Tropically deformed LG-models

Recall Givental's construction of LG-model

of toric variety X_Σ: $M = \mathbb{Z}^n$, $N = M^\vee$, $M_\mathbb{R} = M \otimes \mathbb{R}$,

$N_\mathbb{R} = N \otimes \mathbb{R}$, $\Sigma \subset M_\mathbb{R}$ complete fan. $T_\Sigma := \mathbb{Z}^{\Sigma(1)}$

generated by t_ρ for $\rho \in \Sigma(1)$. We have map

$r : T_\Sigma \to M$, $t_\rho \mapsto m_\rho$ (primitive int. generator of ρ)

assume: X_Σ smooth $\Longrightarrow r$ surjective ($\cancel{*}$)

sequence: $0 \to K_\Sigma \longrightarrow T_\Sigma \longrightarrow M \to 0$

dually: $0 \to N \to T_\Sigma^\vee \longrightarrow \mathrm{Hom}(K_\Sigma, \mathbb{Z}) \simeq \mathrm{Pic}(X_\Sigma) \to 0$ (*)

(notice $T_\Sigma^\vee \simeq PL(\Sigma) \simeq \mathrm{Div}(X_\Sigma)$)

tensor (*) by $\otimes_\mathbb{Z} \mathbb{C}^*$:

$$1 \to N \otimes \mathbb{C}^* \longrightarrow \mathrm{Hom}_\mathbb{Z}(T_\Sigma, \mathbb{C}^*) \to \underbrace{\mathrm{Pic}(X_\Sigma) \otimes \mathbb{C}^*}_{\overset{!!}{M_\Sigma} = \mathrm{Spec}\, \mathbb{C}[K_\Sigma]} \longrightarrow 1$$

$\kappa : \mathrm{Hom}(T_\Sigma, \mathbb{C}^*) \simeq \mathrm{Spec}\, \mathbb{C}[T_\Sigma] = (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow M_\Sigma \leftarrow$ abstract torus

Consider $\mathcal{M}_\Sigma$ as complex manifold, then

$$\widetilde{\mathcal{M}}_\Sigma = \mathrm{Pic}(X_\Sigma) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \mathcal{M}_\Sigma \text{ universal cover}$$

$$D \otimes y \longmapsto D \otimes e^y$$

Put $\check{X}_\Sigma = \mathrm{Hom}(T_\Sigma, \mathbb{C}^*) \times_{\mathcal{M}_\Sigma} \widetilde{\mathcal{M}}_\Sigma$

Let $W_0 = \sum_{g \in \Sigma(1)} z^{t_g} \in \mathbb{C}[T_\Sigma] \subset \mathcal{O}_{\check{X}_\Sigma}$

<u>Case $X_\Sigma = \mathbb{P}^2$:</u> $T_\Sigma = \mathbb{Z}^3 \xrightarrow[\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}]{r} M \cong \mathbb{Z}^2$

$\Rightarrow \mathrm{rk}\, K_\Sigma = 1$, $\kappa : \mathrm{Spec}\, \mathbb{C}[T_\Sigma] = (\mathbb{C}^*)^3 \longrightarrow \mathcal{M}_\Sigma \cong \mathbb{C}^*$

non-canonical

If we choose gen. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ of $\ker(r)$, then

$$\kappa : (\mathbb{C}^*)^3 \longrightarrow \mathbb{C}^*, \left(z^{t\binom{1}{0}}, z^{t\binom{0}{1}} z^{t\binom{-1}{-1}} \right) =: (x_0, x_1, x_2) \longmapsto x_0 x_1 x_2$$

$$\check{X}_\Sigma = \left\{ (x_0, x_1, x_2, y) \in (\mathbb{C}^*)^3 \times \mathbb{C} \;\middle|\; x_0 x_1 x_2 = e^y \right\}$$

$$W_0 = x_0 + x_1 + x_2 : \check{X}_\Sigma \longrightarrow \mathbb{C}$$

<u>Tropical discs and deformed potentials</u>: assume

rk $M = 2$ from here on (apparently relevant for

definition of multiplicity after Mikhalkin).

tropical discs: $\overline{\Gamma} = (\overline{\Gamma}^{[0]}, \overline{\Gamma}^{[1]}, w)$ weighted, connected graph,

no bivalent vertices, $V_{out} \in E$ univalent + unique flag (V_{out}, E_{out})

put $\Gamma' = (\overline{\Gamma}^{[0]} \setminus \overline{\Gamma}^{[0]}_{\infty} \cup \{V_{out}\}, \overline{\Gamma}^{[1]})$. Γ' is under-

stood as top. space, and we assume

$b_1(\Gamma) = 0$.

<u>Def.</u>: d-pointed tropical disc in $M_{\mathbb{R}}$ is continuous

map $h : \Gamma' \to M_{\mathbb{R}}$ + inclusion $\{p_1, \ldots, p_d\} \hookrightarrow \Gamma'^{1}_{\infty} \setminus E_{out}$

$(w(E) = 0 \iff E = E_{p_i})$ s.t.

1.) $w(E) = 0 \Rightarrow h(E) = \mathrm{const}$, else $\mathrm{slope}(h(E)) \in \mathbb{Q}$

2.) balancing condition at all $V \in \Gamma'^{[0]} \setminus \{V_{out}\}$

modulo homeo $\Phi : \Gamma_1' \to \Gamma_2'$ resp. marking + weights s.t. $h_1 = h_2 \circ \Phi$.

Def.: (multiplicity) assume Γ has no vertices of valency >3 (i.e. only 1 and 3), put

$\forall V \in \Gamma^{[0]}$: $\mathrm{mult}_V(h) = w(E_i)w(E_j)|\det(m_{E_i}|m_{E_j})|$

where E_1, E_2, E_3 are adjacent to V, $i,j \in \{1,2,3\}$, $m_{E_i} \in M = \mathbb{Z}^2$ is primitive integral vector parallel to E_i, and where $w(E_i) \neq 0$ (otherwise $\mathrm{mult}_V(h) := 1$)

well-defined due to balancing cond.

$$\mathrm{mult}(h) := \sum_{V \in \Gamma^{[0]}} \mathrm{mult}_V(h)$$

Def.: (trop. discs in fans, degree) $\Sigma \subset M_{\mathbb{R}} = \mathbb{R}^2$ complete, polyhedral fan, recall $r: T_\Sigma = \mathbb{Z}^{\Sigma(1)} \to M$.

h tropical disc in $M_{\mathbb{R}}$ is in $\Sigma \overset{\text{Def}}{\Longleftrightarrow} \forall E \in \Gamma_\infty^{[1]} \setminus E_{out}$ $h(E) = \text{const}$ or parallel to some $\rho \in \Sigma(1)$.

$$\Delta(h) = \sum_{\rho \in \Sigma(1)} \underset{\substack{\uparrow \\ \#\{E \in \Gamma_\infty^{[1]} \mid E \text{ parallel to } \rho, \text{ unmarked}\}}}{d_\rho} \cdot \underset{\nwarrow \text{generator}}{t_\rho} \in T_\Sigma$$

$$|\Delta(h)| = \sum_{\rho \in \Sigma(1)} d_\rho$$

Fix $P_1, \dots, P_k \in M_{\mathbb{R}}$; fix $Q \in M_{\mathbb{R}}$

$(P_1, \dots, P_k, Q)$ should be in general position

<u>Def</u>. A tropical disc in $(X_\Sigma, P_1, \dots, P_k)$ with boundary Q is a d-pointed tropical disc in Σ, where $d \leq k$, where $h(V_{out}) = Q$ and where $\exists\ 1 \leq i_1 < \dots < i_d \leq k$ s.t. $h(p_j) = P_{i_j}$ $\forall j \in \{1, \dots, d\}$

Maslov-index: $MI(h) := 2(|\Delta(h)| - d)$

<u>Lemma:</u> $P_1, \dots, P_k, Q$ in general position.

$\{$tropical discs h in $(X_\Sigma, P_1, \dots, P_k)$, boundary $= Q$, $MI(h) = 2n\}$

is $n-1$-dim. polyhedral complex

<u>Idea of proof</u>: Fix comb.-type of h (i.e. homeom. type of $\bar{\Gamma} + \{p_i\}, w, V_{out}, \{m_E\}$). Assume (generic?) Γ' has only trivalent vertices $+ V_{out}$

$\exists\ |\Delta(h)| + d$ unbounded edges

$\Longrightarrow |\Delta(h)| + d - 1$ bounded edges (incl. E_{out})

(follows from $\chi(\bar{\Gamma}) = 1 - b_1(\bar{\Gamma}) \overset{!}{=} \# \bar{\Gamma}^{[0]} - \# \bar{\Gamma}^{[1]}$)

$\rightsquigarrow$ space of trop. discs of fixed comb. type has open cell with closure $\mathbb{R}_{\geq 0}^{|\Delta(h)|+d-1} \times M_{\mathbb{R}}$. $\exists$ finite nb. of comb. types

$\Longrightarrow \dim \mathcal{M}_{\Delta,d}^{disc}(X_\Sigma) = |\Delta(h)| + d - 1 + 2 = |\Delta(h)| + d + 1$

fixing points $P_{i_1}, \dots, P_{i_d}$ generically removes

$2d$ degrees of freedom $\Rightarrow$ $\dim = |\Delta(h)| - d + 1$

Hence $\{$trop. discs $h, \ldots, MI(h) = 2|\Delta(h)| - d \overset{!}{=} 2n\}$

has dimension $n + 1 - 2 \overset{=\dim M_{\mathbb{R}}}{} \underset{=n-1}{(\text{position of } Q)}$

in part: $\#\{$trop discs $h, \ldots, MI(h) = 2\} < \infty$

<u>Def</u>: $-$ $h: (\Gamma', p_1, \ldots, p_d) \to M_{\mathbb{R}}$ with $MI(h) = 2$

$$\mathrm{Mono}(h) := \mathrm{mult}(h) \cdot \underbrace{z^{\Delta(h)}}_{\in \mathbb{C}[T_\Sigma]} \cdot u_{I(h)}$$

where $I(h) := \{i \mid h(p_j) = P_i \text{ for some } j\} \subset \{1, \ldots, k\}$

$\mathrm{Mono}(h) \in \mathbb{C}[T_\Sigma] \otimes R_k$ with $R_k = \dfrac{\mathbb{C}[u_1, \ldots, u_k]}{(u_1^2, \ldots, u_k^2)}$

$-$ k-pointed (perturbed) Landau-Ginzburg potential:

$$W_k(Q) := y_0 + \sum_h{}' \mathrm{Mono}(h) \in \mathbb{C}[T_\Sigma] \otimes R_k[[y_0]]$$

sum over all tropical discs in $(X_\Sigma, P_1, \dots, P_k)$ with boundary Q, Maslov index $= 2$

<u>examples:</u> – $\Sigma \subset M_\mathbb{R}$ any fan, $k=0 \Rightarrow \forall h$, have $d=0$ $\Rightarrow |\Delta(h)| = 1$, i.e. one unbounded edge (parallel to $\rho \in \Sigma(1)$) starting at Q. Since $\nexists$ vertices other than Q, mult $= 1$ (no contribution from Q, $\prod_V \mathrm{mult}_V(h)$ is empty) $\Rightarrow W_0(Q) = y_0 + \sum_{\rho \in \Sigma(1)} z^{t_\rho}$

– <u>$\Sigma = \mathbb{P}^2$</u>: $k=1 \Rightarrow d \in \{0, 1\}$

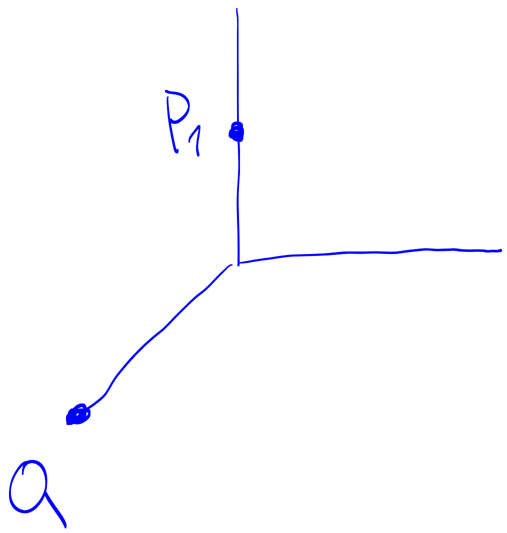
P_1 •

Q •

if <u>$d=0$</u> $\Rightarrow W_0(Q) = y_0 + x_0 + x_1 + x_2$

<u>$d=1$</u>: $MI(h) = 2 \Rightarrow |\Delta(h)| = 2$

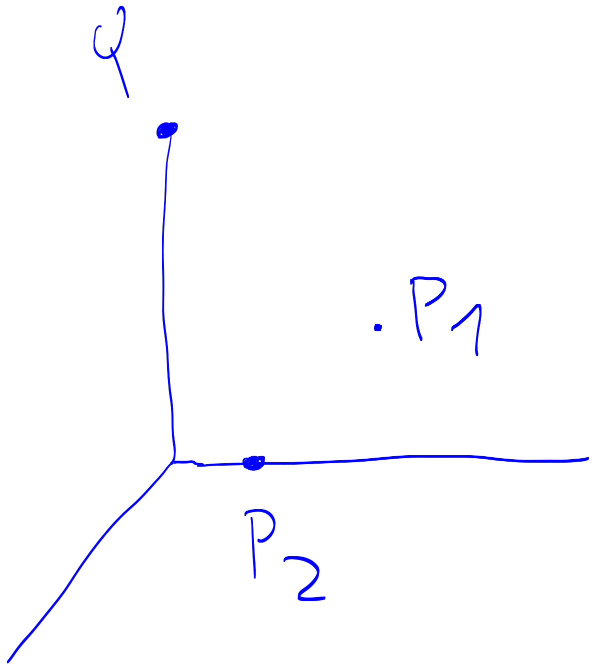
hence $|\Delta(a)| + d = 2+1 = 3$ unbounded edges, one marked with P_1, one ending at Q

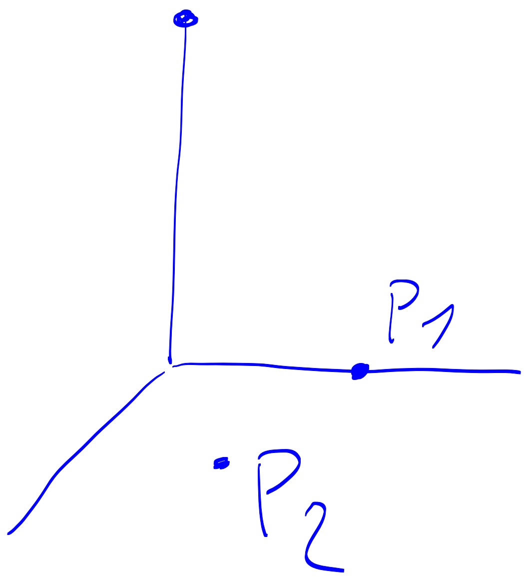


$\rightsquigarrow W_1(Q) = W_0(Q) + u_1 x_1 x_2$

<u>$k=2$:</u> $d \in \{0, 1, 2\}$

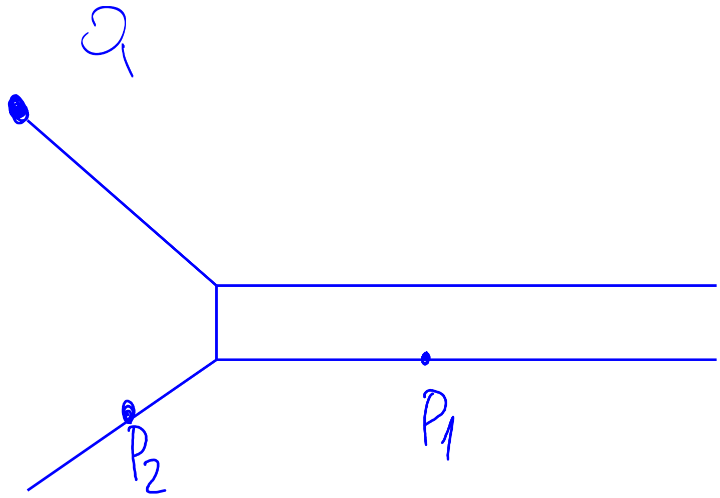
$d=1$





$$W_1(Q) = W_0(Q) + u_1 x_0 x_1 + u_2 x_0 x_1$$

$d = 2$:



$$W_2(Q) = W_1(Q) + u_1 u_2 x_0 x_1^2$$