Tropical descendant invariants for $\mathbb{P}^2$

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1. Introduction.

In today's series of talks, we want to build the B-model tropically for smooth and proper toric surfaces; the first talk was dedicated to the introduction of the Gromow-Witten potential in a greater generality (we already saw in in the case of the projective plane but Givental's construction works for any smooth proper toric variety) and of its relevant deformations defined in terms of tropical disks of Maslov index two.

In that talk, we restrict to the case of the projective plane; we do the next step: we introduce a tropical J function by just copying the expression we obtained in the third series of talks and we therefore just need to define (tropical) descendant Gromov-Witten invariants. In general, descendant Gromov-Witten invariants encode incidence conditions and tangency conditions; nevertheless, tangency does not make sense with tropical curves since they do not have vertices at infinity. To solve this problem, we will introduce the notion of *tropical disk* which is a sort of tropical curve with an additional (univalent) vertex supposed to represent the boundary of the curve. The link between these and actual (marked) rational tropical curves will be the process of splitting the curve into (marked) tropical disks at a certain point; this will allow us to define tropical descendant Gromov-Witten invariants.

2. Rational tropical curves and tropical disks.

For this section, we consider $M := \mathbb{Z}^2$ and we fix a fan $\Sigma \subseteq M_{\mathbb{R}}$ such that the associated toric variety X_Σ is projective.

We start by defining the moduli set we are interested in.

Definition **(2.0.1.)** — *Let $(P_1, \dots, P_r) \in M_{\mathbb{R}}^r$ be generic points, $S \subseteq M_{\mathbb{R}}$ be a subset and $\Delta \in \mathrm{H}_2(X_\Sigma, \mathbb{Z})$. For all $n \le r$ and $\nu \ge 0$, we define the moduli set $\mathscr{M}_{\Delta,n}^{\mathrm{trop}}(X_\Sigma, P_1, \dots, P_r, \psi^\nu S)$ as the set whose elements are $(n+1)$-pointed tropical curves $h : (\Gamma, p_1, \dots, p_n, x) \to M_{\mathbb{R}}$ of degree Δ in X_Σ satifying the four following conditions:*

1. *For all $j \in [\![1, n]\!]$, $h(p_j) = P_{i_j}$ for $1 \le i_1 < \cdots < i_n \le r$.*
2. *The edge E_x is attached to a vertex v_x of Γ and we have $\mathrm{val}(v_x) = \nu + 3$.*
3. *The point $h(x)$ belong to S.*
4. *The weight of each unbounded edge of Γ is either zero or one.*

These moduli sets carry some natural polyhedral structures.

Lemma **(2.0.2.)** — *If $(P_1, \dots, P_r) \in M_{\mathbb{R}}^r$ are generic points, then:*

1. *$\mathscr{M}_{\Delta,n}^{\mathrm{trop}}(X_\Sigma, P_1, \dots, P_r, \psi^\nu M_{\mathbb{R}})$ is a polyhedral complex of dimension $|\Delta| - n - \nu$.*

2. *If C is a generic translate of a tropical curve in $M_{\mathbb{R}}$, then $\mathscr{M}^{\mathrm{trop}}_{\Delta,n}(X_\Sigma, P_1, \dots, P_r, \psi^\nu C)$ is a polyhedral complex of dimension $|\Delta| - n - \nu - 1$.*

3. *If Q is a generic point in $M_{\mathbb{R}}$, then $\mathscr{M}^{\mathrm{trop}}_{\Delta,n}(X_\Sigma, P_1, \dots, P_r, \psi^\nu Q)$ is a polyhedral complex of dimension $|\Delta| - n - \nu - 2$.*

Proof. — See [Gro11, Lemma 5.11.]. □

We now describe the splitting of a rational tropical curve at a vertex into a series of tropical disks.

Proposition **(2.0.3.)** — *Let $(P_1, \dots, P_r) \in M^r_{\mathbb{R}}$ be generic points and $S \subseteq M_{\mathbb{R}}$. If $h \in \mathscr{M}^{\mathrm{trop}}_{\Delta,n}(X_\Sigma, P_1, \dots, P_r, \psi^\nu S)$, we denote by $(\Gamma'_i)_{1 \leq i \leq \nu+2}$ the collection of closures of the connected components of $\Gamma \setminus E_x$. For each $i \in [\![1, \nu+2]\!]$, we consider*

$$h_i := h|_{\Gamma'_i} : \Gamma'_i \to M_{\mathbb{R}}$$

which is [1] a tropical disk marked by the points p_i such that $E_{p_i} \subseteq \Gamma'_i$. We distinguish three cases:

1. *If $S = M_{\mathbb{R}}$ and $n = |\Delta| - \nu$, then one of the two following cases happens:*
   (a) *The edge E_x does not share a vertex with any of the edges E_{p_i} and in that case, the Maslov index of h_i is two for all i but 2 and for these two i, the Maslov index of h_i is zero.*
   (b) *The edge E_x shares a vertex with one of the edges E_{p_i} and in that case, the Maslov index of h_i is two for all i.*

2. *If S is a generic translate of a tropical curve in $M_{\mathbb{R}}$ and $n = |\Delta| - \nu - 1$, then the Maslov index of h_i is two for all i except one and for this i, the Maslov index of h_i is zero.*

3. *If S is a generic point of $M_{\mathbb{R}}$ and $n = |\Delta| - \nu - 2$, then the Maslov index of h_i is two for all i.*

Rather than proving this proposition, let us check it on some examples.

Example — Let us consider the tropical 4-pointed conic in $\mathbb{P}^2$ with an extra point x given by the following graph:

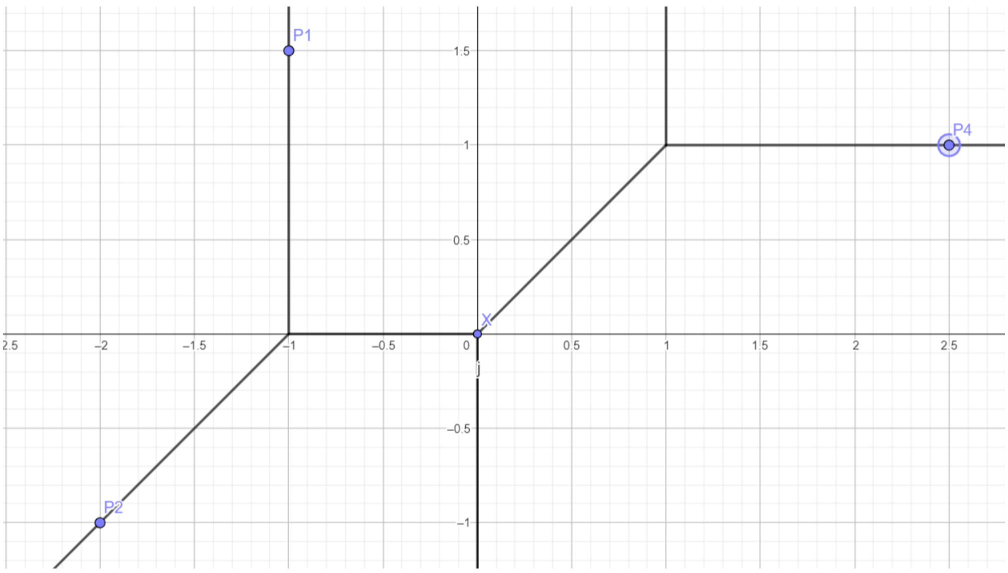


FIGURE 1: An element h of $\mathscr{M}^{\mathrm{trop}}_{\Delta_2,4}(P_1, \dots, P_4, \psi^1 X)$ [2].

[1] If E_x and E_{p_i} share a common vertex, then we eliminate the edge E_{p_i} and we have tropical disks $h_1, \dots, h_{\nu+1}$ since this phenomenon can happen at most once: indeed, $h(p_i) \neq h(p_j)$ if $i \neq j$.

[2] The point P_3 is $(1, \frac{3}{2})$.

If we split this tropical curve at x, we get three tropical disks h_1, h_2 and h_3 where the name are given from left to right. Their respective Maslov indices are given by:

$$\mathrm{MI}(h_1) = 2(3-2) = 2\ ;\ \mathrm{MI}(h_2) = 2(1-0) = 2 \text{ and } \mathrm{MI}(h_3) = 2(3-2) = 2.$$

This is indeed what the third case of the proposition 2.0.3 predicts. Now, let us move the point x in a tropical line L (drawn in green on the figure) ; even if the picture is misleading, the bounded green sections are also part of our tropical curve h.

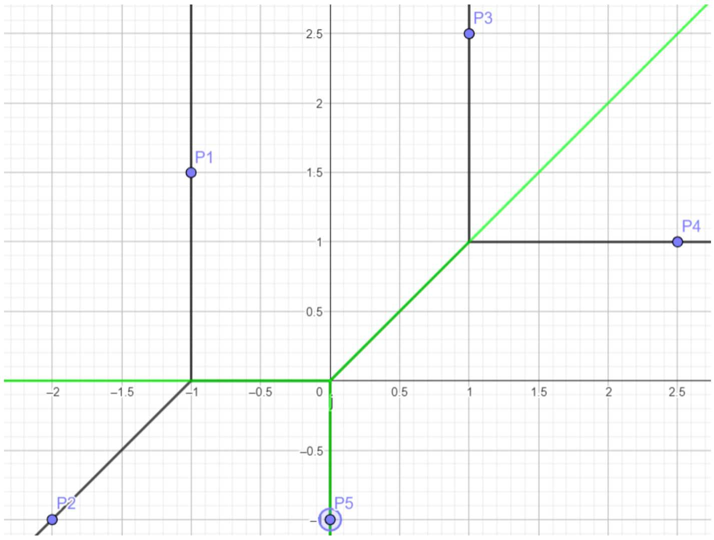


FIGURE 2: An element h of $\mathscr{M}^{\text{trop}}_{\Delta_2,5}(P_1,\dots,P_5,\psi^1 L)$.

Note that because the vertex v_x corresponding to x must have valency 4, x can only be one of the marked points or one of the endpoints of the bounded edges of h. We treat the two cases separately:

1. **If x is, say, the origin**, then if we split h at x, we get 3 tropical disks h_1, h_2 and h_3, their respective Maslov indices are given by:

$$\mathrm{MI}(h_1) = 2(3-2) = 2,\ \mathrm{MI}(h_2) = 2(1-1) = 0 \text{ and } \mathrm{MI}(h_3) = 2(3-2) = 2$$

   which is what was predicted by the proposition 2.0.3.

2. **If x is one of the marked points**, then if we split h at x, we get 2 tropical disks h_1 and h_2 whose respective Maslov indices are given by

$$\mathrm{MI}(h_1) = 2(1-0) = 2 \text{ and } \mathrm{MI}(h_2) = 2(4-4) = 0$$

   which is also what was predicted by the proposition 2.0.3.

To end this example, let us see what happens in the case where x is allowed to move inside the whole plane. We now work with six marked points.

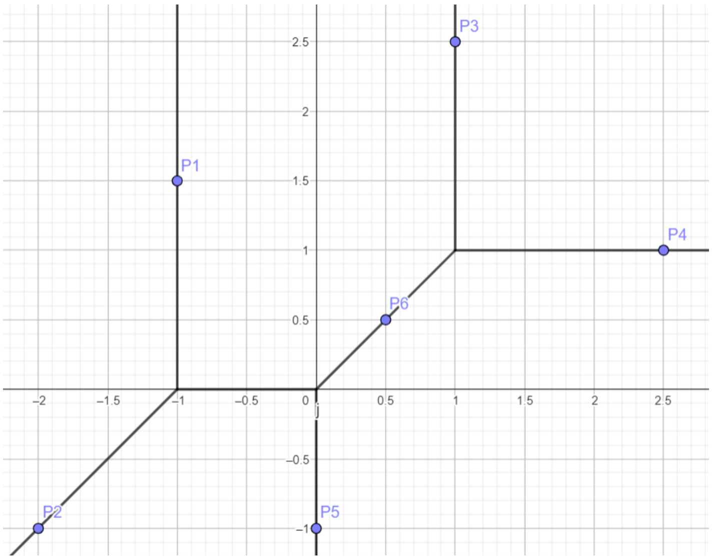


FIGURE 3: An element h of $\mathscr{M}^{\mathrm{trop}}_{\Delta_2,6}(P_1,\ldots,P_6,\psi^1 M_{\mathbb{R}})$.

As in the previous example, the same two cases can happen, we treat them separately.

1. **If x is, say, the origin**, then splitting h at x yields three tropical disks h_1, h_2 and h_3 whose respective Maslov indices are given by

$$\mathrm{MI}(h_1) = 2(3-2) = 2,\ \mathrm{MI}(h_2) = 2(1-1) = 0 \text{ and } \mathrm{MI}(h_3) = 2(3-3) = 0$$

which is what the proposition 2.0.3 predicted.

2. **If x is one of the marked points**, then splitting h at x yields two tropical disks h_1 and h_2 whose respective Maslov indices are given by

$$\mathrm{MI}(h_1) = 2(4-3) = 2 \text{ and } \mathrm{MI}(h_2) = 2(3-2) = 2$$

which is also what proposition 2.0.3 predicted.

Let us now give a proof of the splitting proposition.

Proof. — Since the moduli set $\mathscr{M}^{\mathrm{trop}}_{\Delta,n}(X_\Sigma, P_1,\ldots,P_k,\psi^\nu S)$ is a zero-dimensional polyhedral complex, the tropical disks h_i cannot be deformed keeping their boundary fixed, therefore by [Gro11, Lemma 5.6.], the Maslov indices of the tropical disks h_i must be at most 2 for all i. If we denote by n_i the number of marked points on h_i, we have:

$$\frac{1}{2}\sum_i \mathrm{MI}(h_i) = \sum_i \left(|\Delta(h_i)| - n_i\right) = \begin{cases} |\Delta(h)| - (n-1) \text{ in the case 1. (b)} \\ |\Delta(h)| - n \text{ otherwise} \end{cases}$$

and this quantity equals

- ν in the case 1. (a).
- $\nu+1$ in the cases 1. (b) and 2.
- $\nu+2$ in the case 3.

Since there are $\nu+2$ tropical disks except in the case 1. (b) — where there are $\nu+1$ disks — the result follows. □

3. Tropical descending invariants for $\mathbb{P}^2$.

> In the case of gravitational descendant, however, it is somewhat more difficult to motivate these formulæ. (...) The remaining formulas we give are more mysterious, and have no known justification outside of the mirror symmetry arguments given in this chapter.
>
> M. Gross, *Mirror symmetry and tropical geometry.*

We now restrict to the case of the projective plane. In that case, the rays of the fan Σ are generated by the vectors

$$m_0 := (-1,-1),\ m_1 := (1,0) \text{ and } m_2 := (0,1)$$

and we denote by $t_i \in T_\Sigma$ the preimage of m_i under r. For all $d \geq 0$, we consider the degree

$$\Delta_d := d(t_0 + t_1 + t_2) \in T_\Sigma$$

which corresponds to $d[L] \in \mathrm{H}_2(\mathbb{P}^2, \mathbb{Z})$ where L is a line.

We now (dreadfully) define the tropical descending Gromov-Witten invariant in that case. In order to do this, we make a series of definitions.

Definition **(3.0.1.)** — *Let Q and $(P_i)_i$ be generic points of $M_\mathbb{R}$ and let L be a tropical line with vertex Q in $\mathbb{P}^2$. If h is a tropical curve h in $\mathbb{P}^2$ with a marked point x, we consider the following numbers:*

- *For all $i \in [\![0, 2]\!]$, $n_i(x)$ is the number of unbounded rays that share a common vertex with E_x and that are mapped by h to a ray in the direction m_i.*
- $\mathrm{Mult}_x^0(h) := \dfrac{1}{n_0(x)! \cdot n_1(x)! \cdot n_2(x)!}$.
- $\mathrm{Mult}_x^1(h) := -\dfrac{\sum_{j=0}^2 \sum_{i=1}^{n_j(x)} \frac{1}{i}}{n_0(x)! \cdot n_1(x)! \cdot n_2(x)!}$.
- $\mathrm{Mult}_x^2(h) := \dfrac{(\sum_{j=0}^2 \sum_{i=1}^{n_j(x)} \frac{1}{i})^2 + \sum_{j=0}^2 \sum_{i=1}^{n_j(x)} \frac{1}{i^2}}{2 \cdot n_0(x)! \cdot n_1(x)! \cdot n_2(x)!}$.

Remark. — The number $\mathrm{Mult}_x^0(h)$ will appear because the symmetric group $\mathfrak{S}_{n_0(x)} \times \mathfrak{S}_{n_1(x)} \times \mathfrak{S}_{n_2(x)}$ acts on the set of unbounded edges of Γ with vertex v_x (the way the different factors act depend on the direction of the edge) but h is invariant under the action of this group; therefore, combinatorially, we have to multiply by $\mathrm{Mult}_x^0(h)$. The other nevertheless remain a mystery.

Before we give the actual definition of the tropical descendant Gromov-Witten invariants, let us recall the tropical version of Bezout's theorem for tropical curve in the projective plane.

3.0.2. — For all $i \in [\![1, 2]\!]$, let us consider C_i a tropical curve of degree d_i in $\mathbb{P}^2$. Let us suppose that C_1 and C_2 intersect at a finite number of points (this can always be achieved if we translate one of the two curves — in tropical geometry, this procedure is called *stable intersection*. For $x \in C_1 \cap C_2$, we define the *intersection multiplicity* of C_1 and C_2 at x by:

$$i_x(C_1, C_2) := w(E_1) \cdot w(E_2) \cdot |m_1 \wedge m_2|$$

where E_i is an edge of C_i containing x and m_i is a primitive tangent vector to E_i at x and where we have identified $\bigwedge^2 M$ with $\mathbb{Z}$. Note that, in that context, the direction of the vectors m_i do not matter. Now, the tropical Bezout theorem states that:

$$i(C_1, C_2) := \sum_{x \in C_1 \cap C_2} i_x(C_1, C_2) = d_1 d_2.$$

We finally come to the actual definition of the tropical descendant Gromov-Witten invariants.

Definition **(3.0.3.)** (Tropical descending Gromov-Witten invariants.) — *Let Q and $(P_i)_i$ be generic points of $M_{\mathbb{R}}$, $d \geq 0$ and $v \geq 0$ and let L be a tropical line with vertex Q in $\mathbb{P}^2$.*

- *If $h \in \mathscr{M}^{\text{trop}}_{\Delta_d, 3d-2-v}(P_1, \dots, P_{3d-2-v}, \psi^v Q)$, we define the* multiplicity *of h by*

$$\text{Mult}(h) := \text{Mult}^0_x(h) \cdot \prod_{v \in \Gamma^{[0]} \wedge v \notin E_x} \text{Mult}_V(h).$$

and also the tropical descendant Gromov-Witten invariant

$$\langle P_1, \dots, P_{3d-2-v}, \psi^v Q \rangle^{\text{trop}}_{0,d} := \sum_h \text{Mult}(h)$$

where the sum is over $h \in \mathscr{M}^{\text{trop}}_{\Delta_d, 3d-2-v}(P_1, \dots, P_{3d-2-v}, \psi^v Q)$.

- *If $h : (\Gamma, p_1, \dots, p_{3d-1-v}, x) \to M_{\mathbb{R}}$ is a marked rational tropical curve such that $h(p_i) = P_i$ for all i and that satisfies one of the following two conditions:*

  1. *$h \in \mathscr{M}^{\text{trop}}_{\Delta_d, 3d-1-v}(P_1, \dots, P_{3d-1-v}, \psi^v L)$ and such that E_x is the only unbounded edge of Γ that shares a vertex with E_x and that is mapped to the connected component of $L \setminus \{Q\}$ containing $h(x)$. We suppose that this connected component is $Q + \mathbb{R}^\times_+ m_i$ and we define the* multiplicity *of such a h by:*

  $$\text{Mult}(h) := |m(h_j) \wedge m_i| \cdot \text{Mult}^0_x(h) \cdot \prod_{v \in \Gamma^{[0]} \wedge v \notin E_x} \text{Mult}_v(h)$$

  where $m(h_j) := w(E_{j,\text{out}}) m^{\text{prim}}(h_j)$ where $m^{\text{prim}}(h_j) \in M$ is a primitive vector tangent to $h_i(E_{j,\text{out}})$ pointing away from $h(x)$ and $j \in [\![1, v+2]\!]$ is the only one (by proposition 2.0.3) such that $\text{MI}(h_j) = 0$. For the first factor, we have used the identification $\bigwedge^2 M \cong \mathbb{Z}$.

  2. *$v \geq 1$ and $h \in \mathscr{M}^{\text{trop}}_{\Delta_d, 3d-1-v}(P_1, \dots, P_{3d-1-v}, \psi^{v-1} Q)$. In that case, we define the* multiplicity *of h by*

  $$\text{Mult}(h) := \text{Mult}^1_x(h) \cdot \prod_{v \in \Gamma^{[0]} \wedge v \notin E_x} \text{Mult}_v(h),$$

  we define the tropical descendant Gromov-Witten invariant

$$\langle P_1, \dots, P_{3d-1-v}, \psi^v L \rangle^{\text{trop}}_{0,d} := \sum_h \text{Mult}(h)$$

where the sum is over h as in the second bullet.

- *Let $h : (\Gamma, p_1, \dots, p_{3d-v}, x) \to M_{\mathbb{R}}$ be a marked tropical rational curve such that $h(p_j) = P_j$ for all j and satisfies one of the four following conditions:*

  1. *$h \in \mathscr{M}^{\text{trop}}_{\Delta_d, 3d-v}(P_1, \dots, P_{3d-v}, \psi^v M_{\mathbb{R}})$ does not share a vertex with any of the E_{p_i}'s and E_x is the only unbounded edge of Γ that shares a vertex with E_x and that is mapped to the connected component of $M_{\mathbb{R}} \setminus L$ containing $h(x)$. By the proposition 2.0.3, there are exactly two integers $(j_1, j_2) \in [\![1, v+2]\!]^2$ such that $\text{MI}(h_{j_i}) = 0$. In that case, we define the* multiplicity *of h by*

$$\mathrm{Mult}(h) := |m(h_{j_1}) \wedge m(h_{j_2})| \cdot \mathrm{Mult}_x^0(h) \cdot \prod_{v \in \Gamma^{[0]} \wedge v \notin E_x} \mathrm{Mult}_v(h).$$

2. *$h \in \mathscr{M}^{\mathrm{trop}}_{\Delta_d, 3d-\nu}(P_1, \dots, P_{3d-\nu}, \psi^\nu M_{\mathbb{R}})$ and E_x shares a vertex with E_{p_i}. In addition, the edges E_x and E_{p_i} are the only unbounded edges of Γ mapping to the connected component of $M_{\mathbb{R}} \setminus L$ containing $h(x)$. In that case, we define the* multiplicity *of h by:*

$$\mathrm{Mult}(h) := \mathrm{Mult}_x^0(h) \cdot \prod_{v \in \Gamma^{[0]} \wedge v \notin E_x} \mathrm{Mult}_v(h).$$

3. *$\nu \geq 1$, $h \in \mathscr{M}^{\mathrm{trop}}_{\Delta_d, 3d-\nu}(P_1, \dots, P_{3d-\nu}, \psi^{\nu-1} L)$ and E_x is the only unbounded edge of Γ mapping into the connected component of $L \setminus \{Q\}$ containing $h(x)$. By the proposition 2.0.3, there exists exactly one $j \in [\![1, \nu+1]\!]$ such that $\mathrm{MI}(h_j) = 0$. If we write the connected component of $L \setminus \{Q\}$ containing $h(x)$ as $Q + \mathbb{R}_+^\times m_i$, the* multiplicity *of h is defined by*

$$\mathrm{Mult}(h) := |m(h_j) \wedge m_i| \cdot \mathrm{Mult}_x^1(h) \cdot \prod_{v \in \Gamma^{[0]} \wedge v \notin E_x} \mathrm{Mult}_v(h).$$

4. *$\nu \geq 2$ and $h \in \mathscr{M}^{\mathrm{trop}}_{\Delta_d, 3d-\nu}(P_1, \dots, P_{3d-\nu}, \psi^{\nu-2} Q)$. In that case, the* multiplicity *of h is defined by:*

$$\mathrm{Mult}(h) := \mathrm{Mult}_x^2(h) \cdot \prod_{v \in \Gamma^{[0]} \wedge v \notin E_x} \mathrm{Mult}_v(h),$$

we define the tropical descendant Gromov-Witten invariant

$$\langle P_1, \dots, P_{3d-\nu}, \psi^\nu M_{\mathbb{R}} \rangle^{\mathrm{trop}}_{0,d} := \sum_h \mathrm{Mult}(h)$$

where the sum is over h as in the third bullet.

Remark. — When $\nu = 0$, the first item of the definition above gives back the number of rational (torically transverse) curves of degree d passing through $3d - 1$ points.

Now comes a very important point which (finally) allows us to define our tropical descendant Gromov-Witten invariants.

Theorem **(3.0.4.)** — *If Q and the P_i's are chosen generically, the tropical descendant Gromov-Witten invariants do not depend on the points.*

Definition **(3.0.5.)** — *We define the* tropical descendant Gromov-Witten numbers*:*

- $\langle T_2^{3d-2-\nu}, \psi^\nu T_2 \rangle^{\mathrm{trop}}_{0,d} := \langle P_1, \dots, P_{3d-2-\nu}, \psi^\nu Q \rangle^{\mathrm{trop}}_{0,d}$.
- $\langle T_2^{3d-1-\nu}, \psi^\nu T_1 \rangle^{\mathrm{trop}}_{0,d} := \langle P_1, \dots, P_{3d-1-\nu}, \psi^\nu L \rangle^{\mathrm{trop}}_{0,d}$.
- $\langle T_2^{3d-\nu}, \psi^\nu T_0 \rangle^{\mathrm{trop}}_{0,d} := \langle P_1, \dots, P_{3d-\nu}, \psi^\nu M_{\mathbb{R}} \rangle^{\mathrm{trop}}_{0,d}$.
- *If $m + i + \nu \neq 3d$, $\langle T_2^m, \psi^\nu T_i \rangle^{\mathrm{trop}}_{0,d} := 0$.*

All these choice have been made for generically chosen points.

Now, we can define the tropical J function by analogy with the classical case:

$$J^{\mathrm{trop}}_{\mathbb{P}^2} := e^{\frac{y_0 T_0 + y_1 T_1}{\hbar}} \cup \left(T_0 + \sum_{i=0}^{2} \Big(\frac{y_2 \delta_{2,i}}{\hbar} + \sum_{d=1}^{+\infty} \sum_{\nu=0}^{+\infty} \frac{\langle T_2^{3d+i-2-\nu}, \psi^\nu T_{2-i} \rangle^{\mathrm{trop}}_{0,d}}{\hbar^{\nu+2}} e^{d y_1} \frac{y_2^{3d+i-2-\nu}}{(3d+i-2-\nu)!} \Big) T_i \right)$$

which is also written in coordinates

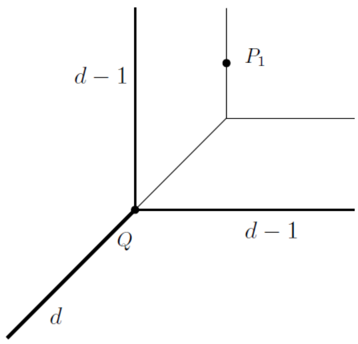
$$J_{\mathbb{P}^2}^{\text{trop}} =: \sum_{i=0}^{2} J_i^{\text{trop}} T_i.$$

Let us now try to compute some of the tropical descending Gromov-Witten invariant.

3.0.6. — We now compute the number $\langle \psi^{3d-2} T_2 \rangle_{0,d}^{\text{trop}}$, i.e. $\nu = 3d-2$. In that case, only one tropical curve h contribute to the sum: the one having d unbounded edges of weight one in each of the three directions. We therefore get:

$$\langle \psi^{3d-2} T_2 \rangle_{0,d}^{\text{trop}} = \text{Mult}(h) = \text{Mult}_x^0(h) = \frac{1}{(d!)^3}.$$

Now, we want to compute the same number for $\nu = 3d-3$, i.e. $\langle T_2, \psi^{3d-3} T_2 \rangle_{0,d}^{\text{trop}}$. In that case, if we fixe $P_1 \in M_{\mathbb{R}}$, a tropical curve h contributing to this number and will have a vertex of valency $3d-3+3 = 3d$ at Q and will split into $3d-1$ tropical disks with Maslov index two with boundary Q. Therefore, the picture is as follows:

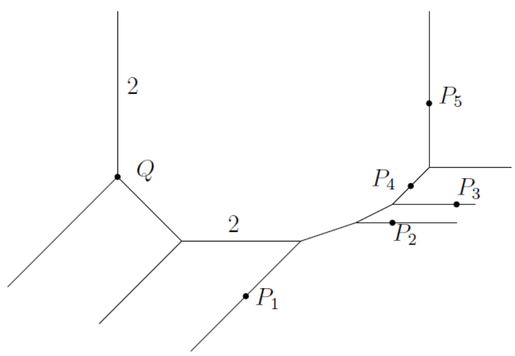


where the numbers refer to the number of edges and not to the weights. As a consequence:

$$\langle T_2, \psi^{3d-3} T_2 \rangle_{0,d}^{\text{trop}} = \frac{1}{d! \cdot (d-1)! \cdot (d-1)!}.$$

Let us finish with an example.

Example — We consider the tropical cubic h represented by



If we split h at Q, we get four tropical disks of Maslov index two since the vertical line with vertex Q counts twice. From vertices different from Q, the contibution to the mulitplicity is 4 since each endpoint of the weight two horizontal bounded edge will bring a 2. We also have

$$n_0(x) = 1,\ n_1(x) = 0 \text{ and } n_2(x) = 2.$$

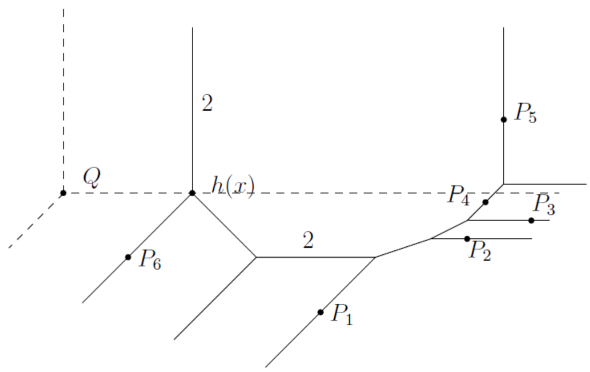
We can thus compute the multiplicities and we get:

$$\mathrm{Mult}_x^0(h) = \frac{1}{2}, \mathrm{Mult}_x^1(h) = -\frac{5}{4} \text{ and } \mathrm{Mult}_x^2(h) = \frac{17}{8}.$$

We can now compute the respective contribution of h to $\langle P_1, \dots, P_5, \psi^2 Q\rangle_{0,3}^{\mathrm{trop}}$, to $\langle P_1, \dots, P_5, \psi^3 L\rangle_{0,3}^{\mathrm{trop}}$ and to $\langle P_1, \dots, P_5, \psi^4 M_{\mathbb{R}}\rangle_{0,3}^{\mathrm{trop}}$. For the first one, we get $\frac{1}{2} \cdot 4 = 2$; for the second one, we are in the second case so we get $4 \cdot (-\frac{5}{4}) = -5$ and for the last one, we are in the fourth case and we therefore get $4 \cdot \frac{17}{8} = \frac{17}{2}$.

Let us finish with an example where other cases appear.

Example — We consider the same tropical cubic as before (with an additional point for the moduli set to be 0-dimensional) but we now let $h(x)$ vary in the tropical line L represented in dashed lines.



We want to understand the contibution of h to $\langle P_1, \dots, P_6, \psi^2 L\rangle_{0,3}^{\mathrm{trop}}$ and to $\langle P_1, \dots, P_6, \psi^3 M_{\mathbb{R}}\rangle_{0,3}^{\mathrm{trop}}$. If we split h at x, we still get four tropical disks but now, since we have this new marked point P_6, the tropical disk marked with P_6 has Maslov index zero. In addition, we have

$$n_0(x) = n_1(x) = 0 \text{ and } n_2(x) = 2.$$

Note that we do not get $n_0(x) = 1$ since the potential contributor to this number does not contribute since on the graph, it is not an unbounded edge: it has endpoints E_x and E_{p_6}. We can compute

$$\mathrm{Mult}_x^0(h) = \frac{1}{2} \text{ and } \mathrm{Mult}_x^1(h) = -\frac{3}{4}.$$

For the contribution to $\langle P_1, \dots, P_6, \psi^2 L\rangle_{0,3}^{\mathrm{trop}}$, we are in the first case; the tropical disk with Maslov index zero is the diagonal line with P_6 as a marked point, the contribution is therefore equal to:

$$\left|\det\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\right| \cdot \frac{1}{2} \cdot 4 = 2.$$

For the contribution to $\langle P_1, \dots, P_6, \psi^3 M_{\mathbb{R}}\rangle_{0,3}^{\mathrm{trop}}$, we are in the third case and the contribution is

$$1 \cdot (-\frac{3}{4}) \cdot 4 = -3.$$

References

[Gro11] M. Gross. Tropical geometry and mirror symmetry. Number 114 in CBMS Regional Conference Series in Mathematics. AMS, 2011.