Toric Geometry and Toric Degenerations

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Toric varieties are algebraic varieties admitting a dense open subset isomorphic to an algebraic torus, with an action of this torus on the variety extending the natural action of the torus on itself. Each toric variety can be constructed from an underlying convex polyhedral complex, and standard algebro-geometric properties of toric varieties can often be restated as properties of polyhedral complexes. This makes toric varieties an especially important class of examples of algebraic varieties, as we may calculate on and verify statements concerning toric varieties through the combinatorics of convex geometry. On the other hand, being toric is a highly non-generic property of algebraic varieties; for instance, all singularities of a toric variety are rational. The key examples of toric varieties are affine space and projective space.

There are multiple equivalent definitions one may adopt when dealing with toric varieties. In this talk, we will introduce two further definitions of toric varieties to the one given above. Both will involve constructing the toric variety in question from convex polyhedra. We will first introduce some basic concepts from convex geometry.

1 Toric varieties from fans

1.1 Monoids

Monoids are simple algebraic objects consisting of a set and an associative commutative operation, and an identity. The fundamental example of a monoid is $(\mathbb{N}^k, +)$. Monoids play a key role in the theory of log geometry, and a more thorough treatment will be undertaken during our later discussion of this topic.

Definition 1.1. A *monoid* is a pair $(P, +)$, where P is a set and $+$ is an operation

$$+ : P \times P \to P$$

that is commutative and associative, and for which an identity $0_P \in P$ exists.

Given a monoid $(P, +)$, define the *monoid ring* of P over a field $\Bbbk$ as

$$\Bbbk[P] := \bigoplus_{p \in P} \Bbbk z^p,$$

with $\Bbbk$-bilinear multiplication extending

$$z^p \cdot z^{p'} = z^{p+p'}.$$

A *homomorphism of monoids* from P to Q is a map $f : P \to Q$ such that $f(0_P) = 0_Q$ and such that $f(p + p') = f(p) + f(p')$ for all $p, p' \in P$.

A monoid P is *finitely generated* if there exists a surjective homomorphism of monoids $\mathbb{N}^k \to P$ for some $r \geq 0$.

Note. See that if a monoid P is finitely generated, then the monoid ring of P over any field $\Bbbk$ is finitely generated as a $\Bbbk$-algebra.

1.2 Cones and fans

Our first definition of toric variety will begin with a cone generated over some lattice, from which we may obtain a monoid and construct an algebra as above. For simplicity we choose our lattice to be trivial. The dual lattice will also turn out to play an important role from this perspective: denote

$$M = \mathbb{Z}^n, \quad N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}).$$

Definition 1.2. A *polyhedron* in $M_{\mathbb{R}}$ is a finite intersection of closed half-spaces of $M_{\mathbb{R}}$. A *face* of a polyhedron σ is an intersection of σ with a hyperplane H for which σ is contained within a closed half-space defined by H. A *lattice polyhedron* is an intersection of closed half-spaces defined over $\mathbb{Q}$. A *polytope* is a compact polyhedron.

A *strictly convex rational polyhedral cone* in $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ is a lattice polyhedron with exactly one vertex $0 \in M_{\mathbb{R}}$.

Definition 1.3. If σ is a strictly convex rational polyhedral cone in $M_{\mathbb{R}}$, the *dual cone* $\sigma^\vee$ is the strictly convex rational polyhedral cone in $N_{\mathbb{R}}$ defined by

$$\sigma^\vee := \{n \in N_{\mathbb{R}} \mid \langle n, m \rangle \geq 0 \ \ \forall\, m \in \sigma\}.$$

Lemma 1.4 (Gordon's lemma)**.** *Let σ be a strictly convex rational polyhedral cone in $M_{\mathbb{R}}$. Then the monoid $\sigma^\vee \cap N$ is finitely generated.*

Proof. See [1], Proposition 1.2.1.

□

Definition 1.5. A *fan* in $M_{\mathbb{R}}$ is a set Σ of strictly convex rational polyhedral cones in $M_{\mathbb{R}}$ satisfying:

(1) if $\sigma \in \Sigma$ and $\tau \subseteq \sigma$ is a face of σ, then $\tau \in \Sigma$,

(2) If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face of σ_1 and of σ_2.

Denote $|\Sigma|$ for the *support* of the fan Σ, defined as the union of the elements of Σ.

Definition 1.6. Given fans Σ_1 in $M_{\mathbb{R}}$ and Σ_2 in $M'_{\mathbb{R}}$, where $M' = \mathbb{Z}^m \subseteq \mathbb{R}^m$, a *morphism of fans* from Σ_1 to Σ_2 is a group homomorphism $\varphi : M \to M'$ such that

$$\forall \sigma_1 \in \Sigma_1, \ \ \exists \sigma_2 \in \Sigma_2 \ \text{ such that } \ \varphi_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2.$$

1.3 Toric varieties

Definition 1.7. Let σ be a strictly convex rational polyhedral cone in $M_{\mathbb{R}}$. For any field $\Bbbk$, the monoid ring $\Bbbk[\sigma^\vee \cap N]$ is a finitely generated $\Bbbk$-algebra and therefore

$$X_\sigma := \operatorname{Spec} \Bbbk[\sigma^\vee \cap N]$$

is an algebraic variety over $\Bbbk$. We will call this an *affine toric variety* for reasons which will soon become apparent.

Example 1.8. 1. $\sigma = 0 \subseteq \mathbb{R}^n$. Then $\sigma^\vee \cap N = N \subseteq \mathbb{R}^n$, so

$$X_0 = \operatorname{Spec}\,(\Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_n]/(x_i y_i - 1)) = \operatorname{Spec} \Bbbk[z_1^{\pm 1}, \ldots, z_n^{\pm 1}],$$

so X_0 is the n-dimensional algebraic torus.

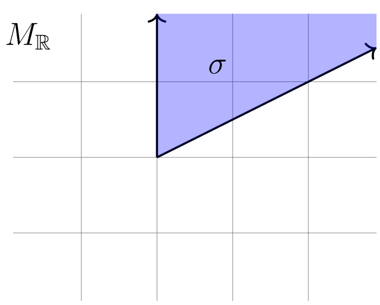
2. $\sigma = \mathbb{R}_{\geq 0} e_1 + \ldots + \mathbb{R}_{\geq 0} e_n \subseteq \mathbb{R}^n$. Then $\sigma^\vee \cap N = \mathbb{N}^n \subseteq \mathbb{R}^n$, so

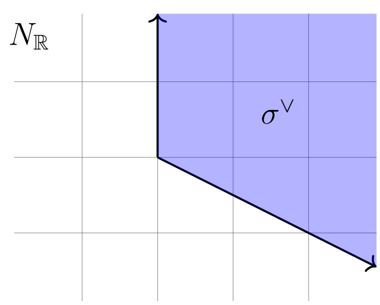
$$X_\sigma = \operatorname{Spec} \Bbbk[z_1, \ldots, z_n] = \mathbb{A}^n_{\Bbbk}.$$

3. $\sigma = \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0}(e_1 + r e_2) \subseteq \mathbb{R}^2$, for r some positive integer. Then $\sigma^\vee = \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0}(r e_1 - e_2) \subseteq \mathbb{R}^2$. The monoid $\sigma^\vee \cap N$ is generated by e_1, e_2 and $r e_1 - e_2$. Therefore

$$X_\sigma = \operatorname{Spec}\,(\Bbbk[x, y, z]/(yz - x^r)) = \mathbb{V}(yz - x^r).$$

The pictures below show the cone and dual cone for the case $r = 2$.





Definition 1.9. Let Σ be a fan in $M_{\mathbb{R}}$. For each $\tau, \sigma \in \Sigma$ such that τ is a face of σ, we obtain a natural inclusion $\sigma^\vee \subseteq \tau^\vee$, inducing a natural inclusion of monoid rings and thus in turn an open embedding $X_\tau \subseteq X_\sigma$. We may therefore glue together the varieties X_σ (à la [3], Exercise II.2.12) to obtain an algebraic variety X_Σ over $\Bbbk$, which we will call a *toric variety*.

Note. To prove separatedness of this scheme, we need to show that $X_{\sigma_1 \cap \sigma_2} \to X_{\sigma_1} \times X_{\sigma_2}$ is a closed embedding for all $\sigma_1, \sigma_2 \in \Sigma$. This follows from the so-called separation lemma for polyhedral cones (see [1], Proposition 1.2.3).

Example 1.10.

$$\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \rho_1, \rho_2, \rho_3, 0\} \subseteq \mathbb{R}^2,$$

where

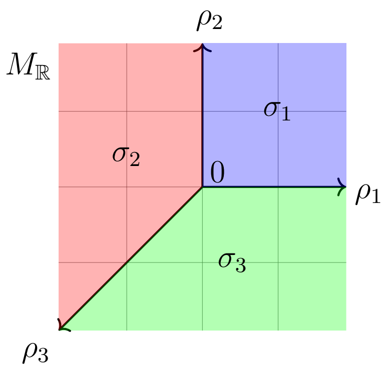
$$\rho_1 = \mathbb{R}_{\geq 0} e_1, \quad \rho_2 = \mathbb{R}_{\geq 0} e_2, \quad \rho_3 = \mathbb{R}_{\geq 0}(-e_1 - e_2), \quad \sigma_i = \rho_i + \rho_{i+1}.$$

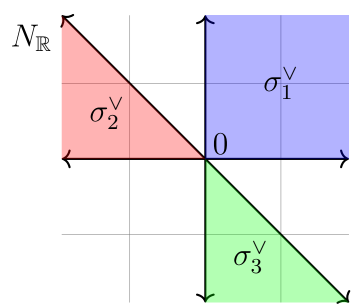
Note that $|\Sigma| = \mathbb{R}^2$. Then

$$X_{\sigma_1} = \operatorname{Spec} \Bbbk[x, y], \quad X_{\sigma_2} = \operatorname{Spec} \Bbbk[y^{-1}, xy^{-1}], \quad X_{\sigma_3} = \operatorname{Spec} \Bbbk[x^{-1}, yx^{-1}],$$

$$X_{\rho_1} = \operatorname{Spec} \Bbbk[x^{\pm 1}, y], \quad X_{\rho_2} = \operatorname{Spec} \Bbbk[x, y^{\pm 1}], \quad X_{\rho_3} = \operatorname{Spec} \Bbbk[xy, (xy^{-1})^{\pm 1}],$$

so we see that $X_\Sigma = \mathbb{P}^2_{\Bbbk}$.





Any morphism of fans $\varphi : \Sigma_1 \to \Sigma_2$ as in Definition 1.6 induces a morphism $X_\sigma \to X_{\sigma'}$ for any $\sigma \in \Sigma$, $\sigma' \in \Sigma'$ by definition. These clearly patch together to give a morphism

$$\varphi : X_\Sigma \to X_{\Sigma'}.$$

Note. The converse is also true, in that any morphism of toric varieties (by which I mean a morphism of algebraic varieties that preserves the torus action which we will later define) is induced by a morphism of the underlying fans. So we see that not only are toric varieties determined completely by their fans, but so are morphisms of toric varieties. i.e., we have an equivalence of categories between the category of toric varieties (over some field $\Bbbk$) and the category of fans.

As promised, many geometric properties of toric varieties (morphisms, resp.) may be restated in terms of the underlying fans (morphisms of fans, resp.). We won't have the time to prove any of these properties, but here are a few examples. See [1] for details.

1. X_Σ is smooth if and only if every cone in the fan is *standard*, meaning that it is generated by a subset of a basis of M.

2. A morphism $\varphi : X_\Sigma \to X_{\Sigma'}$ is proper if and only if $\varphi_\mathbb{R}^{-1}(|\Sigma'|) = |\Sigma|$.

3. A morphism $\varphi : X_\Sigma \to X_{\Sigma'}$ is birational if and only if $\varphi : M \to M'$ is an isomorphism. Thus a morphism φ is proper and birational if and only if φ is an isomorphism of lattices and Σ is a refinement of Σ'. Note also that for any toric variety there is a refinement of the underlying fan that induces a resolution of singularities.

A toric variety X_Σ contains an open subset isomorphic to an algebraic torus in a natural way; any fan Σ must contain the cone 0, being the intersection of all of the cones in Σ. We can clearly see from Definition 1.9 that there is an open embedding $X_0 \to X_\Sigma$, and we showed in Example 3.1. that X_0 is equal to the n-dimensional algebraic torus. Now there is a natural action of the torus X_0 on X_Σ obtained by gluing together the maps induced by the ring homomorphisms

$$\Bbbk[\sigma^\vee \cap N] \to \Bbbk[N] \otimes_\Bbbk \Bbbk[\sigma^\vee \cap N]; \quad z^n \mapsto z^n \otimes z^n.$$

Note that this group action clearly extends the action of X_0 on itself.

The cones in the fan Σ are then in one-to-one correspondence with the orbits of the action of X_0 on X_Σ, through the mapping that sends a cone $\sigma \in \Sigma$ to the orbit

$$O_\sigma := X_\sigma \backslash \bigcup_{\tau \in \Sigma, \tau \subsetneq \sigma} X_\tau.$$

Denote by D_σ the closure of this orbit. If σ is a one-dimensional cone (i.e., a ray), then D_σ is a (Weil) divisor on X_Σ. In fact every reduced effective Weil divisor invariant under the torus action is equal to one of these divisors.

Example 1.11. $\sigma = \mathbb{R}_{\geq 0} e_1 + \ldots + \mathbb{R}_{\geq 0} e_n \subseteq \mathbb{R}^n$. Denote, for each $I \subset [n]$,

$$\sigma_I = \sum_{i \in I} \mathbb{R}_{\geq 0} e_i, \text{ with } \sigma_\phi := 0,$$

and write $\Sigma = \{\sigma_I \mid I \subseteq [n]\}$. Then Σ is clearly a fan and $X_\Sigma = X_\sigma = \mathbb{A}^n_\Bbbk$. The action of the torus is given by coordinate-wise multiplication. Now

$$O_{\sigma_I} = \{(x_1, \ldots, x_n) \in \Bbbk^n \mid x_i = 0 \; \forall i \in I, \; x_i \neq 0 \; \forall i \notin I\} \text{ and } D_{\sigma_I} = \left\{ \prod_{i \in I} x_i = 0 \right\}.$$

2 Toric varieties from polytopes

We have seen that projective space can be constructed using the theory of cones and fans developed in the previous section, by gluing together affine charts. We are well aware however that projective space can be constructed in a more canonical manner by applying the Proj construction to a graded algebra. It is therefore natural to ask whether a more canonical construction for projective toric varieties exists. This is what we will discuss in the final section of this talk.

Let Δ be any lattice polyhedron in $N_{\mathbb{R}}$ with at least one vertex. We define the *cone over* Δ to be

$$C(\Delta) = \overline{\{(rn, r) | n \in \Delta, \, r \geq 0\}} \subseteq N_{\mathbb{R}} \oplus \mathbb{R}.$$

This is a rational polyhedral cone, and $\Bbbk[C(\Delta) \cap (N \oplus \mathbb{Z})]$ is a finitely generated graded $\Bbbk$-algebra, with the grading generated by $\deg z^{(n,d)} = d$. So we may apply the Proj functor to obtain a variety

$$\mathbb{P}_\Delta := \operatorname{Proj} \Bbbk[C(\Delta) \cap (N \oplus \mathbb{Z})].$$

This is projective over $\operatorname{Spec} \Bbbk[C(\Delta) \cap (N \oplus \{0\})]$.

Example 2.1. $\Delta = \operatorname{Conv}\{(0,0), (1,0), (0,1)\} \subseteq \mathbb{R}^2$. Then

$$\Bbbk[C(\Delta) \cap (N \oplus \mathbb{Z})] \simeq \Bbbk[x_0, x_1, x_2],$$

given the standard grading, under the identification $z^{(0,0,1)} \leftrightarrow x_0$, $z^{(1,0,1)} \leftrightarrow x_1$, $z^{(0,1,1)} \leftrightarrow x_2$. Therefore we see that

$$\mathbb{P}_\Delta \simeq \mathbb{P}^2_{\Bbbk},$$

which is indeed projective over $\operatorname{Spec} \Bbbk[C(\Delta) \cap (N \oplus \{0\})] = \operatorname{Spec} \Bbbk$.

$\mathbb{P}_\Delta$ is in fact a toric variety in the sense of the previous section; for each face $\sigma \subseteq \Delta$ of the polyhedron Δ, the *normal cone to* Δ *along* σ is defined to be

$$N_\Delta(\sigma) = \{m \in M_{\mathbb{R}} \,|\, \langle \sigma, m \rangle = \text{pt}, \langle n, m \rangle \geq \langle n', m \rangle \, \forall n \in \Delta, n' \in \sigma\}.$$

Then the *normal fan* to the polyhedron Δ is

$$\Sigma_\Delta := \{N_\Delta(\sigma) \,|\, \sigma \text{ is a face of } \Delta\}.$$

This is a fan in $M_{\mathbb{R}}$ as you may expect.

Theorem 2.2. $\mathbb{P}_\Delta \simeq X_{\Sigma_\Delta}$.

Example 2.3. Returning to the previous example, the faces of the cone Δ are

$$p_1 = (0,0), \; p_2 = (1,0), \; p_3 = (0,1), \; l_1 = \overrightarrow{p_3 p_1}, \; l_2 = \overrightarrow{p_1 p_2}, \; l_3 = \overrightarrow{p_2 p_3}, \; \Delta,$$

and the respective normal cones along these faces are (recalling the notation we introduced earlier)

$$\sigma_1, \; \sigma_2, \; \sigma_3, \; \rho_1, \; \rho_2, \; \rho_3, \; 0.$$

So Σ_Δ equals the fan Σ that we wrote down earlier. The theorem above thus gives us another way of seeing that $X_\Sigma \simeq \mathbb{P}^2_{\Bbbk}$.

Now that we have all the basic technical tools from toric geometry that we will need, we introduce a certain method of constructing families of toric varieties that will play a vital role in later sections of the reading group.

Example 2.4 (Mumford Degeneration)**.** Let Δ be a lattice polytope in $N_{\mathbb{R}}$, $\mathscr{P}$ a polyhedral decomposition of Δ into lattice polyhedra, and $\varphi : (\Delta, \mathscr{P}) \to \mathbb{R}$ a piecewise-linear convex function with integral slopes. This data induces a lattice polyhedron

$$\widetilde{\Delta} := \{(n, r) \in N_{\mathbb{R}} \oplus \mathbb{R} \,|\, n \in \Delta, \, r \geq \varphi(n)\}$$

in $N_{\mathbb{R}} \oplus \mathbb{R}$. Then $\mathbb{P}_{\widetilde{\Delta}}$ is projective over

$$\operatorname{Spec} \Bbbk[C(\widetilde{\Delta}) \cap (N \oplus \mathbb{Z} \oplus \{0\})] = \operatorname{Spec} \Bbbk[\{0\} \oplus \mathbb{N} \oplus \{0\}] = \mathbb{A}^1_{\Bbbk}.$$

We will write $\pi : \mathbb{P}_{\widetilde{\Delta}} \to \mathbb{A}^1_{\Bbbk}$ for this projective map (induced by $t \mapsto z^{(0,1,0)}$).

Now see that a proper face of $\tilde{\Delta}$ is either a "horizontal face", projecting homeomorphically onto a face of Δ, or a "vertical" face, projecting non-homeomorphically onto a face of Δ. The normal cone of a vertical face $\breve{\sigma}$ is $N_\Delta(\sigma) \oplus \{0\} \subseteq M_{\mathbb{R}} \oplus \mathbb{R}$, while the normal cone to a *maximal* horizontal face τ is the ray generated by $(-m_\tau, 1)$, where m_τ is the slope of φ on the projection of τ.

$\pi^{-1}(0)$ is the union of all of the toric strata that vanish at $z^{(0,1,0)}$, so is by the above the union of the toric divisors of $\mathbb{P}_{\widetilde{\Delta}}$ corresponding to the maximal horizontal faces. Therefore

$$\pi^{-1}(0) = \bigcup_{\sigma \in \mathscr{P}_{\max}} \mathbb{P}_\sigma,$$

with gluing data prescribed by $(\mathscr{P}, \varphi)$.

For the generic fibres of the family, notice that the localisation of the ring $\Bbbk[C(\widetilde{\Delta}) \cap (N \oplus \mathbb{Z} \oplus \mathbb{Z})]$ at $z^{(0,1,0)}$ is isomorphic to $\Bbbk[C(\Delta \times \mathbb{R}) \cap (N \oplus \mathbb{Z} \oplus \mathbb{Z})]$, which shows that

$$\mathbb{P}_{\widetilde{\Delta}} \backslash \pi^{-1}(0) \simeq \mathbb{P}_\Delta \times_{\Bbbk} \mathbb{G}_m.$$

So we finally see that π is a degeneration of toric varieties, whose central fibre is a union of toric varieties with intersection data prescribed by $(\mathscr{P}, \varphi)$ and whose generic fibre is isomorphic to $\mathbb{P}_\Delta$.

References

[1] William Fulton, *Introduction to Toric Varieties*

[2] Mark Gross, *Tropical Geometry and Mirror Symmetry*

[3] Robin Hartshorne, *Algebraic Geometry*