Tropical B-model statements

recall: Kähler moduli space $\mathcal{M}_\Sigma \cong \mathrm{Pic}(X_\Sigma) \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq \operatorname{Spec} \mathbb{C}[K_\Sigma]$,

where $0 \to K_\Sigma \to T_\Sigma = \mathbb{Z}^{\Sigma(1)} \to M \to 0$

$$\widehat{\mathcal{M}}_\Sigma \simeq \mathrm{Pic}(X_\Sigma) \otimes_{\mathbb{Z}} \mathbb{C}$$

kth-order thickening: $\widehat{\mathcal{M}}_{\Sigma,k} := (\widehat{\mathcal{M}}_\Sigma, \mathcal{O}_{\widehat{\mathcal{M}}_\Sigma} \otimes R_k \otimes \mathbb{C}[[y_0]])$

recall: $R_k = \dfrac{\mathbb{C}[u_1, \ldots, u_k]}{(u_1^2, \ldots, u_k^2)}$. Pulling back

$\kappa: \check{X}_\Sigma \to \widehat{\mathcal{M}}_\Sigma$ via $\widehat{\mathcal{M}}_{\Sigma,k} \twoheadrightarrow \widehat{\mathcal{M}}$, we obtain

"kth order thickened mirror family": $\kappa: \check{X}_{\Sigma,k} \to \widehat{\mathcal{M}}_{X,k}$

have $\Omega^1_{\check{X}_{\Sigma,k}/\widehat{\mathcal{M}}_{\Sigma,k}} \simeq \mathcal{O}_{\check{X}_{\Sigma,k}} \otimes_{\mathcal{O}_{X_\Sigma}} \Omega^1_{X_\Sigma/\mathcal{M}_\Sigma} \simeq M \otimes \mathcal{O}_{\check{X}_{\Sigma,k}}$

$$d\log(m) := \frac{dz^{\overline{m}}}{z^{\overline{m}}} \longleftarrow m \otimes 1$$

$\overline{m} \in T_\Sigma \longmapsto m \in M$

<u>$X_\Sigma = \mathbb{P}^2$</u>: $\check{X}_\Sigma = \{(x_0, x_1, x_2, y) \in (\mathbb{C}^*)^3 \times \mathbb{C} \mid x_0 x_1 x_2 = e^y\} \longrightarrow \mathbb{C}$

$(x_0, x_1, x_2, y) \mapsto y$; $\mathcal{O}_{\mathcal{M}_{\Sigma,k}} \simeq \mathcal{O}_{\mathbb{C},y} \otimes R_k \otimes \mathbb{C}[[y_0]]$

$\kappa^*(y_0) = y_0, \ \kappa^*(y_1) = y_1, \ \kappa^*(u_i) = u_i$

$\Omega^1_{\check{X}_{\Sigma,\hbar}/\widetilde{M}_{\Sigma,\hbar}} \simeq \mathcal{O}_{\check{\Sigma},\hbar}\, d\log(e_1) \oplus \mathcal{O}_{\check{X},\hbar}\, d\log(e_2)$ $\Omega := d\log e_1 \wedge d\log(e_2) \in \Omega^2_{\check{X}_{\hbar}/\widetilde{M}_{\Sigma,\hbar}}$

Recall that on $\widetilde{M}_\Sigma \times \mathbb{C}^*_\hbar$ we have local system with fibre over $(y, \hbar)$

$$H_2\left(\kappa^{-1}(y), \ \mathrm{Re}\left(W_0(Q)/\hbar\right) \ll 0\right)$$

See this as local system (denoted by R) on $\widetilde{M}_{\Sigma,\hbar} \times \mathbb{C}^*$, since $\widetilde{M}_\Sigma \simeq \widetilde{M}_{\Sigma,\hbar}$ as topological spaces.

recall construction of basis of R: let

$$\log : (\mathbb{C}^*)^2 \longrightarrow \mathbb{R}^2$$
$$(x_1, x_2) \longmapsto (\log|x_1|, \log|x_2|)$$

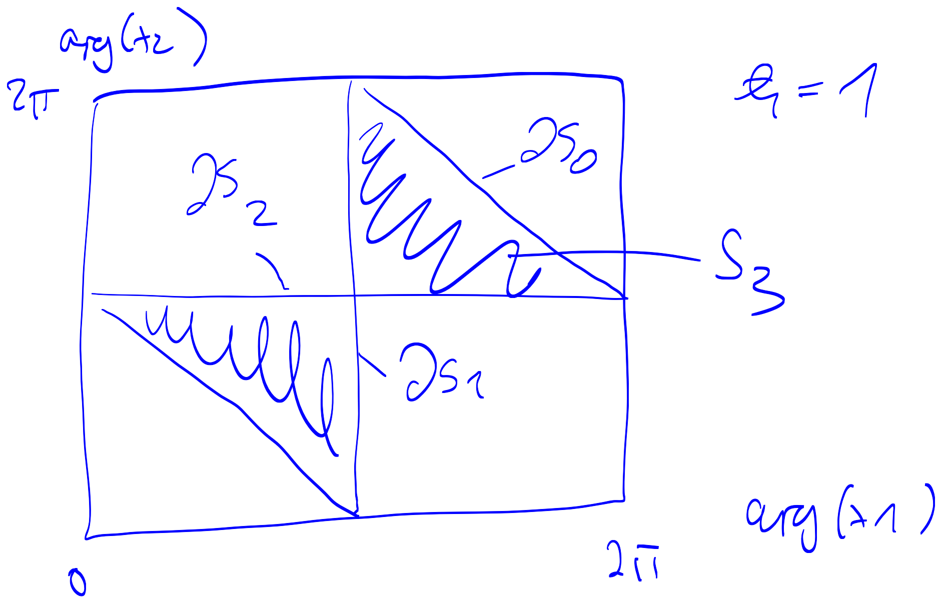
take $\Xi_0 := \log^{-1}(0,0)$, $S_0 = \left\{ (x_1,x_2) \in \mathrm{Log}^{-1}(\varphi_0) \,\middle|\, \arg \frac{1}{x_1 x_2} = \pi + \arg q \right\}$

$S_1 = \left\{ (x_1, x_2) \in \mathrm{Log}^{-1}(\varphi_1) \,\middle|\, \arg x_1 = \pi + \arg(e_1) \right\}$

$S_2 = \left\{ (x_1, x_2) \in \mathrm{Log}^{-1}(\varphi_2) \,\middle|\, \arg x_2 = \pi + \arg(e_1) \right\}$

have $\partial S_i \subset \Xi_0$, let $S_3 \subset \Xi_0$ s.t.

$$\partial S_3 = \partial S_0 \cup \partial S_1 \cup \partial S_2$$



Put $\Xi_1 = S_0 \cup S_1 \cup S_2 \cup S_3$

Put $\Xi_2 = \{(x_1, x_2) \in (\mathbb{C}^*)^2 \mid x_1, x_2 \in \mathbb{R}_{>0}\}$ (for $q = 1$)

<u>recall also from Henry's talk / working session:</u>

$\left(\int_{\Xi_i} e^{W_{0,q}} \Omega\right)_{i=0,1,2}$ is fundamental solution

to DE: $\left((q\partial_q)^3 + 3 \cdot q^3\right) \Psi = 0$ (*)

on the other hand, the coefficients of

$1, \alpha, \alpha^2$ of $\hbar^{-3\alpha} \sum_{d=0}^{\infty} \hbar^{-3d} \prod_{i=1}^{d} \frac{1}{(\alpha+i)^3}$

are also sol'ns to $(*)$. Therefore, we can write:

$$\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{W_0/\hbar} \Omega = \hbar^{-3\alpha} \sum_{d=0}^{\infty} \hbar^{-3d} \cdot \prod_{i=1}^{d} \frac{1}{(\alpha+i)^3} \mod \alpha^3$$

moreover, let $\tilde{\mathcal{M}} = \mathrm{Spf}\, \mathbb{C}[[t_0, t_2]] \times \tilde{\mathcal{M}}_{\Sigma}$, $\check{X} = \{x_0 x_1 x_2 = e^{t_1}\} \subset \tilde{\mathcal{M}} \times (\mathbb{C}^*)^3$

let $W = t_0 + W_0 + t_2 W_0^2$ (universal unfolding of W_0),

then

$$\boxed{\begin{gathered} \sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{W/\hbar} f \cdot \Omega = \hbar^{-3\alpha} \sum_{i=0}^{2} \varphi_i(\underline{t}, \hbar^{-1}) \cdot (\alpha\hbar)^i \\ \text{for some } f \in \mathcal{O}_{\check{X}}^{\vee} \text{ s.t. } f|_{\pi^{-1}(0)\times\mathbb{C}} \equiv 1 \\ \varphi_i(\underline{t}, \hbar) = \delta_{0,i} + \sum_{j=1}^{\infty} \varphi_{i,j}(\underline{t})\, \hbar^{-j} \\ \text{and } (\varphi_{i,1}(\underline{t}))_{i=0,\dots,2} \text{ are } \underline{\text{flat}} \text{ coordinates on } \tilde{\mathcal{M}} \end{gathered}}$$

Theorem: We have

$$\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{W_k/\hbar} \cdot \Omega = \hbar^{-3\alpha} \sum_{i=0}^{2} \varphi_i \cdot (\alpha \hbar)^i$$

$$\varphi_i(y_0, y_1, u_1, \dots, u_k, \hbar) = \delta_{0,i} + \sum_{j=1}^{\infty} \varphi_{ij}(y_0, y_1, u_1, \dots, u_k)\hbar^{-j}$$

and
$$\left.\begin{aligned} \varphi_{0,1} &= y_0 \\ \varphi_{1,1} &= y_1 \\ \varphi_{2,1} &= \sum_{i=1}^{k} u_i =: y_2 \end{aligned}\right\} \text{flat coordinates}$$

and $\varphi_i = J_i^{trop}(y_0, y_1, y_2)$

Corollary: The mirror symmetry statement "blue box" is equivalent to the equality

$$J_{\mathbb{P}^2} = J_{\mathbb{P}^2}^{trop}$$