Counting tropically-induced log curves

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In the previous talk we gave a method to construct a marked parameterized tropical curve from a marked torically transverse log curve. The aim of this talk is to calculate exactly how many torically transverse log curves induce a particular parameterized tropical curve. We will show that this number equals the multiplicity of the marked parameterized tropical curve h, in the case that h is simple, that the number of marked points equals $|\deg h|-1$, and under an extra condition on the weights of h that we will describe shortly.

We are given as in the last talk the data $P_1, \dots, P_s \in M$, with $s = |\Delta| - 1$, $\mathscr{P}$ a good polyhedral decomposition of $M_{\mathbb{R}}$, and general points $Q_1, \dots, Q_s \in \mathbb{G}(\widetilde{M})$, giving points $q_i = \sigma_i(0) \in X_0$. Let

$$h : (\Gamma, x_1, \dots, x_s) \to M_{\mathbb{R}}$$

be a marked parameterized tropical curve with $h(x_i) = P_i$, with h simple of genus 0 and degree Δ.

To count the number of marked torically transverse log curves whose associated tropical curve is h, we will first count the number of morphisms of schemes that can underly such a log curve. More specifically, we define:

Definition 0.1. A *torically transverse pre-log curve* in X_0 is a stable map $f : C \to X_0$ with C a curve, such that for every vertex $v \in \mathscr{P}$, $f^{-1}(D_v) \to D_v$ is torically transverse, and such that, if $x \in C$ such that $f(x) \in \mathrm{Sing}(X_0)$,

1. x is a double point of C, contained in two distinct irreducible components C_1 and C_2 of C, and $f(C_i) \subseteq D_{v_i}$ for $i = 1, 2$, for $v_1, v_2 \in \mathscr{P}$ distinct vertices connected by some edge $\omega \in \mathscr{P}$.

2. The multiplicities of the intersection of C_i at x with $D_\omega \subseteq D_{v_i}$ agree, for $i = 1, 2$.

Note. Identically to in the last talk, we may construct a marked parameterized tropical curve from a marked torically transverse pre-log curve. For $f^\dagger$ a torically transverse log curve (f a torically transverse pre-log curve, respectively), we will write $h_{f^\dagger}$ (h_f, repsectively) for the associated marked parameterized tropical curve.

Note. See that, by Proposition 4.9 of [2], the underlying morphism of schemes of a torically transverse log curve is a torically transverse pre-log curve.

Definition 0.2.

$$N_h^{\log} = \left| \left\{ f^\dagger : C^\dagger \to X_0^\dagger \text{ a t.t. log curve in } X_0^\dagger \text{ with } h_{f^\dagger} = h \text{ and } f(x_i) = q_i \right\} / \sim \right|,$$
$$N_h^{\text{pre-log}} = \left| \{ f : C \to X_0 \text{ a t.t. pre-log curve in } X_0 \text{ with } h_f = h \text{ and } f(x_i) = q_i \} / \sim \right|,$$
$$N_h^{\log/f} = \left| \left\{ f^\dagger : C^\dagger \to X_0^\dagger \text{ a t.t. log curve in } X_0^\dagger, \text{ strict over } C^o, \text{ with scheme morphism } f \right\} / \sim \right|,$$

for f a marked torically transverse pre-log curve with $h_f = h$, and $C^o = C \backslash f^{-1}(\mathrm{Sing}(X_0) \cup \partial X_0)$. The strictness condition will be discussed later.

For these numbers to be non-zero, we know that we must have that the affine length of $h(E)$ is divisible by the weight $w(E)$ for every edge E in $\widetilde{\Gamma}$, where $\widetilde{\Gamma}$ is the subdivision of Γ for which for a point $y \in \widetilde{\Gamma}$, $h(y)$ is a vertex of $\mathscr{P}$ if and only if y is a vertex of $\widetilde{\Gamma}$ or y is contained within a marked edge (by Proposition 4.9.(4) of [2]). Henceforth we will assume that h satisfies this property.

The aim of this talk is to show the following theorem.

Theorem 0.3. *Under the above assumptions for h, $N_h^{\log} = \mathrm{Mult}(h)$.*

The proof will proceed by calculating both $N_h^{\text{pre-log}}$ and $N_h^{\log/f}$ (the latter of which will actually be independent of f), and showing that the product of these two numbers equals $\mathrm{Mult}(h)$.

We begin with a study of irreducible components of torically transverse curves.

1 Lines on toric surfaces

Definition 1.1. Let Y be a projective toric surface. A *line* on Y is a non-constant, torically transverse map $\varphi : \mathbb{P}^1 \to Y$ such that $|\varphi^{-1}(\partial Y)| \leq 3$ and $|\varphi^{-1}(D)| \leq 1$ for each toric divisor D of Y.

Let $u_1, \ldots, u_p$ be the primitive generators of the rays in the fan defining Y corresponding to the toric divisors D_i such that $\varphi^{-1}(D_i) \neq \emptyset$, and let w_i be the order of tangency of φ with D_i. One may check, similarly to in the last talk, that the balancing condition holds for $(\boldsymbol{u}, \boldsymbol{w})$. $(\boldsymbol{u}, \boldsymbol{w})$ will be called the *type* of the line φ. $\mathcal{L}_{(\boldsymbol{u},\boldsymbol{w})}$ will be the set of all lines of type $(\boldsymbol{u}, \boldsymbol{w})$ in Y up to isomorphism.

Example 1.2. Let $f : C \to X_0$ be a torically transverse pre-log curve with $h_f = h$ and $f(x_i) = q_i$. For $V \in \widetilde{\Gamma}^{[0]}$, define C_V to be the union of all irreducible components of C in the equivalence class corresponding to V (recall our construction of h_f). Then $\varphi_V := f|_{C_V} \to D_{h(V)}$ is non-constant and torically transverse, since f is. Since h is simple, there are at most 3 points in C_V mapping into $\mathrm{Sing}(X_0)$, since such points are in correspondence with bounded edges of adjacent to V by Definition 0.1.(1), and each such point maps into a different components of $\mathrm{Sing}(X_0)$. Moreover, by toric transversality any non-contracted component of C_V must have at least two intersections with $\mathrm{Sing}(X_0)$ under

f. Therefore, by the stability of f, C_V is irreducible so φ is a line under the above definition. Its type is given by the primitive tangent vectors adjacent to V and the associated weights.

We now classify all lines on Y, by noticing that either $p = 2$ or $p = 3$ by the balancing condition.

Lemma 1.3. *Assume that $p = 2$. Let E be the sublattice of M generated by u_1, and let $g : Y \to \mathbb{P}^1$ be the map induced by the morphism of lattices $M \to M/E$. Then a line φ of type $(\boldsymbol{u}, \boldsymbol{w})$ has image $g^{-1}(p)$ for some $p \in \mathbb{P}^1 \backslash \{0, \infty\}$, and φ is a w_1-fold branched cover, totally branched precisely at the two points $g^{-1}(p) \cap \partial Y$.*

Proof. The projection $M \to M/E$ induces a morphism of fans from Σ_Y to the fan in $(M/E) \otimes_{\mathbb{Z}} \mathbb{R}$ defining $\mathbb{P}^1$. The toric boundary of Y may then be written $\partial Y = D_1 \cup D_2 \cup g^{-1}(0) \cup g^{-1}(\infty)$. By toric transversality, $\varphi(\mathbb{P}^1)$ is disjoint from $g^{-1}(0)$ and $g^{-1}(\infty)$, so $g \circ \varphi$ is a constant map, $\neq 0, \infty$. Finally, $\varphi^{-1}(\partial Y)$ consists of precisely two points, by the above expression for ∂Y and by our hypothesis on φ, so φ is totally branched at ∂Y.

□

Corollary 1.4. $\mathcal{L}_{(\boldsymbol{u},\boldsymbol{w})} \simeq \mathbb{G}(M/E)$.

The lines for $p = 3$ can also be classified:

Corollary 1.5 ([2], Lemma 4.19, Corollary 4.20)**.** *Assume that $p = 3$. Assume for simplicity that the fan defining Y has only three rays. Then*

$$\mathcal{L}_{(\boldsymbol{u},\boldsymbol{w})} \simeq \left\{ \text{t.t. linear embeddings } \mathbb{P}^1 \to \mathbb{P}^2 \right\} \overset{\text{(non-canon.)}}{\simeq} \mathbb{G}(M).$$

2 Pre-log and log curve counting

We associate to $h : \Gamma \to M_{\mathbb{R}}$ another tropical curve $h : \widehat{\Gamma} \to M_{\mathbb{R}}$, by removing all marked edges and resulting bivalent vertices from Γ. Denote by $E_1, \ldots, E_s$ the distinguished edges of Γ, so that the endpoint of E_{x_i} lies in E_i. Choose orientations of the edges of $\widehat{\Gamma}$, and denote for an edge E of $\widehat{\Gamma}$ $\partial^- E$ and $\partial^+ E$ for the two adjacent vertices accordingly (or just $\partial^- E$ for E non-compact). Finally, choose $u_{(\partial^- E, E)} \in M$ for the primitive tangent vector to $h(E)$ pointing from $h(\partial^- E)$ to $h(\partial^+ E)$.

Proposition 2.1. *The map*

$$\Phi : \mathrm{Hom}(\widehat{\Gamma}^{[0]}, M) \to \left(\prod_{E \in \widehat{\Gamma}^{[1]} \backslash \widehat{\Gamma}^{[1]}_\infty} M/\mathbb{Z}u_{(\partial^- E, E)} \right) \times \left(\prod_{i=1}^{s} M/\mathbb{Z}u_{(\partial^- E_i, E_i)} \right)$$

$$H \mapsto \left(\left(H(\partial^+ E) - H(\partial^- E) \right)_E, \left(H(\partial^- E_i) \right)_i \right)$$

is an inclusion of lattices of index $N_h^{\text{pre-log}}$.

Proof. If $H \in \ker(\Phi_{\mathbb{R}})$, then $H(\partial^+ E) - H(\partial^- E) \in \mathbb{Z}u_{(\partial^- E, E)}$ for all compact edges E and $H(\partial^- E) \in \mathbb{Z}u_{(\partial^- E_i, E_i)}$ for all i. This implies that $h + H$ is a tropical curve, such that P_i is contained within the span of the image of E_i. Assuming that H is sufficiently close to the origin, $h + H$ is a tropical curve passing through P_i. But h is rigid, since as shown in [2] Lemma 1.20, the number of genus zero tropical curves of degree Δ through P_i is finite. Therefore $H = 0$. This shows, by rescaling, that the kernel of Φ is trivial, proving the first statement of the proposition.

To calculate the index, we introduce another morphism of lattices which will turn out to have the same index as Φ. For $V \in \widetilde{\Gamma}^{[0]} \backslash \widehat{\Gamma}^{[0]}$, let $E(V)$ be the edge of $\widehat{\Gamma}$ containing V in its interior. For E an edge in $\widetilde{\Gamma}$, write $\widehat{E}$ for the edge of $\widehat{\Gamma}$ containing E. Define

$$\Phi'' : \prod_{V \in \widehat{\Gamma}^{[0]}} M \times \prod_{V \in \widetilde{\Gamma}^{[0]} \backslash \widehat{\Gamma}^{[0]}} M/\mathbb{Z}u_{(\partial^- E(V), E(V))} \to \prod_{E \in \widetilde{\Gamma}^{[1]} \backslash \widetilde{\Gamma}^{[1]}_\infty} M/\mathbb{Z}u_{(\partial^- \widehat{E}, \widehat{E})} \times \prod_{i=1}^{s} M/\mathbb{Z}u_{(\partial^- E_i, E_i)},$$

mapping an element $(m_V)_V$ to $((m_{\partial^+ E} - m_{\partial^- E})_E, (m_{V_i})_i)$. Now, by Corollaries 1.4 and 1.5, the associated morphism of algebraic tori is isomorphic to the morphism

$$\Phi' : \prod_{V \in \widetilde{\Gamma}^{[0]}} \mathcal{L}_{(\boldsymbol{u}, \boldsymbol{w})} \to \prod_{E \in \widetilde{\Gamma}^{[1]} \backslash \widetilde{\Gamma}^{[1]}_\infty} \mathbb{G}(M/\mathbb{Z}u_{(\partial^- \widehat{E}, \widehat{E})}) \times \prod_{i=1}^{s} \mathbb{G}(M/\mathbb{Z}u_{(\partial^- E_i, E_i)}),$$

which acts on an element $(\varphi_V)_{V \in \widetilde{\Gamma}^{[0]}}$ in the domain as follows.

For $E \in \widetilde{\Gamma}^{[1]} \backslash \widetilde{\Gamma}^{[1]}_\infty$, write $p^\pm = \varphi_{\partial^\pm E}(C_{\partial^\pm E}) \cap D_\omega$, where $\omega \in \mathscr{P}$ is the edge joining $h(\partial^- E)$ and $h(\partial^+ E)$. By toric transversality, $p^\pm$ lie in the big torus of D_ω. Then take the component of the image of $(\varphi_V)_V$ under Ψ' to be p^+/p^-.

For $i \in \{1, \ldots, s\}$, let $V_i \in \widetilde{\Gamma}^{[0]}$ be the vertex adjacent to E_{x_i}. By Lemma 1.3, the image of φ_{V_i} is $g_i^{-1}(r_i)$ for some $r_i \in \mathbb{P}^1 \backslash \{0, \infty\}$, where $g_i : D_{P_i} \to \mathbb{P}^1$ induced by $M \to M/\mathbb{Z}u_{(\partial^- E_i, E_i)}$. We take the component of the image of $(\varphi_V)_V$ under Φ' to be $(g_i(q_i))/r_i$ (recalling that $q_i \in \mathrm{Int}(D_{P_i})$).

It is now clear that the index of the map Φ'' equals the cardinality of the set $\Phi^{-1}(1, \ldots, 1)$ (i.e. the degree of the morphism Φ'). We show firstly that this cardinality equals $N_h^{\log}$.

Indeed, if $f : C \to X_0$ is a marked torically transverse pre-log curve with $h_f = h$, then we obtain an element $(\varphi_V = f|_{C_V})_V \in \prod_{V \in \widetilde{\Gamma}^{[0]}} \mathcal{L}_{(\boldsymbol{u}, \boldsymbol{w})}$ as in Example 1.2. For any $E \in \widetilde{\Gamma}^{[1]} \backslash \widetilde{\Gamma}^{[1]}_\infty$, $\varphi_{\partial^- E}$ and $\varphi_{\partial^+ E}$ intersect D_ω in the same point, i.e. $p^- = p^+$. For any $i \in \{1, \ldots, s\}$, the line φ_{V_i} passes through q_i, so $(g_i(q_i))/r_i = 1$. Therefore we see that $\Phi'((\varphi_V)_V) = (1, \ldots, 1)$.

Conversely, if $(\varphi_V)_V \in \prod_{V \in \widetilde{\Gamma}^{[0]}} \mathcal{L}_{(\boldsymbol{u}, \boldsymbol{w})}$ is such that $\Phi'((\varphi_V)_V) = (1, \ldots, 1)$, then we may glue the φ_V to obtain a torically transverse pre-log curve $f : C \to X_0$, which we mark by choosing a point $x_i \in C_{V_i}$ mapping under φ_{V_i} to q_i, and observing that two choices of x_i give two isomorphic marked pre-log curves. It is clear that if $(\varphi_V)_V \neq (\varphi'_V)_V \in \prod_{V \in \widetilde{\Gamma}^{[0]}} \mathcal{L}_{(\boldsymbol{u}, \boldsymbol{w})}$, then the pre-log curves f and f' obtained respectively as above are non-isomorphic.

So it now suffices only to show that the index of Φ'' equals the index of Φ. We do this by considering the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(\widehat{\Gamma}^{[0]}, M) & \xrightarrow{\Phi} & \left(\prod_{E\in\widehat{\Gamma}^{[1]}\setminus\widehat{\Gamma}^{[1]}_\infty} M/\mathbb{Z}u_{(\partial^- E, E)}\right) \times \left(\prod_{i=1}^s M/\mathbb{Z}u_{(\partial^- E_i, E_i)}\right) \\
\downarrow \Psi_1 & & \downarrow \Psi_2 \\
\prod_{V\in\widehat{\Gamma}^{[0]}} M \times \prod_{V\in\widetilde{\Gamma}^{[0]}\setminus\widehat{\Gamma}^{[0]}} M/\mathbb{Z}u_{(\partial^- E(V), E(V))} & \xrightarrow{\Phi''} & \prod_{E\in\widetilde{\Gamma}^{[1]}\setminus\widetilde{\Gamma}^{[1]}_\infty} M/\mathbb{Z}u_{(\partial^-\widehat{E},\widehat{E})} \times \prod_{i=1}^s M/\mathbb{Z}u_{(\partial^- E_i, E_i)},
\end{array}$$

where

$$\Psi_1((m_V)_{V\in\widehat{\Gamma}^{[0]}}) = \left((m_V)_{V\in\widehat{\Gamma}^{[0]}}), (m_{\partial^- E(V)})_{V\in\widetilde{\Gamma}^{[0]}\setminus\widehat{\Gamma}^{[0]}}\right)$$

and Ψ_2 is the natural inclusion. One may check that Φ'' induces an isomorphism between the cokernels of Ψ_1 and Ψ_2, implying by the snake lemma that $\mathrm{coker}(\Psi) \simeq \mathrm{coker}(\Psi'')$.

□

Now we wish to count the number of torically transverse log curves with a fixed underlying torically transverse pre-log curve. To achieve this, we have to impose a strictness condition (see Definition 0.2), that ensures for instance that all log marked points of $f^\dagger$ are mapped into the toric boundary of X_0. We have the following formula for the number $N_h^{\log/f}$.

Proposition 2.2. *Let $f : C \to X_0$ be a torically transverse pre-log curve with $f(x_i) = q_i$ and $h_f = h$. Then*

$$N_h^{\log/f} = \prod_{E\in\widehat{\Gamma}^{[1]}\setminus\widehat{\Gamma}^{[1]}_\infty} w(E) \prod_{i=1}^s w(E_i).$$

Proof. ([2], Proposition 4.23.) I'd like to sketch it here when I have the time.

□

Proof of Theorem 0.3. The previous two propositions show that

$$N_h^{\log} = \mathrm{index}(\Phi) \prod_{E\in\widehat{\Gamma}^{[1]}\setminus\widehat{\Gamma}^{[1]}_\infty} w(E) \prod_{i=1}^s w(E_i).$$

To prove the theorem it thus remains to show that

$$\mathrm{index}(\Phi) \prod_{E\in\widehat{\Gamma}^{[1]}\setminus\widehat{\Gamma}^{[1]}_\infty} w(E) \prod_{i=1}^s w(E_i) = \mathrm{Mult}(h).$$

We proceed via induction on the number of vertices of $\widehat{\Gamma}$.

If the number of vertices is one, then $\widehat{\Gamma}$ has no compact edges, three non-compact edges, two of which are marked. Writing u_1 and u_2 for the primitive vectors of the marked edges, Φ is the map

$$\Phi : M \to M/\mathbb{Z}u_1 \times M/\mathbb{Z}u_2,$$

the index of which is $|u_1 \wedge u_2|$. So

$$\mathrm{index}(\Phi) w_1 w_2 = w_1 w_2 |u_1 \wedge u_2| = \mathrm{Mult}(h)$$

by definition.

In general, choose E an non-compact unmarked edge. Removing E and $\partial^- E$ gives two connected components $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$, with new non-compact edges $E^{(1)}$ and $E^{(2)}$ repsectively.

Case 1. *Both $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ have vertices.* We obtain new tropical curves h_1 and h_2. let $u_1, \ldots, u_{s-2}$ be primitive tangent vectors to the images under h of the bounded edges of $\widehat{\Gamma}$, $v_1, \ldots, v_s$ those for the marked edges, such that $u_1, \ldots, u_{l-2}$ are associated to $\widehat{\Gamma}_1$, u_{l-1} to $E^{(1)}$, u_l to $E^{(2)}$, $u_{l+1}, \ldots, u_{s-2}$ to $\widehat{\Gamma}_2$, and such that $v_1, \ldots, v_l$ are associated to $\widehat{\Gamma}_1$, $v_{l+1}, \ldots, v_s$ to $\widehat{\Gamma}_2$.

Let Φ_1 and Φ_2 be the maps defined in 2.1 for h_1 and h_2. Writing $B^{(1)}$ and $B^{(2)}$ for the targets of Φ_1 and Φ_2 for ease of notation, the map Φ becomes

$$\begin{aligned}\Phi : \mathrm{Hom}(\widehat{\Gamma}_1^{[0]}, M) \times \mathrm{Hom}(\widehat{\Gamma}_2^{[0]}, M) \times \mathrm{Hom}(\{V\}, M) &\to B^{(1)} \times B^{(2)} \times M/\mathbb{Z}u_{l-1} \times M/\mathbb{Z}u_l \\ (H_1, H_2, H') &\mapsto (\Phi_1(H_1), \Phi_2(H_2), H_1(V^{(1)}) - H'(V), H_2(V^{(2)}) - H'(V)),\end{aligned}$$

where $V^{(1)} \in \widehat{\Gamma}_1^{[0]}$ and $V^{(2)} \in \widehat{\Gamma}_2^{[0]}$ are vertices adjacent to $E^{(1)}$ and $E^{(2)}$.

Therefore we see that

$$\mathrm{index}(\Phi) = \mathrm{index}(\Phi_1)\mathrm{index}(\Phi_2)\mathrm{index}(M \to M/\mathbb{Z}u_{l-1} \times M/\mathbb{Z}u_l).$$

Applying the inductive hypothesis to h_1 and h_2 gives

$$\mathrm{index}(\Phi) \prod_{E \in \widehat{\Gamma}^{[1]} \setminus \widehat{\Gamma}^{[1]}_\infty} w(E) \prod_{i=1}^{s} w(E_i) = \mathrm{Mult}(h_1)\mathrm{Mult}(h_2) w(E^{(1)}) w(E^{(2)}) |u_{l-1} \wedge u_l| = \mathrm{Mult}(h).$$

Case 2. *$\widehat{\Gamma}_2$ consists of just an unbounded edge.* In this case, Φ takes the form

$$\begin{aligned}\Phi : \ \mathrm{Hom}(\widehat{\Gamma}_1^{[0]}, M) \times \mathrm{Hom}(\{V\}, M) &\to B^{(1)} \times B^{(2)} \times M/\mathbb{Z}u_{s-2} \times M/\mathbb{Z}v_s \\ (H_1, H') &\mapsto (\Phi_1(H_1), H_1(V^{(1)}) - H'(V), H'(V)),\end{aligned}$$

so

$$\mathrm{index}(\Phi) = \mathrm{index}(\Phi_1)\mathrm{index}(M \to M/\mathbb{Z}u_{s-2} \times M/\mathbb{Z}v_s).$$

Applying the inductive hypothesis to h_1 gives

$$\mathrm{index}(\Phi) \prod_{E \in \widehat{\Gamma}^{[1]} \setminus \widehat{\Gamma}^{[1]}_\infty} w(E) \prod_{i=1}^{s} w(E_i) = \mathrm{Mult}(h_1) w(E^{(1)}) w(E_s) |u_{s-2} \wedge v_s| = \mathrm{Mult}(h).$$

□

References

[1] William Fulton, *Introduction to Toric Varieties*

[2] Mark Gross, *Tropical Geometry and Mirror Symmetry*

[3] Robin Hartshorne, *Algebraic Geometry*