Tropical hypersurfaces.

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1. Introduction.

Tropical geometry appeared in various fields of mathematics: computer science for network optimization (Gaubert), real geometry (Viro), enumertive geometry (as we will see later on) and also in physics under the name of *Maslov dequantization*: we renormalize the usual ring laws and make the basis of the logarithm goes to 0, i.e.

$$\log_\hbar(\hbar^a + \hbar^b) \underset{\hbar \to 0}{\longrightarrow} \min(a,b) =: a \oplus b$$

and

$$\log_\hbar(\hbar^a \cdot \hbar^b) \underset{\hbar \to 0}{\longrightarrow} a + b =: a \odot b.$$

We will work with the triple $\mathbb{R}^{\text{trop}} := (\mathbb{R}, \oplus, \odot)$. We usually add a neutral element for the addition and consider the (idempotent) semi-ring [1] of *tropical numbers* or the *min-plus algebra*; it is defined by:

$$\mathbb{T} := (-\infty, +\infty].$$

Note the tropicalization of the usual ring laws of $\mathbb{R}$ is valuative. Another convention is also sometimes used where the minimum is replaced by a maximum and $-\infty$ is replaced by $+\infty$; with that convention, the tropicalization of the ring laws of $\mathbb{R}$ behaves like a non-archimedean absolute value. These two points of view are of course equivalent since they are identified through $x \mapsto -x$. Note that in any case, we have $\mathbb{T}^\times = \mathbb{R}$ hence $\mathbb{T}$ is even a *semi-field* which is non-archimedean. In this talk, we will mostly restrict to this group of non-zero tropical numbers.

Let us give a (pocket) dictionnary between the tropical world and the world of affine geometry.

Tropical algebraic geometry	**Affine geometry**
Tropical torus $(\mathbb{T}^\times)^n$	Vector space $\mathbb{R}^n$
$(\mathbb{T}^\times)^n$-torsor	Affine space of dimension n
Monomials	Integral linear forms
Monomial maps	Integral linear maps
Polynomials	Piecewise linear concave functions
Hypersurfaces	Codimension one polyhedral subspaces in $\mathbb{R}^n$

Maybe the terminology "dictionnary" is a little misleading because the correspondance is not one-to-one (as we will see, for example, with polynomials and piecewise linear concave functions; it should be understood as a correspondance of concepts and not really of single objects.

[1] The difference between a ring and a semi-ring is that a semi-ring need not to have additive inverses; in fact, here $+\infty$ is the only invertible element for the tropical addition.

2. Tropical polynomials and polyhedral geometry.

2.1. Tropical polynomials.

In this section, we want to see how algebraic geometry over $\mathbb{R}^{\text{trop}}$ looks like. To work in a coordinate-free way, we consider the following setup:

- M is a lattice of rank n in some real vector space and $N := \text{Hom}(M, \mathbb{Z})$.
- $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$.

We have the usual duality pairing

$$\langle \cdot, \cdot \rangle : M \otimes_{\mathbb{Z}} N \to \mathbb{Z}.$$

Definition **(2.1.1.)** — *A* tropical (Laurent)[2] polynomial *is an element of* $\mathbb{R}^{\text{trop}}[x_1, \ldots, x_n]$. *More explicitely, a tropical (Laurent) polynomial f can be written as follows:*

$$f(x_1, \ldots, x_n) := \min_{(i_1, \ldots, i_n) \in S} \Big(a_{i_1, \ldots, i_n} + \sum_{j=1}^{n} i_j x_j \Big)$$

where $S \subseteq N$ is a finite subset called the support *of f.*

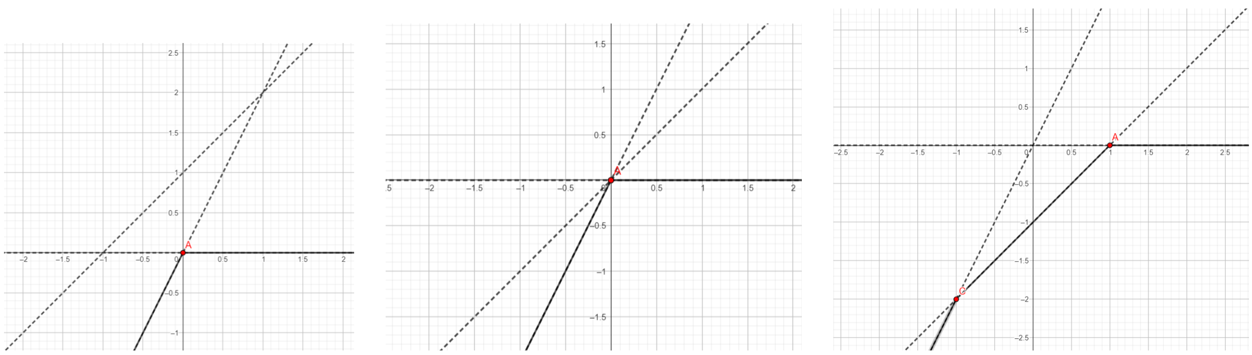
Remark. — The *tropical function* $f : M_{\mathbb{R}} \to \mathbb{R}$ defined by f is piecewise integral affine and concave.

As polynomials are in general not the same things as polynomials functions, different tropical polynomial can lead to the same tropical function $M_{\mathbb{R}} \to \mathbb{R}$. Let us detail another confusing point.

2.1.2. — We consider the three univariate tropical polynomials:

$$f_1 := 0 \oplus 1 \odot x \oplus x^{\odot 2},\ f_2 := 0 \oplus x \oplus x^{\odot 2} \text{ and } f_3 := 0 \oplus (-1) \odot x \oplus x^{\odot 2}.$$

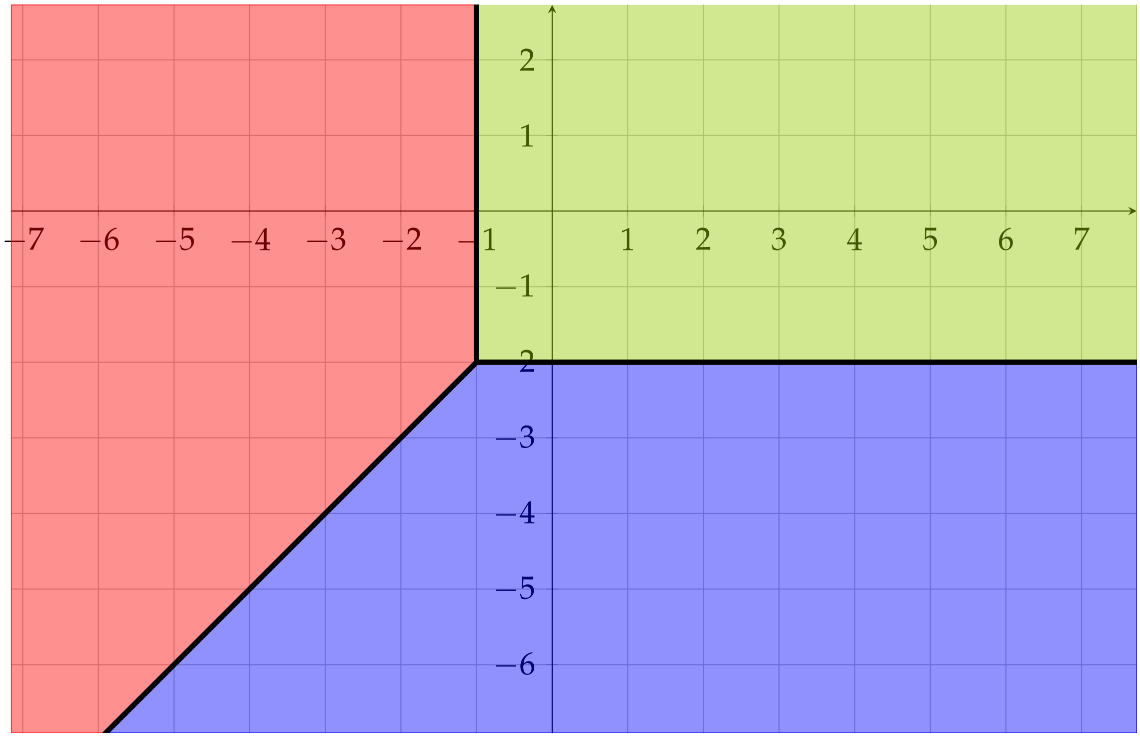
If we draw (in plain black on the pictures below) the graphs of the associated functions, we see that f_1 and f_2 define the same tropical function (whereas f_2) even though they are not equal as tropical polynomials.



Let us now take an example in two variables (we cannot really draw the graph of the associated tropical function because it is three-dimensional). The black part below is the analogue of the red part above — the non-differentiability locus of the tropical function.

Example (Tropical line.) — The tropical line corresponding to $P(x, y) := 2 \odot x \oplus 3 \odot y \oplus 1$ is $\min(x + 2, y + 3, 1)$; if we draw the corresponding regions in the real plane, we get the following regions:

[2] We will usually omit this precision.



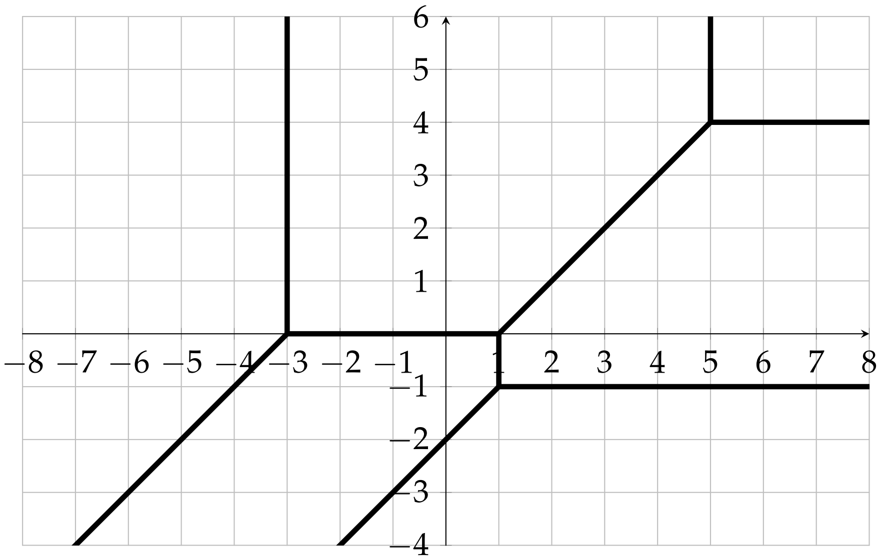
where the green (resp. blue, resp. red) region is the region where $P(x,y)$ is equal to 1 (resp. $x+2$, resp. $y+3$). The black region correspond to the domain where the minimum is attained (at least) twice, it will play a particular role in the sequel.

Let us see how a tropical conic looks like.

Example — We consider the tropical polynomial f of degree 2 defined by

$$f(x,y) := 5 \odot x^{\odot 2} \oplus 2 \odot x \odot y \oplus 4 \odot y^{\odot 2} \oplus 3 \odot y \oplus 2 \odot x \oplus 7.$$

We obtain the following picture:



Now, we would like to consider the zero locus of f but this does not make much sense (especially on $\mathbb{T}$, the 0 is $-\infty$) hence if we extend f to a map $\mathbb{T}^n \to \mathbb{T}$, then the zero locus would be the empty set unless $f = -\infty$ where we get $\mathbb{T}^n$; we thus have to find a better definition. Now, in $\mathbb{R}$, 0 is distinguished from other real numbers by the fact that it is "idempotent for the addition" hence something analoguous to zero (which will hopefully gives a more interesting locus) should be idempotent for $\oplus$; therefore, we put the following definition for the zero locus of a tropical polynomial. [3]

Definition **(2.1.3.)** (Tropical hypersurface.) — *The* tropical hypersurface *associated to a tropical polynomial $f := \oplus_{u \in S} a_u x^u$ is the locus $V(f)$ of $M_{\mathbb{R}}$ of points for which the minimum is attained at least twice. In other words, we have:*

$$V(f) := \{x \in M_{\mathbb{R}} \mid \exists (u,v) \in N^2, u \neq v \text{ and } f(x) = a_u + \langle u, x \rangle = a_v + \langle v, x \rangle \}.$$

Example — In the univariate example above, the tropical hypersurface of the f_i's is drawn in red. In the two bivariate examples above, the tropical hypersurface defined by f is drawn in black. This is much a more interesting definition and note that it has some link with f: it's its non-differentiability locus.

2.2. Polyhedral structure on tropical hypersurfaces.

We defined $V(f)$ as a subset of $M_{\mathbb{R}}$ but it has much more strucutre, namely a polyhedral structure. Let us introduce some polyhedral geometry notions to be able to talk about this extrastructure.

Definition **(2.2.1.)** —

- *A* polyhedron *in $M_{\mathbb{R}}$ is a finite intersection of closed half-spaces i.e., a subset of $M_{\mathbb{R}}$ of the form $P := \{x \in M_{\mathbb{R}} \mid \forall i \in [\![1, n]\!], \langle x, u_i \rangle \leq a_i\}$ where $n \geq 0$, $(u_1, \ldots, u_n) \in M_{\mathbb{R}}^n$ and $(a_1, \ldots, a_n) \in \mathbb{R}^n$.*
- *The* dimension *of a polyhedron is the dimension of the vector subspace it spans.*
- *We say that a subset F of a polyhedron P is a* face *of P if there exists an (affine) hyperplane H such that $F = P \cap H$ and P is contained in one of the two half-spaces defined by H.*
- *A* facet *is a face of codimension 0.*
- *The* relative boundary *∂P of a polyhedron P is the union of all its proper faces.*
- *The* relative interior *of a polyhedron P is the complement in P of the relative boundary of P.*
- *A* lattice *polyhedron is a polyhedron where the a_i in the definition lie in $\mathbb{Q}$ and all vertices of P lie in M.*
- *A* polytope *is a compact polyhedron.*

Remark. — We usually write $\tau \prec \sigma$ to say that τ is a face of σ.

Example — The square $P := [0,1]^2$ is a polytope in $\mathbb{R}^2$. Its proper faces are defined by intersection with the hyperplanes of equation $x = 0$, $y = 0$, $x = 1$ or $y = 1$. We see that its boundary is equal to its topological boundary and its interior is equal to its topological interior. This polytope is in fact even a lattice polytope.

Let us now describe the polyhedral structure on tropical hypersurfaces.

2.2.2. — Let us fix a tropical polynomial $f = \oplus_{u \in S} a_u x^u$. For all $u \in S$, we consider the rational polyhedron:

$$\sigma_u := \{x \in M_{\mathbb{R}} \mid f(x) = a_u x^u\}.$$

[3]Using *multivalued tropical addition*, there is a way to make this definition stick to the usual definition of zero locus.

In other words, the σ_u's divide $M_{\mathbb{R}}$ into linearity domains of f. Note that $\sigma_u \cap \sigma_v$ is either empty or either a face of both σ_u and σ_v (defined by the intersection $a_u + \langle x, u\rangle = a_v + \langle x, v\rangle$).

The collection of polyhedra $\mathscr{S}_f := \{\tau \prec \sigma_u \text{ for some } u \in S\}$ is a *polyhedral decomposition* of $M_{\mathbb{R}}$ in the following sense.

Definition **(2.2.3.)** — *A* polyhedral decomposition [4] *of a subset $S \subseteq M_{\mathbb{R}}$ is a finite collection $\mathscr{P}$ of polyhedra in $M_{\mathbb{R}}$ such that:*

1. *The polyhedra in $\mathscr{P}$ cover S.*
2. *The face of a polyhedron in $\mathscr{P}$ is a polyhedron in $\mathscr{P}$.*
3. *The intersection of two polyhedra of $\mathscr{P}$ is a face of these two polyhedra — that could well be empty.*

The union of the polyhedra in $\mathscr{P}$ is called the support *of $\mathscr{P}$ and is denoted by $|\mathscr{P}|$.*

Let us see how this decomposition looks like on our three univariate examples.

Example — Let us write this in a table to make it easier to read.

	f_1	f_2	f_3
σ_0	$\mathbb{R}_+$	$\mathbb{R}_+$	$[1, +\infty)$
σ_1	$\varnothing$	$\{0\}$	$[-1, 1]$
σ_2	$\mathbb{R}_-$	$\mathbb{R}_-$	$(-\infty, -1]$

We see that the cells σ_u depend on the tropical polynomial and not only on the associated function. On that example, we see that nonetheless $\mathscr{S}_f$ only depends on the associated tropical function.

Remark. — We can even say that $V(f)$ is a *purely $(n-1)$-dimensional* polyhedral decomposition of $M_{\mathbb{R}}$, meaning that all the maximal cells of $\mathscr{S}_f$ have dimension $n-1$. The polyhedral decomposition $\mathscr{S}_f$ constructed here is the decomposition that Gross denotes by $\check{\mathscr{P}}$ and the map that he denotes by φ is probably the Legendre transform of (the tropical function associated to) f in the sense of convex geometry.

2.3. The reduced representation of a tropical polynomial.

Let us clarify a little bit the gap between tropical polynomials and tropical functions. In this subsection, we will define the *reduced form* of a tropical polynomial which will satisfy the following property: two tropical polynomials induce the same tropical function if and only if they have the same reduced form.

Definition **(2.3.1.)** — *Let $f := \oplus_{u \in S} a_u x^u$ be a tropical polynomial.*

- *The* reduced support *of f is defined by*

$$S_{\text{red}} := \{u \in S \mid \dim(\sigma_u) = n\}.$$

- *The* reduced form *of f is the tropical polynomial defined by*

$$f_{\text{red}} := \bigoplus_{u \in S_{\text{red}}} a_u x^u.$$

[4] We also say that $\mathscr{P}$ is a *polyhedral complex* in $M_{\mathbb{R}}$.

Remark. — The tropical functions defined by f and f_{red} are the same since the σ_u with dimension n cover $M_{\mathbb{R}}$, hence any $x \in M_{\mathbb{R}}$ will lie in some σ_u with dimension n and by definition $f(x) = a_u + \langle u, x \rangle = f_{\text{red}}(x)$. Conversely, if two tropical polynomials induce the same tropical function, they have the same reduced support since the latter can be described as the set of the differentials $\mathrm{d}f_x$ for x varying in $M_{\mathbb{R}} - V(f)$ since $\mathrm{d}f_x = u$ for all $x \in \mathrm{Int}(\sigma_u)$. Now, the value $f(x)$ determines the coefficient a_u uniquely on the cell σ_u; therefore, these two tropical polynomials have the same reduced form.

Let us see what it gives on our univariate examples.

Example — The reduced form of f_1 and f_2 should be equal and they are indeed both equal to $0 \oplus x^{\odot 2}$. The reduced form of f_3 is equal to f_3 itself.

Remark. — In the book of Gross, f_{red} is equal to (the tropical function associated to) $\check{\varphi}$. The dual decomposition of f is the decomposition that Gross denotes by $\mathscr{P}$ in his book.

3. The dual decomposition of the Newton polytope.

In the previous section, we discussed a polyhedral decomposition of $M_{\mathbb{R}}$ whose $(n-1)$ skeleton (i.e., the union of cells of dimension at most $n-1$) was $V(f)$. Now, we will associate to f a polytope in the dual space $N_{\mathbb{R}}$ and will construct a polyhedral decomposition of the latter which will be dual to the polyhedral decomposition constructed in the previous section.

Definition **(3.0.1.)** (Newton polytope.) — *Let $f := \oplus_{u \in S} a_u x^u$ be a tropical polynomial.*
The Newton polytope *of f is the integral polytope* $\mathrm{NP}(f)$ *defined by*

$$\mathrm{NP}(f) := \mathrm{Conv}(S) \subseteq N_{\mathbb{R}}.$$

Example — For our three univariate examples, the Newton polytope is equal to $[0, 2]$. For the first bivariate example, we get the right triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$.

Let us now construct a polyhedral subdivision of $\mathrm{NP}(f)$.

3.0.2. — Let $f := \oplus_{u \in S} a_u x^u$ be a tropical polynomial. Let us consider $\overline{\mathrm{NP}}(f)$ defined as the upper convex hull of the set

$$\widetilde{S} := \{(u, a_u) \in N \times \mathbb{R} \mid u \in S\}.$$

In other words, we have:

$$\overline{\mathrm{NP}}(f) := \{(u, \lambda) \in N_{\mathbb{R}} \times \mathbb{R} \mid \exists (u, \mu) \in \mathrm{Conv}(\widetilde{S}), \mu \leq \lambda\}.$$

The projection of the faces of $\overline{\mathrm{NP}}(f)$ gives a polyhedral decomposition of $\mathrm{NP}(f)$ called the *dual subdivision* of f.

Let us describe all these objects on our three univariate examples.

Example — Let us describe them successively.

- For f_1, $\overline{\mathrm{NP}}(f)$ is equal to $[0, 2] \times \mathbb{R}_+$ and the dual decomposition splits the Newton polynomial of f_1 in 1 facet $[0, 2]$ and two 0-cells $\{0\}$ and $\{1\}$.
- For f_2, $\overline{\mathrm{NP}}(f)$ is equal to $[0, 2] \times \{0\}$ and the dual decomposition is the same as the one for f_1.

- For f_3, $\overline{\mathrm{NP}}(f)$ is equal to the union of $[0,2] \times \mathbb{R}_+$ with the triangle with vertices $(0,0)$, $(2,0)$ and $(1,-1)$. In that case, the dual decomposition is different: it has still this facet but it has three 0-cells : $\{0\}$, $\{1\}$ and $\{2\}$.

The fact that the dual decomposition of f_1 and f_2 are similar is no surprise: this decomposition only depends on the tropical function and not on the tropical polynomial.

On this example, we also notice an inclusion-reversing duality between the cells of $\mathscr{S}_f$ and the dual subdivision of f which holds in general and can be formulated as follows:

3.0.3. — If σ is a cell of $\mathscr{S}_f$, then we consider the set of monomials which are minimal on σ i.e.,

$$A_\sigma := \{u \in S \mid \sigma \subseteq \sigma_u\}$$

and we denote by D_σ the convex hull of A_σ in $N_\mathbb{R}$. Now, the duality can be written as follows: the map $\sigma \mapsto D_\sigma$ is an inclusion-reversing bijection between cells of $\mathscr{S}_f$ and the dual subdivision of f. Moreover, for all cell σ of $\mathscr{S}_f$, we have $\dim(\sigma) + \dim(D_\sigma) = n$ and $\mathrm{L}(\sigma)^\perp = L(D_\sigma)$ where L stands for the span as a real vector space.

4. Weights and the balancing condition.

In this section, we want to prove that in fact, a tropical hypersurface has more structure than being a polyhedral decomposition of $N_\mathbb{R}$: it is a *weighted* polyhedral decomposition of $N_\mathbb{R}$ which satisfies the *balancing condition*.

Let us first introduce all the terminology needed about weights.

Definition **(4.0.1.)** —

- *A* weight function *on a polyhedral complex $\mathscr{P}$ is a function $w : \mathscr{P} \to \mathbb{Z}$ that associates to any cell σ of $\mathscr{P}$ with maximal dimension an integer $w(\sigma)$ called the* weight *of σ.*
- *A* weighted polyhedral complex *is a polyhedral complex with a weight function.*

We now come to the definition which will allow us to formulate an important property of tropical hypersurfaces.

Definition **(4.0.2.)** — *Let $\mathscr{P}$ be a positively weighted polyhedral complex in $M_\mathbb{R}$.*
We say that $\mathscr{P}$ is balanced *if for all cell τ of codimension 1, we have*

$$\sum_{\tau \prec \sigma} w(\sigma) v_{\sigma/\tau} \in \mathrm{L}(\tau)$$

where $\mathrm{L}(\tau)$ is the subspace of $M_\mathbb{R}$ spanned by τ and $v_{\sigma/\tau} \in M$ satisfies $(\mathrm{L}(\sigma) \cap M) = (\mathrm{L}(\tau) \cap M) \oplus \mathbb{Z} v_{\sigma/\tau}$ and is pointing from τ in the direction of σ.

Remark. — One can dually formulate this balancing condition as follows: let $u_{\sigma/\tau} \in M/(\mathrm{L}(\tau) \cap M)$ such that $v_{\sigma/\tau} + \mathrm{L}(\tau) = u_{\sigma/\tau}$, the balancing condition now rewrites as:

$$\sum_{\tau \prec \sigma} w(\sigma) u_{\sigma/\tau} = 0 \in M_\mathbb{R}/\mathrm{L}(\tau).$$

Now, let us describe how we put weights on tropical hypersurfaces. We first need a definition from lattice theory.

Definition **(4.0.3.)** — *The* index *of $m \in N$ is the biggest $r > 0$ such that there exists $m' \in N$ satisfying $m = rm'$. We say that m is* primitive *if it has index 1.*

Remark. — If we choose a basis, the index of $m := (m_1, \ldots, m_n)$ is just $\gcd(m_1, \ldots, m_n)$.

4.0.4. — Let f be a tropical polynomial. If τ is a codimension one cell of $\mathscr{S}_f$ then, it is the face of two facets σ_1 and σ_2 which will correspond to two linearity domains of f. We denote respectively by u_1 and u_2 the exponent of f on σ_1 and σ_2. We now define the weight of τ as the index of $u_1 - u_2$. Note that this is well-defined since the index of an element is equal to the index of its opposite.

Now, we state an important theorem that we will prove only for $n = 2$ since it is the only case that we will really use.

Theorem **(4.0.5.)** — *If f is a tropical polynomial, then $V(f)$ is a balanced polyhedral complex.*

Proof. — If we pick a codimension one cell v of $\mathscr{S}_f$ (a vertex in that case), then locally around v, the subdivision $\mathscr{S}_f$ looks like the normal fan of D_v by duality. In particular, for each edge τ containing v, the primitive generator $\nu_{\tau/v}$ is orthogonal to the dual edge D_τ in D_v (by duality again) and $w(\tau)$ is the integer length of D_τ by definition. Therefore, summing all vectors $w(\tau)\nu_{\tau/v}$ in clockwise order gives zero since the boundary of D_σ is "closed", in the sense of curves i.e., its complement has two connected components. □

We will now verify this on some examples. First, note that if f is a univariate tropical polynomial, then its tropical hypersurface is just a finite number of points hence the balancing condition is empty. Let us start with the example of the tropical line.

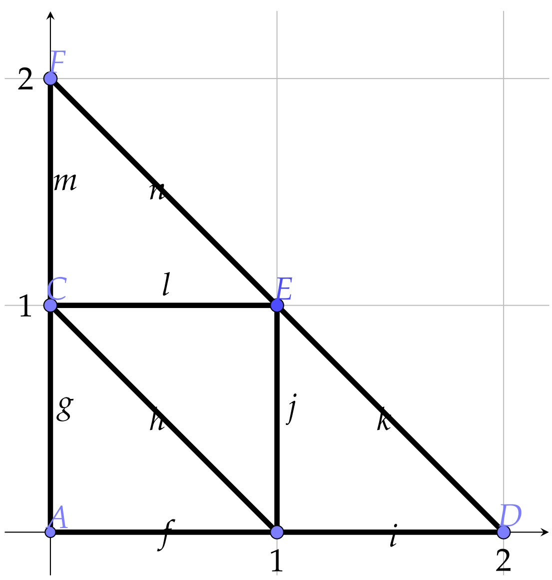
Example — In that case, the Newton polytope is the right triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. There is only one codimension one cell in $\mathscr{S}_f$, let us check the balancing condition there. The weights appearing in the sum are all equal to 1 hence we have:

$$1 \cdot (1,0) + 1 \cdot (0,1) + 1 \cdot (-1,-1)$$

which is indeed equal to zero.

Let us now try to check it on the example of the tropical conic.

Example — In that case, the Newton polytope is the right triangle with vertices $(0,0)$, $(2,0)$ and $(0,2)$. There are four codimension one cells in $\mathscr{S}_f$, which appears on the Newton polytope as four triangles. Here is the subdivision of the Newton polytope induced by the dual decomposition:



Let us treat (for example) the case of the vertex $(-3,0)$. It's the face of three facets which all have weight 1. We thus indeed have:

$$(1,0)+(0,1)+(-1,-1)=(0,0).$$

All the other vertices are similar since all the weight of the facets will be equal to 1.

Let us try to work-out an example with three variables.

Example — We consider the tropical polynomial $f(x,y,z) := x \oplus y \oplus z \oplus 0$. Let us check the balancing condition for the codimension one cell τ of $\mathscr{S}_f$ given by the equation $x=y=z$. We denote by $\sigma_{x,y}$ (resp. $\sigma_{x,z}$, resp. $\sigma_{y,z}$) the facets admitting τ as a face — they are respectively defined by the equation $x=y$, $x=z$ and $y=z$. Now we have:

$$\mathrm{L}(\tau)=\mathrm{Span}((1,1,1)).$$

We choose the following vectors:

$$\nu_{\sigma_{x,y}/\tau}=(1,1,-2),\ \nu_{\sigma_{y,z}/\tau}=(-2,1,1) \text{ and } \nu_{\sigma_{x,z}/\tau}=(1,-2,1).$$

Now, for τ the balancing condition is indeed vertified since

$$(1,1,-2)+(1,-2,1)+(1,1,-2)=(0,0,0)\in \mathrm{L}(\tau).$$

It is nevertheless a little disappointing since we do not use the full power of the sum belonging in $\mathrm{L}(\tau)$ — which is non-zero in that case, whereras before! Here, the vector were too well-chosen: we choose normal vectors to $(1,1,1)$. Let us choose less well-chosen vectors, for instance:

$$\nu_{\sigma_{x,y}/\tau}=(1,1,3),\ \nu_{\sigma_{y,z}/\tau}=(3,1,1) \text{ and } \nu_{\sigma_{x,z}/\tau}=(1,3,1).$$

With these vectors, we have:

$$(1,1,3)+(1,3,1)+(1,1,3)=(5,5,5)\in \mathrm{L}(\tau)$$

and they are relative primitive generators.