on this component. Now, move to another adjacent component, say the bottom right one. To obtain a tropical function whose hypersurface has a half-line going right from $(1,1)$, we consider a linear function that is zero on this half-line, less than zero below it, and more than zero above it, for example $x_2 - 1$. Of course, any multiple of $x_2 - 1$ will be zero on half half-line; however, we want the tropical hypersurface to have weight 1 on the half-line to match with the marked tropical curve. Hence, we have to take 1 as constant coefficient. The function so far is now $\min\{0, 1\cdot(x_2 - 1)\}$. Move from the lower component to the left component: we now want a function that equals $x_2 - 1$ on the half-line $\{(1,1) + r(-1,1) \mid r \in [0,\infty\rangle\}$, that is bigger than

$x_2 - 1$ to the right of it, and is smaller than $x_2 - 1$ to the left of it. Clearly, any multiple of $x_1 - 1$ satisfies this. Since the half-line has weight 1, we take the scalar multiple $1 \cdot (x_1 - 1)$.

Now, we have the tropical function $\min\{0, x_1 - 1, x_2 - 1\}$, but we still have to check we get the last half-line separating the left and upper-right components: Luckily, $x_1 - 1$ equals 0 on the line going up from $(1, 1)$, so this works out. However, this is not lucky at all, but guaranteed by the balancing condition of the marked tropical. Hence, turning the tropical function into the tropical polynomial $0 \oplus [(-1) \odot x_1] \oplus [(-1) \odot x_2]$, we get the result.

The general case goes analogously: (i) start at a certain component with the function 0; (ii) cross a line segment to another component; (iii) take a linear function that equals the previous function along the line segment, is bigger than that function on the previous component and smaller than that function on the new component; (iv) use the weight of the line segment to determine the required scalar multiple; (v) add the new function to the minimum of functions obtained so far; then choose another line segment and repeat steps (ii)—(v). The balancing condition ensures the function is well-defined. □

4. Moduli space of marked tropical curves in a toric variety

Just as stable curves can be distinguished on the number of irreducible components and the distribution of the marked points over the irreducible components, marked tropical curves can be distinguished by their *combinatorial type*.

Definition 4.1. The *combinatorial type* of a marked tropical curve $h = (\overline{\Gamma}, w, \{x_i\}, h)$ is the tuple $(\overline{\Gamma}, w, \{x_i\}, \{m_{(V,E)}\})$ where $(\overline{\Gamma}, w, \{x_i\})$ is the marked graph underlying h and where $\{m_{(V,E)}\}$ is a set of primitive vectors in M tangent to $h(E)$ pointing away from $h(V)$, one for each flag (V, E) with $V \in \overline{\Gamma}^{[0]} \setminus \overline{\Gamma}^{[0]}_{\infty}$ and $E \in \overline{\Gamma}^{[1]}$ connected to V. A *combinatorial type* is a *combinatorial equivalence class*; the combinatorial type of a given marked tropical curve h is denoted by $[h]$.

Example 4.2. The combinatorial type of the marked tropical curve in (1.3) is simply forgetting the location of the vertices of $h(\Gamma)$, only the direction of the edges under an embedding is remembered. ◊

Definition 4.3. The *moduli space of k-marked, genus-g tropical curves of degree* Δ *lying in* X_Σ is denoted $\mathcal{M}_{g,k}(\Sigma, \Delta)$. The subset of those tropical curves of a given combinatorial type is denoted $\mathcal{M}^{[h]}_{g,k}(\Sigma, \Delta)$.

Proposition 4.4 ([Gross], Proposition 1.17)**.** We have $\mathcal{M}_{g,k}(\Sigma, \Delta) = \bigsqcup_{[h]} \mathcal{M}^{[h]}_{g,k}(\Sigma, \Delta)$ with $[h]$ running over all possible combinatorial types.

For fixed combinatorial type $[h]$, $\mathcal{M}^{[h]}_{g,k}(\Sigma, \Delta)$ is the interior of a polyhedron of dimension *greater or equal to*

$$D^{[h]}_{g,k} = (3 - \dim M_{\mathbb{R}})(g - 1) + k + e - \mathrm{ov}(\overline{\Gamma}),$$

where $\mathrm{ov}(\overline{\Gamma}) = \sum_{V \in \overline{\Gamma}^{[0]} \setminus \overline{\Gamma}^{[0]}_{\infty}} \big(\mathrm{Valency}(V) - 3\big)$ and e is the number of *unmarked*, unbounded edges in $\overline{\Gamma}^{[1]}_{\infty}$.

Proof (sketch). The idea of this proof is the observation that a marked tropical curve is obtained from a combinatorial type by fixing the location of the vertices. However, once we fix the location of one of the vertices, the vertices connected to it must lie in the direction dictated by the combinatorial type: hence these vertices will be determined by fixing the distance to the first vertex. Subsequently, all vertices connected to these vertices must lie in the direction dictated by the combinatorial type, so again their position is determined by fixing the distance from the other vertices.

Hence the moduli space is some subset of $M_{\mathbb{R}} \times \mathbb{R}_{>0}^{\#(\Gamma^{[1]} \setminus \Gamma_{\infty}^{[1]})}$, where $M_{\mathbb{R}}$ is the location of the vertex and where each positive real number determines the length of a compact edge (i.e. edge in $\Gamma^{[1]} \setminus \Gamma_{\infty}^{[1]}$). The number of edges is determined using the topological Euler characteristic of the underlying graph $\chi(\overline{\Gamma}) = 1 - g = \#\overline{\Gamma}^{[0]} - \#\overline{\Gamma}^{[1]}$. From this the

On the other hand, the lengths of the compact edges are not independent of each other: for every cycle in the graph, the line segments need to start and end at the same vertex, forcing the moduli space to lie in some polyhedron of lower dimension. □

Remark 4.5. If $\dim(M_{\mathbb{R}}) \geq 3$ and $g \geq 1$ there exist marked tropical curves that are neither trivalent nor limits of trivalent curves.

Definition 4.6. If $\dim\big(\mathcal{M}_{g,k}^{[h]}(\Sigma, \Delta)\big) > D_{g,k}^{[h]}$, then $[h]$ is called *superabundant*. Else, $[h]$ is called *regular*.

If $g \geq 1$, there exist superabundant combinatorial types.

5. Simple tropical curves in a two-dimensional toric variety

Restrict to the case $\dim(M_{\mathbb{R}}) = 2$:

Definition 5.1. A *simple tropical curve* (in two dimensions) is a marked tropical curve $h = (\overline{\Gamma}, w, \{x_i\}, h)$ such that:

- **(i)** The underlying (topological) graph Γ is trivalent, i.e. every $V \in \overline{\Gamma}^{[0]} \setminus \overline{\Gamma}_{\infty}^{[0]}$ has valency 3.
- **(ii)** Distinct vertices of Γ are mapped to distinct points under h.
- **(iii)** There are no disjoint edges E_1 and E_2 connected to the same vertex for which $h|_{E_1}$ and $h|_{E_2}$ are non-constant and $h(E_1) \subseteq h(E_2)$.
- **(iv)** Each unmarked $E \in \overline{\Gamma}_{\infty}^{[1]}$ has $w(E) = 1$.

Note that the number e of unmarked, unbounded edges of a simple tropical curve h equals $|\Delta|$. It can be shown that simple tropical curves in two dimensions are always regular (Proposition 2.21 of Mikhalkin).

We now consider tropical curves passing through a general set of points in a toric variety X_Σ:

Definition 5.2. A set of points $(P_1, \ldots, P_k)$, $P_i \in M_{\mathbb{R}}$, is *general* if it lies in some dense open subset of $M_{\mathbb{R}}^k$. We say that a marked tropical curve $h : (\Gamma, \{x_1, \ldots, x_k\}) \to M_{\mathbb{R}}$ *passes through the points* $(P_1, \ldots, P_k)$ if $h(x_i) = P_i$ for all i.

(Recall that $h(x_i) = h(E_{x_i})$.)

Lemma 5.3 ([Gross], Lemma 1.20)**.** Let Σ be a fan in $M_{\mathbb{R}}$, $\dim M_{\mathbb{R}} = 2$, and let $\Delta \in T_\Sigma$. For any given general set of points $(P_1, \ldots, P_k)$ in $M_{\mathbb{R}}$, where $k = |\Delta| - 1$, there exist a *finite number* of marked, genus-0 tropical curves $h : (\Gamma, \{x_1, \ldots, x_k\}) \to M_{\mathbb{R}}$ in X_Σ with $h(x_i) = P_i$. Moreover, these curves are *simple*, and there is *at most one* such curve of a given combinatorial type.

Proof (sketch). First, we observe that there are only finitely many combinatorial types of marked tropical curves lying in X_Σ: The idea is that a combinatorial type corresponds to regular (lattice) subdivision of the Newton polytope defined by the degree Δ; this works through the identification of a marked tropical curve in $M_{\mathbb{R}}$ where $\dim(M_{\mathbb{R}}) = 2$ with a tropical hypersurface as given in Proposition 3.1.

This allows us to consider a fixed combinatorial type $[h]$. Now, we look at Proposition 4.4 and its proof, which tells us for genus 0 that

$$\mathcal{M}^{[h]}_{0,|\Delta|-1}(\Sigma,\Delta) \cong M_{\mathbb{R}} \times \mathbb{R}^{e+|\Delta|-4-(\Gamma)}_{>0}.$$

On the other hand, the evaluation map sending a marked tropical curve h to $(h(x_1),\ldots,h(x_{|\Delta|-1}))$ is an affine linear map to $(M_{\mathbb{R}})^{|\Delta|-1}$: it depends on the location $h(V) \in M_{\mathbb{R}}$ of the first reference vertex and on the lengths of the line segments connecting $h(x_i)$ to $h(V)$.

From this, we conclude that a curve of combinatorial type $[h]$ can only pass through a general set of $|\Delta|-1$ points if the dimension of the moduli space is at least $|\Delta|-1$. Using Proposition 4.4, we conclude that the number of unbound, unmarked edges *must* be $|\Delta|$ and the overvalency must be 0. This means that all unbounded, unmarked edges have weight 1 and Γ is trivalent. Finally, we can conclude that the existence of some marked tropical curve going through (P_i) implies that the evaluation map is a local isomorphism, so that there is at most one going through (P_i) in general position.

Checking that this trivalent curve is simple is then a simple check. □

Hence, we are able to count the number of tropical curves passing through a general set of points. To get a meaningful count, we have to add a multiplicities to each of these curves:

Definition 5.4. The *Mikhalkin multiplicity* of a simple tropical curve in two dimensions is defined as $\mathrm{Mult}(h) = \prod_{V\in\Gamma^{[0]}} \mathrm{Mult}_V(h)$, where $\mathrm{Mult}_V(h)$ is 1 if one of the three connected edges are marked, and where $\mathrm{Mult}_V(h)$ equals

$$w(E_1)w(E_2)\,|m_{E_1}\wedge m_{E_2}| = w(E_2)w(E_3)\,|m_{E_2}\wedge m_{E_3}| = w(E_3)w(E_1)\,|m_{E_3}\wedge m_{E_1}|$$

otherwise. Here m_{E_i} denotes a primitive vector tangent to $h(E_i)$ pointing away from $h(V)$, and $\bigwedge^2 M$ is identified with $\mathbb{Z}$.

The three expressions above are equal due to the balancing condition.

Definition 5.5. $N^{0,\mathrm{trop}}_{\Sigma,\Delta} = \sum_h \mathrm{Mult}(h)$ with h running over all $h \in \mathcal{M}_{0,|\Delta|-1}(\Sigma,\Delta)$ passing through a general set of $|\Delta|-1$ points in $M_{\mathbb{R}}$.

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