Mirror Symmetry for $\mathbb{P}^n$

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So far we have encountered two key examples of semi-infinite variations of Hodge structure, the A-model structure on $\mathbb{P}^n$ and the B-model structure on its "mirror" $((\mathbb{C}^*)^n, W_0)$. In this talk we will try to understand precisely what the statement "Mirror Symmetry for $\mathbb{P}^n$" means, in terms of the semi-infinite variations of Hodge structure that we have built up on both sides.

First I will give a quick summary of some of the data we have constructed on the A- and B-model sides, respectively.

A-model

- $\widetilde{\mathcal{M}}^A := (\mathbb{C}_{y_1}, \mathcal{O}_{\widetilde{\mathcal{M}}^A})$, where local sections are formal power series $\sum f_{i_0 i_2 \ldots i_n} y_0^{i_0} y_2^{i_2} \ldots y_n^{i_n}$, with $f_{i_0 i_2 \ldots i_n}$ holomorphic.
- $\mathbb{J} = S^{-1} : \mathcal{E}^A = H^*(\mathbb{P}^n, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\widetilde{\mathcal{M}}^A}\{\hbar\} \longrightarrow \mathcal{H}^A = H^*(\mathbb{P}^n, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\widetilde{\mathcal{M}}^A}\{\hbar, \hbar^{-1}\}$ inclusion.
- $s_0^A = \mathbb{J}(T_0) = J_{\mathbb{P}^n} \in \mathcal{E}^A$ the *Givental J-function*, a miniversal section for the A-model semi-infinite variation of Hodge structure (meaning that $\mathbb{J}$ is determined completely by $J_{\mathbb{P}^n}$) (this section of $\mathcal{H}^A$ is seen as a section of $\mathcal{E}^A$ by identifying $\mathcal{E}^A$ with its image under $\mathbb{J}$).

B-model

- $\widetilde{\mathcal{M}}^B := (\mathbb{C}_{t_1}, \mathcal{O}_{\widetilde{\mathcal{M}}^B})$, where local sections are formal power series $\sum f_{i_0 i_2 \ldots i_n} t_0^{i_0} t_2^{i_2} \ldots t_n^{i_n}$, with $f_{i_0 i_2 \ldots i_n}$ holomorphic.
- R the local system of $\mathbb{C}$-vector spaces on $\widetilde{\mathcal{M}}^B \times \mathbb{C}^*_\hbar$ whose fibre over $(t_1, \hbar)$ is $H_n(\pi^{-1}(t_1), \mathrm{Re}(W|_{\pi^{-1}(t_1)}/\hbar) << 0\,; \mathbb{C})$ (which always has dimension $n+1$) (W the universal unfolding of W_0). $\mathcal{R}$ the associated locally free sheaf on $\widetilde{\mathcal{M}}^B \times \mathbb{C}^*_\hbar$. $\mathcal{R}^\vee$ its dual.
- [Sections of $\mathcal{R}^\vee$ are given by one-forms $[f\Omega]$, where $\Omega = dx_1 \ldots dx_n / x_1 \ldots x_n$ and f is holomorphic with algebraic fibres, where the associated map $\mathcal{R} \to \mathcal{O}_{\widetilde{\mathcal{M}}^B \times \mathbb{C}^*_\hbar}$ is given by $\Xi \mapsto \int_\Xi e^{W/\hbar} f\Omega$.]

$\mathcal{R}^\vee$ the extension of $\mathcal{R}^\vee$ to $\widetilde{\mathcal{M}}^B \times \mathbb{C}_\hbar$, where we impose that f extends holomorphically over $\hbar = 0$.

- $\mathcal{E}^B$ the $\mathcal{O}_{\widetilde{\mathcal{M}}^B}\{\hbar\}$-module of sections of $\mathcal{R}^\vee$ "near $\hbar = 0$" (i.e., over U, sections $[f\Omega]$ with f holomorphic on some $U \times \{\hbar : |\hbar| < \epsilon\}$).
- $\Xi_0, \ldots, \Xi_n$ local basis of R satisfying formula (2.38) of [2]. We then write

$$[f\Omega] = \sum_{i=0}^{n} \alpha^i \int_{\Xi_i} f e^{W/\hbar}\Omega,$$

under the natural identification of an $(n{+}1)$-dimensional $\mathbb{C}$-vector space with $\mathbb{C}[\alpha]/(\alpha^{n+1})$.

- $\mathcal{H}^B$ is then the free $\mathcal{O}_{\widetilde{\mathcal{M}}^B}\{\hbar, \hbar^{-1}\}$-module generated by the (single-valued) sections $\hbar^{-(n+1)\alpha}\alpha^i$ of $\mathcal{E}^B \otimes_{\mathcal{O}_{\widetilde{\mathcal{M}}^B}\{\hbar\}} \mathcal{O}_{\widetilde{\mathcal{M}}^B}\{\hbar, \hbar^{-1}\}$, for $i = 0, \ldots, n$.
- A miniversal section s_0^B of $\mathcal{E}^B$ (satisfying $[\Omega] \equiv s_0^B \mod \mathcal{H}_-^B$) (defined on some open neighbourhood of $0 \in \widetilde{\mathcal{M}}^B$).

Looking at the lists above, we know precisely what s_0^A is. We'd like firstly to understand a little better what s_0^B should look like, in order to assist us in seeing what results from the "mirror map" between $\widetilde{\mathcal{M}}^A$ and $\widetilde{\mathcal{M}}^B$ mapping s_0^A to s_0^B.

Lemma 1.

$$s_0^B = \hbar^{-(n+1)\alpha} \sum_{i=0}^{n} \varphi_i(\underline{t}, \hbar^{-1})(\alpha\hbar)^i,$$

where

$$\varphi_i(\underline{t}, \hbar^{-1}) = \delta_{0,i} + \sum_{j=1}^{\infty} \varphi_{i,j}(\underline{t})\hbar^{-j},$$

for some $\varphi_{i,j}(\underline{t})$ local sections of $\widetilde{\mathcal{M}}^B$. Moreover, $\varphi_{i,1}(\underline{t})$ form a system of coordinates for $\widetilde{\mathcal{M}}^B$ in some open neighbourhood of $0 \in \widetilde{\mathcal{M}}^B$.

Proof. By the choice of basis $\Xi_0, \ldots, \Xi_n$,

$$[\Omega] = \sum_{i=0}^{n} \alpha^i \int_{\Xi_i} e^{W/\hbar}\Omega \overset{(2.38)}{=} (\xi(\hbar, \alpha) =)\, \hbar^{-(n+1)\alpha} \mod \mathcal{H}_-^B,$$

so that when we write

$$\hbar^{-(n+1)\alpha} \sum_{i=0}^{n} \varphi_i(\underline{t}, \hbar, \hbar^{-1})(\alpha\hbar)^i,$$

we obtain the expression for φ_i as required, since $[\Omega] \equiv s_0^B \mod \mathcal{H}_-^B$. The fact that the $\varphi_{i,1}(\underline{t})$ form a system of coordinates follows by the expression of the Barannikov period map; see below.

□

Now we may define a map of germs

$$m : (\widetilde{\mathcal{M}}^A, 0) \to (\widetilde{\mathcal{M}}^B, 0)\,;\, y_i \mapsto \varphi_{i,1},$$

which is a local isomorphism by the above lemma. This is called the *mirror map*. Using it, we can now give a very concise statement of mirror symmetry for $\mathbb{P}^n$. We're not going to prove it here.

Theorem 2 (Barannikov, [1]). *The mirror map m induces an isomorphism between the A-model semi-infinite variation of Hodge structure and the B-model semi-infinite variation of Hodge structure.*

Note. This means the vector bundles $\mathcal{E}^A$ and $\mathcal{E}^B$ are identified, as well as the connections and pairings and miniversal sections. Note moreover that the gradings and opposite subspaces are identified under this isomorphism of vector bundles.

Now we'll try to unpack some of what this statement means in terms of the data we've collected at this start of these notes. Some conclusions are:

Corollary 3. *Assuming Theorem 2, the following conditions hold:*

1. *Write the Givental J-function as*

$$J(y_0, \ldots, y_n, \hbar^{-1}) = \sum_{i=0}^{n} J_i(y_0, \ldots, y_n, \hbar^{-1}) T_i.$$

Then in the $\mathbb{C}$-vector space $\mathbb{C}[[y_0, \ldots, y_n, \hbar^{-1}]]$,

$$J_i = \varphi_i \quad \textit{for all } 0 \leq i \leq n.$$

2. *Under the mirror map $m : (\widetilde{\mathcal{M}}^A, 0) \to (\widetilde{\mathcal{M}}^B, 0)$, the vector fields E_A and E_B are identified.*

3. $$\int_{\mathbb{P}^n} T_i \cup T_j = (\hbar^{-(n+1)\alpha}(\hbar\alpha)^i, \hbar^{-(n+1)\alpha}(\hbar\alpha)^j)_{\mathcal{E}^B}|_{\hbar=\infty}.$$

Proof. The induced mirror map on the vector bundles $\mathcal{E}^\bullet$ is the composition of the so-called Barannikov period maps that are obtained from the miniversal sections $s_0^\bullet$ with m. Under these local isomorphisms, $(\widetilde{\mathcal{M}}^A, 0)$ and $H^*(\mathbb{P}^n, \mathbb{C})$ are idenitified, by mapping y_i to T_i, and $(\widetilde{\mathcal{M}}^B, 0)$ and $\mathbb{C}\langle \hbar^{-(n+1)\alpha}(\alpha\hbar)^i \rangle$ are identified, by mapping $\varphi_{i,1}$ to $\hbar^{-(n+1)\alpha}(\alpha\hbar)^i$ (since $\psi_B(\underline{t}) = \sum_{i=0}^n \varphi_{i,1}(\underline{t})\hbar^{-(n+1)\alpha}(\hbar\alpha)^i$, see [2], proof of Proposition 2.45). So the mirror map m induces a map

$$m : H^*(\mathbb{P}^n, \mathbb{C}) \to \mathbb{C}\langle \hbar^{-(n+1)\alpha}(\alpha\hbar)^i \rangle\,;\, T_i \mapsto \hbar^{-(n+1)\alpha}(\alpha\hbar)^i.$$

This induces maps

$$\mathcal{E}^A \to \mathcal{E}^B, \quad \mathcal{H}^A \to \mathcal{H}^B,$$

which are precisely the maps that the theorem above claims are isomorphisms. Now:

1. Under the mirror map, $T_i \mapsto \hbar^{-(n+1)\alpha}(\alpha\hbar)^i$ and $s_0^A \mapsto s_0^B$ by Theorem 2. Since $J = s_0^A$, the result follows immediately.

2. This also follows immediately.

3. This follows since the two pairings are equal by Theorem 2, and the left and right hand sides of the equation in 3. are the metrics associated to these pairings (that appear in the associated Frobenius manifold structures).

□

Note. Note also that a stronger converse statement is true, see Proposition 2.45 of [2].

References

[1] Serguei Barannikov, *Semi-infinite Hodge structures and mirror symmetry for projective spaces*

[2] Mark Gross, *Tropical geometry and mirror symmetry*

[3] Claude Sabbah, *Isomononodromic deformations and Frobenius manifolds*

[4] Kyoji Saito, *Period mapping associated to a primitive form*

[5] Christian Sevenheck, *Mirror symmetry, singularity theory and non-commutative Hodge structures*