Deux ou trois choses que je sais d'elle — la preuve de la formule de comptage de Mikhalkin

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...she took her hand and raised her brush. For a moment it stayed trembling in a painful but exciting ecstacy in the air. Where to begin? — that was the question at what point to make the first mark. One line placed on the canvas committed her to innumerable risks, to frequent and irrevocable decisions. All that in idea seemed simple became in practice immediately complex; as the waves shape themselves symmetrically from the cliff top, but to the swimmer among them are divided by streep gulfs, and foaming crests. Still the risk must run; the mark made.

V. Woolf, *To the lighthouse.*

In this document, k will always be an algebraically closed field of characteristic zero.

1. Introduction.

In the middle of the nineties, Mikhalkin found a way to reduce the question of counting (certain) holomorphic curves in the projective plane to counting tropical curves (i.e. certain types of graphs). If Σ is a fan in a real vector space of dimension 2 such that the toric variety X_Σ is smooth, if $\Delta \in \mathrm{H}_2(X_\Sigma, \mathbb{Z})$ is a degree [1] and if $(p_1, \dots, p_{|\Delta|-1}) \in X_\Sigma^{|\Delta|-1}$ are in general position, we consider

$$N^{0,\mathrm{hol}}_{\Delta,\Sigma} := \left|\left\{ f \in \overline{\mathscr{M}}_{0,|\Delta|-1}(X_\Sigma, \Delta) \middle| \, f : (C, x_1, \dots, x_{|\Delta|-1}) \to X_\Sigma \text{ torically transverse, } f(x_i) = p_i \right\}\right|$$

then, his curve counting formula can be formulated as follows:

Theorem **(1.0.1.)** (Mikhalkin's curve counting formula.) — *If Σ is a complete fan in a real vector space of dimension 2 whose associated toric variety X_Σ is smooth, then the number $N^{0,\mathrm{hol}}_{\Delta,\Sigma}$ is finite and we have the equality*

[1] If the toric variety X_Σ is smooth, then $H_2(X_\Sigma, \mathbb{Z}) \cong T_\Sigma$. This tells us that, in that case, we can link the degree of a tropical curve to a homology class, i.e. something that has to do with curve counting.

$$N_{\Delta,\Sigma}^{0,\mathrm{trop}} = N_{\Delta,\Sigma}^{0,\mathrm{hol}}$$

where $N_{\Delta,\Sigma}^{0,\mathrm{trop}}$ is (up to the Mikhalkin multiplicity) the number of genus zero tropical curves in X_Σ passing through $|\Delta| - 1$ general points — that has been defined in the third talk.

The strategy of the proof is to construct a degeneration of the toric variety X_Σ which is in some sense adapted to the general points $P_1, \dots, P_{|\Delta|-1} \in M_\mathbb{R}$ that we have chosen to define the number $N_{\Delta,\Sigma}^{0,\mathrm{trop}}$, in the sense that the degeneration will be induced by a polyhedral decomposition of $M_\mathbb{R}$ whose vertices are exactly the P_i's. To work on the central fiber of this degeneration, we will use logarithmic geometry (the central fiber will be a *log stable curve*) and the nearby fiber will be an ordinary stable curve.

So far, we have seen:

- how to associate a paramet(e)rized tropical curve to a torically transverse pre-log curve via the graph construction (talk 2).
- how to associate Mult(h) torically transverse marked log curves to a marked tropical curve h (talk 3).

In this talk, we will see:

- how to associate a family of log curves (whose central fiber is a torically transverse pre-log curve) to a family of rational curves.
- how to associate a (formal) family of marked rational curves to a torically transverse log curve of genus zero.

2. From the classical world to the logarithmic world.

Let us introduce some terminology.

Definition **(2.0.1.)** — *Let Σ be a fan in a real vector space $M_\mathbb{R}$.*
We say that a finite polyhedral decomposition $\mathscr{P}$ of $M_\mathbb{R}$ is a degeneration *of Σ if it satisfies the two following conditions:*

1. *All the cells of $\mathscr{P}$ have at least one vertex.*
2. *All the cones of Σ are recession cones of cell(s) of $\mathscr{P}$.*

We say that the degeneration is rational *if in addition, all the cells of $\mathscr{P}$ have faces with rational slopes and rational vertices.*

Remark. — Since the polyhedral decomposition is finite, a *rational* degeneration of Σ can be supposed to be *integral*, changing the lattice M by the (bigger) lattice $M[\frac{1}{d}]$ where d is the lcm of all the denominators appearing in the coordinates of the vertices and the slopes.

For convenience, we recall how a degeneration $\mathscr{P}$ of a fan Σ induces a one-parameter degeneration — usually called a *Mumford degeneration* — of the toric variety X_Σ. Let us consider

$$\tilde{M} := M \oplus \mathbb{Z}$$

and $\tilde{N}$ the dual lattice of $\tilde{M}$.

2.0.2. — For all cell σ of $\mathscr{P}$, we consider the cone over it

$$C(\sigma) := \overline{\{(rm, r) \in \tilde{M}_{\mathbb{R}} \mid r \geq 0 \text{ and } m \in \sigma\}}.$$

The collection of the $C(\sigma)$'s for σ a cell of $\mathscr{P}$ form a fan that we denote by $\Sigma_{\mathscr{P}}$. Now, since

$$C(\sigma) \cap (M_{\mathbb{R}} \oplus \{0\}) = \mathrm{Recc}(\sigma)$$

for all cell σ of $\mathscr{P}$, we deduce that Σ can be recovered from $\Sigma_{\mathscr{P}}$ as follows:

$$\Sigma = \{\tau \in \Sigma_{\mathscr{P}} \mid \tau \subseteq (M_{\mathbb{R}} \oplus \{0\})\}.$$

This gives some geometric justification to our terminology: the toric variety $X_{\Sigma_{\mathscr{P}}}$ comes with a morphism

$$\pi : X_{\Sigma_{\mathscr{P}}} \longrightarrow \mathbb{A}^1_k$$

induced on the algebras by the monomial $z^{(0_N,1)}$. In particular, $\pi^{-1}(0)$ is the union of the toric divisors on which this monomial vanishes, which are exactly those corresponding to cones $C(v)$ where v is a vertex of $\mathscr{P}$. Now, $X_{\Sigma_{\mathscr{P}}} \setminus \pi^{-1}(0)$ corresponds to the toric variety X_Σ where Σ is seen as a fan in $\tilde{M}_{\mathbb{R}}$. As a consequence:

$$X_{\Sigma_{\mathscr{P}}} \setminus \pi^{-1}(0) \cong X_\Sigma \times \mathbb{G}_{\mathrm{m},k}$$

which shows that $\pi : X_{\Sigma_{\mathscr{P}}} \to \mathbb{A}^1_k$ is indeed a one-parameter degeneration of X_Σ.

In that section, we start with a one parameter degeneration $\pi : X \to \mathbb{A}^1_k$ of the toric variety X_Σ coming from a rational degeneration $\mathscr{P}$ of Σ.

Let us now formulate the theorem ([Gro11, Theorem 4.24.]) we are going to prove in this section.

Theorem **(2.0.3.)** — *Let us consider the following data:*

- *a one-parameter degeneration $\pi : X \to \mathbb{A}^1_k$ induced by a rational degeneration $\mathscr{P}$ of Σ.*
- *a discrete valuation ring R with residue field k and whose fraction field will be denoted by K.*
- *a dominant morphism $\psi : \mathrm{Spec}(R) \to \mathbb{A}^1_k$ mapping the closed point 0 of $\mathrm{Spec}(R)$ to the origin.*
- *a torically transverse stable map $f^* : (C^*, x_1^*, \ldots, x_\ell^*) \to X \setminus X_0$ such that the square*

$$\begin{tikzcd} (C^*, x_1^*, \ldots, x_\ell^*) \arrow[r, "f^*"] \arrow[d] & X \setminus X_0 \arrow[d, "\pi|_{X \setminus X_0}"] \\ \mathrm{Spec}(K) \arrow[r, "\psi"'] & \mathbb{G}_{\mathrm{m},k} \end{tikzcd} \tag{1}$$

is commutative.

Then, we have the following conclusions:

1. *Possibly after making a finite base change $t \in \mathbb{A}^1_k \mapsto t^e \in \mathbb{A}^1_k$ and replacing R with $S := R[t]/(t^d - \varpi)$ where ϖ is a uniformizing element of R, there exists a refinement $\tilde{\mathscr{P}}$ of $\mathscr{P}$ with integral vertices defining a toric blow-up $\tilde{X}$ of X such that the commutative square (1) extends to a commutative square*

$$\begin{tikzcd} (C, x_1, \ldots, x_\ell) \arrow[r, "f"] \arrow[d] & \tilde{X} \arrow[d, "\pi"] \\ \mathrm{Spec}(S) \arrow[r, "\psi"'] & \mathbb{A}^1_k \end{tikzcd} \tag{2}$$

where the restriction of f to the fiber over the closed point of $\mathrm{Spec}(S)$ is a torically transverse pre-log curve and if we write $\mathbb{A}^1_k := \mathrm{Spec}(k[z])$, then $\psi^*(z)$ is a uniformizing element of S.

2. *Moreover, the diagram (2) induces a diagram of logarithmic schemes*

$$\begin{tikzcd} (C^\dagger, x_1, \dots, x_\ell) \arrow[r, "f^\dagger"] \arrow[d] & \tilde{X}^\dagger \arrow[d, "\pi^\dagger"] \\ \mathrm{Spec}(S)^\dagger \arrow[r, "\psi^\dagger"'] & (\mathbb{A}^1_k)^\dagger \end{tikzcd} \tag{3}$$

where all the logarithmic structures are divisorial: induced by $\partial\tilde{X}$ *for* $\tilde{X}$*, induced by* $\{0\}$ *for* $\mathrm{Spec}(S)$*, induced by* $f^{-1}(\partial\tilde{X}) \subseteq C$ *for* C *and induced by* $\{0\}$ *for* $\mathbb{A}^1_k$*.*

3. *If, in addition, the following conditions are fulfilled:*

   - C_0 *has genus zero.*
   - *the tropical curve associated to the pre-log curve* f_0 *is simple*
   - $f^{-1}\left(\overline{\partial(\tilde{X} - \tilde{X}_0)}\right)$ *is a disjoint union of sections of* $C \to \mathrm{Spec}(S)$

   then, $C_0^\dagger \to \mathrm{Spec}(k)^\dagger$ *is log smooth; in particular,* $f_0^\dagger$ *yields a torically transverse log curve.*

Remark. — The hypothesis of rationality of C_0 and on the associated tropical curve allow us to deduce that no singular point of C_0 is mapped to a singular point of $\tilde{X}_0$ by f. The third hypothesis allows us to prove that at the points of C_0 mapping into $\partial\tilde{X}_0$, the curve C_0 is log smooth.

For time reasons, I will not give the full detailed proof, namely this lemma of toric geometry which will be nevertheless useful for the proof.

Lemma **(2.0.4.)** — *Let* X *be a toric variety and* $W \subsetneq X$ *be a proper closed subset with no irreducible component contained in* ∂X*. If* $\mathrm{codim}(W, X) > c$*, then there exists a toric blow-up* $\varphi : \tilde{X} \to X$ *such that the proper transform of* W *is disjoint from any toric stratum of dimension at most (possibly equal to)* c*.*

We also state (without proof) a stable reduction theorem that will be useful — and that also comes into the proof of the fact that the Deligne-Mumford stack $\overline{\mathscr{M}}_{g,n}(X, \beta)$ is proper.

Proposition **(2.0.5.)** (Stable reduction.) — *Let* S *be a smooth curve over* k *and* $s \in S$*.*
If we have a family of stable maps $f : (C_U, \sigma_1, \dots, \sigma_n) \to X$ *over* $U := S \setminus \{s\}$ *then, there exist an open neighborhood* V *of* s*, a finite morphism* $\pi : V' \to V$ *where* V' *is a smooth curve and* $s' \in V'$ *such that:*

1. *the morphism* π *is étale on* $U' := V' \setminus \{s'\}$*.*
2. *the pull-back family of stable maps* $f' := (C_U \times_U U', \sigma_1', \dots, \sigma_n') \to X$ *over* U' *extends to a family of stable maps over* V'*.*

The proof of the theorem will be cut down in two steps: the proof of (1) and the proof of (2) and (3).

2.1. Extending the family over zero.

In this step of the proof, we will show (1) of the theorem 2.0.3 i.e., after a finite base change on the base and a toric blow-up on the target, we can extend the family over the central fiber and for each irreducible component Z of $\tilde{X}_0$, the map $f|_{f^{-1}(Z)}$ is a torically transverse stable map.

We will first prove that we can reduce to the case where C^* is geometrically irreducible.

2.1.1. — Let us consider the normalization $\tilde{C^*}$ of C^*. Since the nodal points of C^* are closed points and the normalization is a finite morphism, the (finintely many) points in the preimage of the singular locus of C^* are defined over a common finite extension of K which is the fraction field of some $R[t]/(t^d - \varpi)$ — that we will still denote by R and not S for convenience — where ϖ is a uniformizing element of (the old) R.
Now, it is enough to prove that a toric blow-up $\tilde{X}$ exists for each marked curve $(D^*, x^*_{i_1}, \dots, x^*_{i_\ell}, y^*_{j_1}, \dots, y^*_{j_m})$ defined over K where $D^* \subseteq \tilde{C^*}$ is an irreducible component and where the x_i's map to the marked points of C^* and the y_i's map to singular points of C^*. Indeed, it is then enough to glue together the $(D, x_{i_1}, \dots, x_{i_\ell}, y_{j_1}, \dots, y_{j_m}) \to \tilde{X}$ we obtained along the y_i's the same way that C^* is obtained from $\tilde{C^*}$.
Now, if we construct a refinement $\tilde{\mathscr{P}}$ of $\mathscr{P}$ for every irreducible component of C^*, since C^* has finitely many irreducible components, we can find a common refinement of all the $\tilde{\mathscr{P}}$'s which works. We a priori get something with vertices in $M_{\mathbb{Q}}$ but we can making a finite base change $t \in \mathbb{A}^1_k \mapsto t^e \in \mathbb{A}^1_k$ for some $e \geq 1$ to clear denominators and get integral vertices.

From now on, we suppose that C^* is geometrically irreducible.

Let us denoted by W the closure of the image of $f^* : C^* \to X \setminus X_0$. Since C^* is a curve, W is a closed subset of dimension at most 2. In addition, it must also have dimension at least one since the composition $\pi \circ f^* : C^* \to X \setminus X_0 \to \mathbb{G}_{m,k}$ is dominant — since ψ is and by commutativity of the square (1). We now treat these two cases separately.

2.1.2. — **If W has dimension 1.**
In that case, f^* is constant on C^* since it is a family of curves over a one-dimensional base and also because C^* is geomtrically irreducible hence connected. We deduce from this that W is the closure of its generic point $f(C^*)$. Since f^* is torically transverse, W is not contained in ∂X, so we can apply the toric geometry lemma 2.0.4 to W — which has codimension two in X since X has dimension 3 —, we get a toric blow-up $\tilde{X}$ of X such that the proper transform of W is disjoint from any toric stratum of dimension 1 of X. Moreover, we can choose the toric blow-up such that $\tilde{X} \setminus \tilde{X}_0$ is isomorphic to $X \setminus X_0$ since W is disjoint from $\partial(X \setminus X_0)$ — otherwise, we would contradict the fact that f^* is torically transverse. Therefore, we can assume that $\tilde{\mathscr{P}}$ comes from a refinement of $\mathscr{P}$.
Now, since f^* is constant, it factors [2] through $\mathrm{Spec}(K)$

$$\begin{tikzcd}
C^* \arrow[r, "f^*"] \arrow[d] & W \\
\mathrm{Spec}(K) \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

and using the stable reduction theorem (proposition 2.0.5) with $S := \mathrm{Spec}(R)$ and $s = 0$, we obtain a family $(C, x_1, \dots, x_n) \to \mathrm{Spec}(R)$ of stable curves after (possibly) a finite base change on $\mathrm{Spec}(R)$. Since our degeneration π is assumed to be proper, its restriction $\pi|_W : W \to \mathbb{A}^1_k$ to the closed subscheme W of X is proper as well; therefore, the valuative criterion for properness yields a unique morphism $\mathrm{Spec}(R) \to W$ fitting in the following commutative square:

$$\begin{tikzcd}
\mathrm{Spec}(R) \arrow[r, dashed] & W \arrow[d, "\pi|_W"] \\
\mathrm{Spec}(K) \arrow[u, hook] \arrow[ur, "g"] \arrow[r, "\psi|_{\mathrm{Spec}(K)}"'] & \mathbb{A}^1_k
\end{tikzcd}$$

The map $f : C \to \tilde{X}$ is now just constructed as the composition

$$C \to \mathrm{Spec}(R) \to W \hookrightarrow \tilde{X}.$$

[2] On the diagrams, the dashed arrows do not mean that the map is rational but it is simply a way to highlight them.

Since W is disjoint from any one dimensional toric stratum of $\tilde{X}$, f is a stable torically transverse map over any irreducible component of $\tilde{X}_0$ intersecting W and the other irreducible do not contribute because they are disjoint from the schematic image of f.

We now pass to the case where W is of dimension 2. In that case, things become more difficult because W has codimension 1 in X so the toric geometry lemma only tells us that the proper transform of W will be disjoint from the zero-dimensional toric strata of X but for toric transversality, we also need that W is disjoint from all toric strata of X with codimension strictly greater than one i.e., of dimension zero (already controlled by the toric geometry lemma) and dimension one; in fact, in the next paragraph, we will construct our toric blow-up $\tilde{X}$ of X such that it is true.

2.1.3. — If W has dimension 2.

Since f^* is torically transverse, no irreducible component of W is mapped into ∂X, therefore, the toric geometry lemma 2.0.4 guarantees the existence of a toric blow-up $\tilde{X}$ of X such that the proper transform of W in $\tilde{X}$ is disjoint from any zero-dimensional toric stratum of X. As in the previous case, this procedure can be done preserving the complement of the central fiber and we can assume that it comes from a refinement $\tilde{\mathscr{P}}$ of $\mathscr{P}$.

Let τ be an edge of $\tilde{\mathscr{P}}$ and let us denote by D_τ the corresponding one-dimensional stratum of $\tilde{X}$ and by $X_\tau := X_{C(\tau)}$ the corresponding affine subset of $\tilde{X}$. Then, we have

$$X_\tau \cong \mathbb{G}_{\mathrm{m},k} \times V_e$$

where $V_e := \operatorname{Spec}(k[x,y,t]/(xy - t^e))$ where e is the affine length of τ. Now, let us consider

$$C_\tau^* := f^{-1}(X_\tau) \subseteq C^*$$

and consider the composition

$$h^* : C_\tau^* \xrightarrow{f^*} X_\tau \xrightarrow{\mathrm{pr}_2} V_e \ .$$

Now, since h^* is locally of finite type, there exists an open subset $U \subseteq C_\tau^*$ — possibly empty — on which h^* is étale; moreover, if h^* is dominant then, it is non-empty [3]. Let $Z_\tau \subseteq V_e$ be the smallest closed subscheme containing the image of $C_\tau^* \setminus U$ and the images of the marked points of C_τ^*. Since f^* is torically transverse, h^* is and the image of C_τ^* is therefore disjoint from ∂V_e (which is the divisor with equation $t = 0$ in V_e); it implies that no irreducible component of Z_τ is contained in ∂V_e. Hence, by the toric geometry lemma 2.0.4, there exists a toric blow-up $\tilde{V}_e$ of V_e such that the proper transform of Z_τ is disjoint from the zero-dimensional stratum of V_e. In polyhedral terms, this toric blow-up of V_e is given by a subdivision of the edge τ, so we can choose a refinement $\tilde{\mathscr{P}}$ of $\mathscr{P}$ inducing this subdivision on the cell τ.

We do this operation for every edge τ of $\tilde{\mathscr{P}}$ whose corresponding toric stratum intersect W — this terminates because $\mathscr{P}$ has only finitely many cells. This gives us our sought-for toric blow-up.

Now, we apply the stable reduction theorem 2.0.5 to $f^* : (C^*, x_1^*, \dots, x_\ell^*) \to \tilde{X}$ over $\operatorname{Spec}(K)$ and we get a stable map $f : (C, x_1, \dots, x_\ell) \to \tilde{X}$ over $\operatorname{Spec}(R)$ after possibly a finite base change. Note that the image of f is contained in the proper transform of W inside $\tilde{X}$ which, by construction, avoids zero-dimensional toric strata of $\tilde{X}$. In particular, no irreducible component of $f(C_0)$ can be contained in a one-dimensional toric stratum of $\tilde{X}$.

To prove that f is torically transverse, we now have to prove that what we have added (the central fiber) does not destroy the toric transversality of f^*: we need to make us sure that f does not contract an irreducible component of C_0 to a point in a one-dimensional stratum of $\tilde{X}$; this is what we prove in the last paragraph of this section.

[3] Probably since in that case the target is integral.

2.1.4. — Recall that we still suppose that W of f^* is two-dimensional. We proceed by contradiction: let us assume that there exists an irreducible component of C_0 contracted to a point in a one-dimensional toric stratum of $\tilde{X}$ defined by an edge τ of $\tilde{\mathscr{P}}$. As before, we can consider X_τ, $C_\tau := f^{-1}(X_\tau) \subseteq C$ and we have again $X_\tau \cong \mathbb{G}_{\mathrm{m},k} \times V_e$ for some $e \geq 1$. We can also construct, as before, Z_τ which is disjoint from the singular point of V_e.

If h^* were not dominant, we would be done because Z_τ would contain the image of C_τ^*, hence the image of $\tilde{W} \cap X_\tau$ where $\tilde{W}$ is the proper transform of W in $\tilde{X}$. This would in particular imply that $\tilde{W}$ would be disjoint from the one-dimensional stratum of $\tilde{X}$.

We can suppose that $h : C_\tau^* \to V_e$ is dominant.

Let now $\tilde{Z}_\tau \subseteq X_\tau$ be the preimage of Z_τ under the projection $X_\tau \to V_e$ and put $Z'_\tau := f^{-1}(\tilde{Z}_\tau) \subseteq C_\tau^*$. The morphism $f : C_\tau \setminus Z'_\tau \to \tilde{X}$ factors as

$$\begin{tikzcd}
C_\tau \arrow[rr, "f|_{C_\tau}"] & & X_\tau \\
C_\tau \setminus Z'_\tau \arrow[u, hook] \arrow[rr, "f|_{C_\tau \setminus Z'_\tau}"'] \arrow[dr, "f'"'] & & X_\tau \setminus \tilde{Z}_\tau \arrow[u, hook] \\
& \mathrm{Spec}(R) \times_{\mathbb{A}^1_k} X_\tau \setminus \tilde{Z}_\tau \arrow[ur] &
\end{tikzcd}$$

where f' is proper and where the morphisms to $\mathbb{A}^1_k$ are respectively ψ and the restriction of π. Since f' is proper, we know it is finite over the complement of a finite subset $T \subseteq \mathrm{Spec}(R) \times_{\mathbb{A}^1_k} X_\tau \setminus \tilde{Z}_\tau$ on which the fibers of f are not finite. Let us now consider the Stein factorization of f'

$$\begin{tikzcd}
C_\tau \setminus Z'_\tau \arrow[rr, "f'"] \arrow[dr, "f''"'] & & \mathrm{Spec}(R) \times_{\mathbb{A}^1_k} X_\tau \setminus \tilde{Z}_\tau \\
& Y_\tau \arrow[ur, "g"'] &
\end{tikzcd}$$

which means that f'' is proper with connected fibers and g is finite. Since f' is finite over the complement of T, the unicity in Stein factorization implies that f'' is an isomorphism away from $f'^{-1}(T)$. The map $Y_\tau \to \tilde{X}$ therefore glues to $f|_{C \setminus f^{-1}(T)}$ to give a map $g' : C' \to \tilde{X}$. This map is marked as well since we have the compositions $x_i : \mathrm{Spec}(R) \to C \to C'$ and we glue outside Z'_τ which contains the marked points. We now get our contradiction:

the map $g' : (C', x_1, \ldots, x_\ell) \to \tilde{X}$ is stable.

Indeed, it contradicts the fact that f has finite automorphism group since we can do anything on $f'^{-1}(T)$ (infinite fibers of f'); therefore, T must be empty, which means that f' must be finite and in that case, f cannot contract an irreducible component of C_0 to a point because the f' has finite fibers which implies that f has finite fibers.

Now, we would have to prove that $g' : (C', x_1, \ldots, x_\ell) \to \tilde{X}$ is a stable map, see the second to last paragraph of [Gro11, p. 163].

Before we move to the second part of the proof, we conclude the proof of the first point of the theorem 2.0.3 by proving that the pullback of the coordinate of $\mathbb{A}^1_k$ to $\mathrm{Spec}(R)$ is a uniformizing element in R.

2.1.5. — Let us consider the map $\psi : \mathrm{Spec}(R) \to \mathbb{A}^1_k$ in the diagram (2) we have just constructed. Let us denote by z the coordinate on $\mathbb{A}^1_k$ and suppose that $\psi^*(z) \in \mathfrak{m}^d \setminus \mathfrak{m}^{d-1}$ for $d \geq 1$; in particular, $\psi^*(z) = \varpi^d$ where ϖ is a uniformizing element in R. If $d = 1$, we are done. If $d > 1$, we make a degree d base change $t \in \mathbb{A}^1_k \mapsto t^d \in \mathbb{A}^1_k$ and we replace R by

$$\mathrm{Spec}(R) \times_{\mathbb{A}^1_k} \mathbb{A}^1_k = \mathrm{Spec}\Big(R[t]/(t^d - \varpi^d)\Big)$$

which has d irreducible components, each of them being isomorphic to $\mathrm{Spec}(R)$; we pick one of them and we have $\psi^*(z) = \varpi$.

2.2. The restriction to the central fiber is a torically transverse pre-log curve.

Now that we have filled our diagram (1) to a diagram (2), we would like to prove that this procedure was well done and that the central fiber we obtain is logarithmically nice — in that context, it means that the restriction of our stable map f to the central fiber is a *torically transverse pre-log curve* [4].

2.2.1. — Let $x \in C_0$ be a closed point mapping to a singular point of $\tilde{X}_0$. The (local) ring homomorphism $f^\sharp$ induces a homomorphism of complete local $k[\![t]\!]$-algebras

$$\hat{f}^\sharp_x : \hat{\mathscr{O}}_{\tilde{X}, f(x)} \longrightarrow \hat{\mathscr{O}}_{C,x}$$

Now, since f is stable, x is either a smooth point of C_0 or either a double point of C_0. In fact, because of [Gro11, Proposition 4.9.], the first case cannot happen. Indeed if C is smooth over R at x, the logarithmic structure on C induced by $f^{-1}(\partial\tilde{X})$ is log smooth at x. Restricting this logarithmic structure to C_0 yields a curve $C_0^\dagger \to \mathrm{Spec}(k)^\dagger$ which is log smooth at x and a morphism of logarithmic schemes $C_0^\dagger \to \tilde{X}_0^\dagger$ over $\mathrm{Spec}(k)^\dagger$. Now, since $f(x)$ lies in the singular locus of $\tilde{X}_0$, [Gro11, Proposition 4.9.] implies that x must be a double point of C_0.

Therefore, x is a double point of C_0, thus

$$\hat{\mathscr{O}}_{C_0,x} = k[\![x, y]\!]/(xy).$$

This implies that:

$$\hat{\mathscr{O}}_{C,x} = k[\![x, y, t]\!]/(xy - ft^e)$$

for some $e \geq 1$ and some $f \in k[\![x, y, t]\!]$. In fact, we have

$$k[\![x, y, t]\!]/(xy - ft^e) \cong k[\![x, y, t]\!]/(xy - \lambda t^e)$$

as $k[\![t]\!]$-algebras, where $\lambda \in \{0, 1\}$. We now distinguish two cases:

- **If λ is zero**, then locally around x for the étale topology, C is reducible with two smooth components; we can then restrict to one of them and argue as before using [Gro11, Proposition 4.9.] to get a contradiction.

- **If λ is one**, then endowing C with the logarithmic structure defined by $f^{-1}(\partial\tilde{X}) \subseteq C$, we have that $C^\dagger \to \mathrm{Spec}(R)^\dagger$ is log smooth around x, which exactly means that f_0 (or rather $f_0^\dagger$) is a torically transverse pre-log curve.

This terminates the proof of the second step. We don't know yet that the curve f_0 is log smooth, we just have proved it around points of C_0 mapping via f to singular points of $\tilde{X}_0$.

[4] The term *pre-log* just means that the curve is not necessarily log smooth.

2.3. Under some hypothesis, it is even log smooth.

We now want to prove that under some conditions, the torically transverse pre-log curve is in fact a *log* curve, which just means that it is in addition log smooth. In the previous subsection, we have already considered points mapping via f to singular points of $\tilde{X}_0$, we have two other types of points: the points mapping to $\partial\tilde{X}_0$ and the others. Note that these three cases match with Kato's theorem [Gro11, Example 3.26.] about structure of log smooth curves: smooth, simple normal crossing or image of a section.

We treat the two cases in order.

2.3.1. — If $x \in C_0$ is a smooth point mapping to $\partial\tilde{X}_0 \backslash \mathrm{Sing}(\tilde{X}_0)$, then the hypothesis ensuring that $f^{-1}\left(\overline{\partial(\tilde{X} \setminus \tilde{X}_0)}\right)$ is a disjoint union of sections of $C \to \mathrm{Spec}(R)$ implies that $C_0^\dagger \to \mathrm{Spec}(k)^\dagger$ is log smooth at x since it corresponds to the second case in Kato's structure theorem.

2.3.2. — If $x \in C_0$ is a smooth point mapping to a smooth point of $\tilde{X}_0$ such that $f(x) \notin \partial\tilde{X}_0$, then $C_0^\dagger \to \mathrm{Spec}(k)^\dagger$ is log smooth at x since outside from the divisor $f^{-1}(\partial\tilde{X}_0)$ defining the logarithmic structure on X, the morphism $C_0 \to \mathrm{Spec}(k)$ is strict; hence, since x is a smooth point of C_0, it is also log smooth.

We now rule out the case of the singular points (i.e. nodal) mapping into the smooth locus of X_0 thanks to the hypothesis on the induced tropical curve.

2.3.3. — Since the tropical curve $h : \Gamma \to M_{\mathbb{R}}$ associated to the pre-log curve f_0 is simple, we deduce that for any vertex v of $\mathscr{P}$ such that $h^{-1}(v) \neq \emptyset$, two cases are possible:

- **If $h^{-1}(v)$ is a single trivalent vertex of** Γ, then since C_0 is rational, $f^{-1}(D_v)$ is rational as well and is thus a single line.
- **If $h^{-1}(v)$ is a finite number of points in the interior of edges of** Γ, then $f^{-1}(D_v)$ is a disjoint union of bivalent lines.

In the two cases, C_0 does not have double points which are not mapped into the singular locus of X_0 by f.

Now, since C_0 is stable, this implies that all the possible cases have been treated; therefore, the morphism of log schemes $C_0^\dagger \to \mathrm{Spec}(k)^\dagger$ is log smooth.

3. From the logarithmic world to the classical world.

In this section, we go the other way around: we start with points $P_1, \ldots, P_s$ of the lattice M (where $s := |\Delta| - 1$), with a good lattice polyhedral decomposition $\mathscr{P}$ of $M_{\mathbb{R}}$ and general points $Q_1, \ldots, Q_s$ in $\mathbb{G}(\tilde{M})$. These data induces for all $1 \leq i \leq s$, a section $\sigma_i : \mathbb{A}^1_k \to X$ and a point $q_i \in X_0$ such that $q_i := \sigma_i(0)$.

We want to prove the following theorem:

Theorem **(3.0.1.)** — *Let $f_0 : (C_0^\dagger, x_1, \ldots, x_s) \to X_0^\dagger$ be a torically transverse log curve of genus zero with $f(x_i) = q_i$ for all $1 \leq i \leq s$ and with associated tropical curve $h : \Gamma \to M_{\mathbb{R}}$. If h is simple then, there exists a unique marked rational curve $(C_\infty, x_1^\infty, \ldots, x_s^\infty)$ over* $\mathrm{Spec}(k[\![t]\!])$ *sitting in a commutative square*

$$\begin{tikzcd}
(C_\infty, x_1^\infty, \ldots, x_s^\infty) \arrow[r, "f_\infty"'] \arrow[d] & X \arrow[d, "\pi"] \\
\mathrm{Spec}(k[\![t]\!]) \arrow[r, "\psi_\infty"'] & \mathbb{A}^1_k
\end{tikzcd} \tag{4}$$

and satisfying the following three conditions:

1. *The morphism ψ_∞ is induced by the inclusion $k[t] \hookrightarrow k[\![t]\!]$.*

2. *For all $i \in [\![1, s]\!]$, we have the equality $\sigma_i \circ \psi_\infty = f_\infty \circ x_i^\infty$.*

3. *If C_∞ is given the logarithmic structure induced by $f_\infty^{-1}(\partial X) \subseteq C_\infty$ and if $\mathrm{Spec}(k[\![t]\!])$ is given the logarithmic structure pulled back from the one on $\mathbb{A}_k^1$, then the induced morphism of logarithmic schemes*

$$(C_0^\dagger, x_1, \dots, x_s) \longrightarrow X_0^\dagger$$

over $\mathrm{Spec}(k)^\dagger$ coincide with f_0.

The proof of this theorem will be divided in two main parts: in the first one, we show that the logarithmic deformations of $f_0 : C_0^\dagger \to X^\dagger$ are unobstructed (this is easy since C_0 is a curve but one has to see that we can extend the marked points and that they indeed satisfy the second item of the theorem) and we can then lift it to any order; the second part if the proof starts with the formal logarithmic deformation of f_0 (which will be the candidate for C_∞) and shows that it is a marked rational curve and that it is the unique one satisfying the three conditions of the theorem.

For convenience (and since it was probably not introduced before), we recall a result of logarthmic deformation theory that is used in the proof — see [Gro11, Theorem 3.43.].

Proposition **(3.0.2.)** — *Let $f_0^\dagger : (C_0^\dagger, x_1, \dots, x_s) \to X_0^\dagger \hookrightarrow X^\dagger$ be as above. If the morphism*

$$(f_0)_* : \mathscr{T}_{C_0/k}(\log) \hookrightarrow f_0^* \mathscr{T}_{X/\mathbb{A}_k^1}(\log)$$

is injective and if $f_0 \circ x_i = \sigma_i \circ \iota$ for all $1 \le i \le s$ where $\iota : \mathrm{Spec}(k) \hookrightarrow \mathbb{A}_k^1$, then there is a natural map

$$\Xi : \mathrm{H}^0(C_0, \mathscr{N}_{f_0,(x_1,\dots,x_s)}) \longrightarrow \prod_{i=1}^{s} \mathscr{T}_{X/\mathbb{A}_k^1}(\log)_{\sigma_i(0)},$$

it lifts a section of $\mathscr{N}_{f_0,(x_1,\dots,x_s)}$ locally around x_i to a section of $f_0^* \mathscr{T}_{X/\mathbb{A}_k^1}(\log)$ which is then evaluated at x_i to get an element of $f_0^* \mathscr{T}_{X/\mathbb{A}_k^1}(\log)_{\sigma_i(0)}$. Then, given a lift f_{k-1} of f_0 to $(k-1)$-th order (compatible with the marking), there exists a lift of f_0 to the k-th order (compatible with the marking) if Ξ is surjective. Moreover, if a lift exists, the set of such lifts is a torsor under $\mathrm{Ker}(\Xi)$.

In fact, one can prove that the map Ξ in the proposition 3.0.2 is in fact an isomorphism — see [Gro11, pp. 165–168]. Note that the proposition implies that lifts of $f_0^\dagger$ exist but there are unique since Ξ is also injective!

Now, we finish the proof of the theorem 3.0.1.

3.0.3. — From the previous step, we get a commutative square

$$\begin{tikzcd}
(C_\infty, x_1^\infty, \dots, x_m^\infty) \arrow[r, "f_\infty"] \arrow[d] & X \arrow[d, "\pi"] \\
\mathrm{Spec}(k[\![t]\!]) \arrow[r, "\psi_\infty"'] & \mathbb{A}_k^1
\end{tikzcd}$$

where f_∞ is a stable map. The log smoothness of $C_\infty^\dagger \to \mathrm{Spec}(k[\![t]\!])^\dagger$ can be checked locally for the étale topology around double points and marked points of C_∞.

- **If $x \in C_\infty$ is a double point**, we have already observed that as $k[\![t]\!]$-algebras, we had an isomorphism:

$$\hat{\mathscr{O}}_{C_\infty,x} \cong k[\![x, y, t]\!]/(xy - \lambda t^e)$$

for some $e \geq 1$ and $\lambda \in \{0,1\}$. By Kato's classification of log smooth and integral families of curves [Gro11, Example 3.26.], we have that $\lambda = 1$ and e is determined by the requirement $\overline{\mathscr{M}}_{C_0,x} = S_e$. Therefore, the morphism of logarithmic schemes $C_\infty^\dagger \to \mathrm{Spec}(k[\![t]\!])^\dagger$ is log smooth at x.

- **If $x \in C_\infty$ is a marked point**, since $f_k^\dagger : C_k^\dagger \to \mathrm{Spec}(k[t]/(t^{k+1}))^\dagger$ is log smooth, we have that $f_k^{-1}(\overline{\partial(X \setminus X_0)})$ is a disjoint union of sections of $C \to \mathrm{Spec}(k[\![t]\!])$ hence it is also the case for $f_\infty^{-1}(\overline{\partial(X \setminus X_0)})$, we deduce that $f_\infty^\dagger$ is log smooth at x.

Now, if we restrict this logarihtmic structure to C_0, we get another torically transverse log curve $(C_0')^\dagger \to X_0^\dagger$ over $\mathrm{Spec}(k)^\dagger$. Since the logarithmic structure on C_0 is uniquely determined away from the nodes and since at the nodes, they are determined according to Kato's description, we deduce that these two logarihtmic structure in fact coincide. This proves the existence in the third item of the theorem 3.0.1.

Let us now finish the proof by proving the unicity.

3.0.4. — If we have $f_\infty : C_\infty \to X$ as in the theorem 3.0.1, we obtain a commutative square

$$\begin{tikzcd} C_\infty^\dagger \arrow[r, "f_\infty^\dagger"] \arrow[d] & X^\dagger \arrow[d, "\pi^\dagger"] \\ \mathrm{Spec}(k[\![t]\!])^\dagger \arrow[r, "\psi_\infty^\dagger"'] & (\mathbb{A}_k^1)^\dagger \end{tikzcd}$$

where the logarithmic structure on C_∞ is induced by $f_\infty^{-1}(\partial X)$ and the logarithmic structure on $\mathrm{Spec}(k[\![t]\!])$ is pulled-back from the divisorial one on $\mathbb{A}_k^1$. By assumption, the induced logarithmic structure on the central fiber C_0 coincide with the one we started with — associated with f_0. This implies that $C_\infty^\dagger$ is log smooth over $\mathrm{Spec}(k[\![t]\!])^\dagger$ and by unicity of the log smooth lifting in deformation theory at each order, this proves that $C_\infty^\dagger$ is uniquely determined.

Remark. — In fact, the proof of unicity above is not entirely correct: it just proves that the log curve $C_\infty^\dagger$ is unique but it does not say anything about the unicity of the marking. In fact, since the map Ξ of the proposition 3.0.2 is injective, it proves the unicity of the log *marked* curve.

4. The coda of the proof.

In this section, we want to prove that

$$N_{\Delta,\Sigma}^{0,\mathrm{trop}} = N_{\Delta,\Sigma}^{0,\mathrm{hol}}.$$

We have fixed points $P_1, \dots, P_s \in M_\mathbb{Q}$ such that all tropical rational curves passing through these points are simple; in fact, we have rescaled the lattice M such that $P_i \in M$ for all $i \in [\![1,s]\!]$. After having chosen a good polyhedral decomposition $\mathscr{P}$ of $M_\mathbb{R}$, we can rescale the lattice M again in such a way that for every tropical rational curve $h : (\Gamma, x_1, \dots, x_s) \to M_\mathbb{R}$ with $h(x_i) = P_i$ for all $i \in [\![1,s]\!]$, the image of each edge of $\bar{\Gamma}$ has affine length divisible by its weight. Now, [Gro11, Theorem 4.14.] tells us that there exist $\mathrm{Mult}(h)$ torically transverse marked log curves passing though the P_i's whose associated tropical curve is h.

After choosing, for all $i \in [\![1,s]\!]$, a section $\sigma_i : \mathbb{A}_k^1 \to X$ to π and considering $q_i := \sigma_i(0) \in X_0$, [Gro11, Theorem 4.14.] implies that we get $N_{\Delta,\Sigma}^{0,\mathrm{trop}}$ torically transverse marked log rational curves

$$f : (C^\dagger, x_1, \dots, x_s) \longrightarrow X_0^\dagger$$

with $f(x_i) = q_i$ for all $1 \leq i \leq s$. The set of such curves will be denoted by

$$\mathscr{M}^{0,\log}_{\Delta,\Sigma}(\sigma_1, \dots, \sigma_s).$$

On the other hand, if κ stands for the field of Puiseux series over k (i.e., the algebraic closure of $k((t))$), the inclusion $k[t] \hookrightarrow k((t)) \hookrightarrow \kappa$ induces a morphism of k-schemes $\mathrm{Spec}(\kappa) \to \mathbb{A}^1_k$ and we have:

$$X \times_{\mathbb{A}^1_k} \mathrm{Spec}(\kappa) \cong \left(X \times_{\mathbb{A}^1_k} \mathrm{Spec}(k) \right) \times_k \mathrm{Spec}(\kappa) \cong X_\Sigma \times_k \mathrm{Spec}(\kappa).$$

Moreover, each section σ_i defines a κ-valued point of $X_\Sigma \times_k \mathrm{Spec}(\kappa)$ that we still denote by σ_i.

We now want to show the following proposition:

Proposition **(4.0.1.)** — *There is a bijection between*

- *the set* $\mathscr{M}^{0,\log}_{\Delta,\Sigma}(\sigma_1, \dots, \sigma_s)$.
- *the set* $\mathscr{M}^{0,\mathrm{hol}}_{\Delta,\Sigma}(\sigma_1, \dots, \sigma_s)$ *of torically transverse rational curves*

  $$f : (C, x_1, \dots, x_s) \to X_\Sigma \times_k \kappa$$

  over κ *with* $f(x_i) = \sigma_i$ *for all* $1 \leq i \leq s$.

Remark. — Since the first set has cardinality $N^{0,\mathrm{trop}}_{\Delta,\Sigma}$, it will prove Mikhalkin's curve counting formula because the statement does not depend on a specific algebraically closed field of characteristic zero — since $N^{0,\mathrm{hol}}_{\Delta,\Sigma}$ does not.

Proof. — Let us start with a torically transverse log curve $f : (C^\dagger_0, x^0_1, \dots, x^0_s) \to X$ in $\mathscr{M}^{0,\log}_{\Delta,\Sigma}(\sigma_1, \dots, \sigma_s)$. By the theorem 3.0.1, this gives us a (formal) curve

$$f_\infty : (C_\infty, x^\infty_1, \dots, x^\infty_s) \to X$$

over $k[\![t]\!]$. The inclusion $k[\![t]\!] \subseteq \kappa$ induces a scheme morphism $\mathrm{Spec}(\kappa) \to \mathrm{Spec}(k[\![t]\!])$ and we thus get a curve

$$C := C_\infty \times_{k[\![t]\!]} \mathrm{Spec}(\kappa)$$

over κ equipped with a morphism of k-schemes

$$f : C \to X \times_{\mathbb{A}^1_k} \mathrm{Spec}(\kappa) \cong X_\Sigma \times_k \mathrm{Spec}(\kappa)$$

which maps the point x^∞_i to σ_i for all $1 \leq i \leq s$. In other words, $f \in \mathscr{M}^{0,\mathrm{hol}}_{\Delta,\Sigma}(\sigma_1, \dots, \sigma_s)$.

Conversely, if $f \in \mathscr{M}^{0,\mathrm{hol}}_{\Delta,\Sigma}(\sigma_1, \dots, \sigma_s)$, then the curve C is defined over the field $k((\sqrt[d]{t}))$ for some $d \geq 1$ since C is of finite type over κ and κ is the union of these fields. Now, we get a commutative square similar to (1) with $R := k[\![\sqrt[d]{t}]\!]$ and $C^* := C$.

Now, the theorem 2.0.3 (classical to log) implies that up to replacing K by $k((\sqrt[de]{t}))$ for some $e \geq 1$, making a finite base change $\mathbb{A}^1_k \to \mathbb{A}^1_k$ and replacing X by a toric blow-up corresponding to a subdivision of $\mathscr{P}$, we get a diagram similar to (2). Restricting the stable map f to the central fiber yields a torically transverse pre-log curve

$$f_0 : (C^\dagger_0, x^0_1, \dots, x^0_s) \to \tilde{X}^\dagger_0.$$

The tropical curve $h : (\Gamma, x_1, \dots, x_s) \to M_\mathbb{R}$ associated to f_0 passes through the P_i's and since the latter were chosen generically, h must be simple and f_0 is therefore log smooth by the third point in the theorem 2.0.3. Now, this curve belongs to the $N^{0,\mathrm{trop}}_{\Delta,\Sigma}$ such tropical curves; in particular, this log curve gives a unique family which is already defined over $k[\![t]\!]$ and this implies that $de = 1$ and no finite base change is needed. In particular, all the torically transverse log curves defined over κ are in fact defined over $k((t))$. Therefore, we have $h \in \mathscr{M}^{0,\log}_{\Delta,\Sigma}(\sigma_1, \dots, \sigma_s)$. These two assignments are inverse one from another. □

She looked at the steps; they were empty; she lookoed at her canvas; it was blurred. With a sudden intensity, as if she saw it clear for a second, she drew a line there, in the centre. It was done; it was finished. Yes, she thought, laying down her brush in extreme fatigue, I have had my vision.

V. Woolf, *To the lighthouse.*

References

[Gro11] M. Gross. Tropical geometry and mirror symmetry. Number 114 in CBMS Regional Conference Series in Mathematics. AMS, 2011.