Mirror symmetry and tropical geometry reading group.

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SS 2022–23

Introduction:

This reading group aims at exploring the links between tropical geometry and mirror symmetry. We will follow quite closely (up to some action of the symmetric group of some subsets of the (sub)sections) the book [5]. The ultimate goal is to go through the first three chapters of the book. The format would be the following: each talk would be between one hour long, we will meet every two weeks (alternatively in Chemnitz and Leipzig) for two (or more) talks in a row — this depends mostly on the audience and the (hopefully non-empty!) intersection of our availabilities. It would be nice (if you have time!) if everyone could write down their talk in TeX in such a way we can put everything together afterwards.

Current schedule:

03/05	Chemnitz	Talks 1, 2, 3
17/05	Leipzig	Talks 4, 5, 6, 7
31/05	Chemnitz	Talks 8, 9, 10, 11
14/06	Leipzig	Talks 12, 13, 14, 15
28/06	Chemnitz	Talks 16, 17, 18

For each meeting, the timetable will look like this:

10h00 – 11h00	First talk
11h00 – 11h30	Coffee break
11h30 – 12h30	Second talk
12h30 – 14h00	Lunch break
14h00 – 15h00	Third talk
15h00 – 15h30	Coffee break
15h30 – 16h30	Fourth talk

For the first and the last meetings, we will start at 11h00.

Motivational overview:

The story of mirror symmetry starts with the quintic threefold Q_t in $\mathbb{P}^4$ defined by the equation

$$\sum_{i=0}^{4} x_i^5 - tx_0x_1x_2x_3x_4 = 0$$

where $t \in \mathbb{A}^1$. For most t, Q_t is smooth and one can compute the Hodge numbers of Q_t, one gets:

$$h^{1,1}(Q_t) = 1 \text{ and } h^{1,2}(Q_t) = 101.$$

One can construct a diagonal action of a group G on $\mathbb{P}^4$ which restricts on Q_t and resolve the quotient Q_t/G to a Calabi-Yau threefold $\check{Q}_t$ called the *mirror* of Q_t; the Hodge numbers of $\check{Q}_t$ are given by

$$h^{1,1}(\check{Q}_t) = 101 \text{ and } h^{1,2}(\check{Q}_t) = 1.$$

In [1], Candelas, de la Ossa, Green and Parkes conjectured links between curve counting on Q_t and period integrals on $\check{Q}_t$. This suggests the following (vague) conjecture: starting with a Calabi-Yau threefold X, one should be able to construct another Calabi-Yau threefold $\check{X}$ called the *mirror* of X such that the curve counting on X is very much related to period integrals on $\check{X}$. In any case, this already suggest the existence of two worlds which are interrelated:

- The A-model: sympletic world, Gromov-Witten invariants (curve counting).
- The B-model: complex analytic world, period integrals.

In 1996, Strominger, Yau and Zaslow proposed in [6] a conjecture (called the *SYZ conjecture*) formalizing in a geometric way this previous conjecture: in addition of the existence of the mirror $\check{X}$, the latter predicts the existence of a basis B and of two fibrations in special Lagrangian tori f_1 and f_2

$$\begin{tikzcd} X \arrow[dr, "f_1"] & & \check{X} \arrow[dl, "f_2"'] \\ & B & \end{tikzcd}$$

over B whose (generic) fiber are dual to each other — in the sense of tori. The base B is usually an affine manifold (i. e., there exists an atlas and the transition maps are of the form $z \mapsto Az + b$ where $A \in \mathrm{GL}_n(\mathbb{Z})$ and $b \in \mathbb{R}^n$) which is allowed to be singular — the name is a little unfortunate. This base brings into the picture a third world which is the one of affine structures: the tropical world.

For $\mathbb{P}^2$ — which is not Calabi-Yau but Fano, anyway —, a first link between the A-model and the tropical world is Mikhalkin's curve counting formula which relates holomorphic curve counting and tropical curve counting; after introducing all the needed material to navigate in these three worlds and the proof of mirror symmetry for $\mathbb{P}^2$, the proof of this formula will be our first goal. The proof of this formula will make appear another world: the logarithmogeometric world; it will appear as a bridge between the A-model and the tropical world.

An ultimate goal would be to relate the tropical world with the B-model, this would complete the picture.

There is another approach to mirror symmetry developed by Kontsevich and called *homological mirror symmetry* formulating mirror symmetry in term of existence of an equivalence of categories between $\mathrm{D}^b(X)$ and a category of a symplectic nature called the (derived category) of the Fukaya ∞-category of $\check{X}$. We won't talk about this approach in this reading group.

Another part we will not talk about is the so-called *Gross-Siebert program* (chapter 6 of [5]) that try to reconstruct the pair $(X, \check{X})$ from the base B. [1]

Here is a global plan of the seminar; one section per talk:

1 A crash-course in toric geometry. (Henry)

In that talk, we want to get all the toric material needed for the seminar, i.e. we want *at least* the parts 1.2. and 3.1. of [5]. You should define cones, fans, polyhedra, toric varieties and how you construct toric varieties out of a fan. Put the emphasis on the dictionary between toric geometry and polyhedral geometry (smoothness, properness, etc.); if you have enough time, you can maybe discuss the orbit-cone correspondance — it is discussed in [5] but kind of avoided. You should also maybe put the emphasis on the monoid part since it will be used in logarithmic geometry. Give as many examples as you can. Helpful references could be [2] and [3].

2 Tropical hypersurfaces. (Emeryck)

In this talk, follow the section 1.1. of [5], put emphasis on the pictures.

3 Parametrized tropical curves. (Peter)

In that talk, follow the section 1.3. of [5]. You can maybe do some tiny recalls about graphs.

4 Stable pointed curves and stable maps. (Max)

In that talk, follow the section 2.1. of [5]. As in the book, take stacks as a black box (maybe you can give a rough idea about it: we like quotients to exists but the non-triviality of the stabilizer(s) is usually a problem for the quotient to exist as a scheme, hence we have to pass to stacks; in our case, the groups in questions are group of automorphisms[2], passing to stacks will make more functors representable. Since things could seem a little hidden under the carpet here, put the emphasis on the examples and especially on the Gromov-Witten invariant since it will be important in the sequel: this is *the* object for curves counting and it comes into the play in the definition of quantum cohomology. Explain the manifestation of stacky phenomena in Gromov-Witten invariants (when they are not a natural number).

[1] It might be the object of a second act of this seminar.

[2] "La conclusion pratique à laquelle je suis arrivé dès maintenant, c'est que chaque fois que en vertu de mes critères, une variété de modules (ou plutôt, un schéma de modules) pour la classification des variations (globales, ou infinitésimales) de certaines structures (variétés complètes non singulières, fibrés vectoriels, etc.) ne peut exister, malgré de bonnes hypothèses de platitude, propreté, et non singularité éventuellement, la raison en est seulement l'existence d'automorphismes de la structure qui empêche la technique de descente de marcher." A letter from Grothendieck to Serre (1959)

5 Quantum cohomology and quantum differential equation. (Luca)

In that talk, follow the sections 2.1.2. and 2.1.4. of [5]. Introduce the (big) quantum cohomology on a variety with no odd cohomology and the WDVV equation. Compute the quantum product for $\mathbb{P}^2$. Introduce the quantum differential equation and derive a fundamental set of solutions. The map $S(T_i) = s_i$ as defined in 2.1.4 will play an important role in the sequel so should be discussed.

6 Semi-infinite variations of Hodge structures and moving subspace realisation. (Christian)

In that talk, follow the sections 2.1.5. and 2.1.6. of [5]. Define semi-infinite variations of Hodge structure and show how one arises from the quantum cohomology ring. Talk about the moving subspace realisation, again with emphasis on the quantum cohomology example.

7 Frobenius manifolds and semi-infinite variations of Hodge structures. (Christian)

In that talk, follow the sections 2.1.3. and 2.1.7. of [5]. Define the formal notions of pre-Frobenius structures and Frobenius manifolds and describe how a Frobenius manifold arises from the quantum product. In more generality, explain how a Frobenius manifold may be constructed from a miniversal semi-infinite variation of Hodge structure with a flat identity and Euler vector field. (Due to time constraints some of the details may have to be omitted; focus on the quantum cohomology example if necessary.) Define the Givental J-function and derive it for $\mathbb{P}^2$.

8 The twisted de Rham complex and homology. (Yichen)

In that talk, follow the sections 2.2.1. and 2.2.2. of [5]. Now we turn to the B-model. For a smooth variety X and regular function $W : X \to \mathbb{C}$, define the twisted de Rham complex $(\Omega^\bullet_X, d + dW\wedge)$, and discuss the homology groups dual to the hypercohomology groups of this complex. Describe a basis of this homology group for the case $(\check{X}, W) = ((\mathbb{C}^*)^2, x_0 + x_1 + x_2)$, the mirror to $\mathbb{P}^2$.

9 The B-model semi-infinite variation of Hodge structure. (Avi)

In that talk, follow the section 2.2.3. of [5]. Define the universal unfolding of the mirror to $\mathbb{P}^n$ and define a (graded) semi-infinite variation of Hodge structure on this.

10 The B-model Frobenius manifold. (Paul)

In that talk, follow the section 2.2.4. of [5]. Use the theory of talk 7 to define a Frobenius manifold structure on the universal unfolding of the mirror to $\mathbb{P}^n$ from the semi-infinite variation of Hodge structure defined in the previous talk.

11 Mirror symmetry for $\mathbb{P}^n$. (Henry or Emeryck)

In that talk, follow the section 2.2.5. of [5]; this is going to be probably quite short but it's important. Describe exactly what we mean by mirror symmetry in this case.

12 Logarithmic geometry. (?)

Here, the content can be found in the third chapter of [5]. As you can see, there are many things and it is not possible to discuss everything in a one hour talk. I think it's good to discuss the introduction (it shows that logarithmic geometry is helpful to study singular complex algebraic varieties) and one of the leading ideas is that smooth varieties are really well-behaved so we try to define a class of varieties (we have to add a little something: a *logarithmic structure*) which is larger but which somehow behave like a smooth varieties, we will call them *log smooth varieties*.

You should discussed the small part on monoids very quickly; you can also discuss functoriality in the construction of toric varieties and you must discuss the orbit-cone correspondance; I am not that sure that the

discussion on the Picard group of a toric variety is needed. The part 3.1.3. has already been discussed in the first talk.
Concerning logarithmic geometry itself, the definitions that must be discussed are the following: (pre)log structures, log scheme, morphism of log schemes, log structure associated to a prelog structure, finiteness condition (fine log scheme, saturated log scheme, log smoothness, log derivations, log forms (mention it's locally free in the log smooth case).
Now, the examples that must be discussed are the following: the divisorial log structure, the log point, the example 3.12., the toric example (example 3.15.), example 3.24., example 3.26. about local description of log smooth curves, example 3.36. It is not necessary to talk about logarithmic deformation theory.

13 Mikhalkin's curve counting formula and log to tropical. (?)

In that talk, follow the sections 4.1. and 4.2. of [5]. The next few talks will focus on Mikhalkin's curve-counting formula. The proof relates holomorphic curves and tropical curves by passing through log geometry. In this first talk, state Mikhalkin's curve counting formula and begin by explaining how we construct a tropical curve out of a (torically transverse) log curve.

14 The way back: tropical to log. (?)

In that talk, follow the section 4.3. of [5]. The main theorem of this section calculates exactly how many torically transverse marked log curves give a fixed marked tropical curve under the construction given in the previous talk.

15 Wandering: classical to log to classical and Mikhalkin's final act. (?)

In that talk, follow the sections 4.4., 4.5. and 4.6. of [5]. Show how a torically tranverse log curve arises on the central fibre of a degeneration of a toric variety on whose general fibres is given a family of genus zero curves. Show conversely that such a (genus zero) log curve can be extended to a family of curves. Combine these results with those of the previous two talks to deduce Mikhalkin's curve counting formula.

16 The perturbed Landau-Ginzburg potential. (?)

In that talk, follow the section 5.1. of [5]. The next focus for the book is to reformulate the B-model in terms of tropical geometry, as we may now do with relative ease for the A-model using Mikhalkin's formula. Chapter 5 of Gross' book is based upon his paper [4], I recommend reading the opening of this paper for some additional context. The first step is to define the universal unfolding of the Landau-Ginzberg potential.

17 Tropical descendant invariants and the main B-model statement. (?)

In that talk, follow the sections 5.2. and 5.3. of [5]. Define the tropical descendent invariants and the tropical J-function. Use this to state both the formulation of the B-model in terms of tropical geometry and the resulting tropical geometry theoretic mirror symmetry statement (Theorem 5.18 and Corollary 5.20 of [5]).

18 An overview of the proof of the main B-model statement. (?)

In that talk, try to give the ideas of the proof of the main B-model statement, i.e. squeezing the 52 pages of the sections 5.4. and 5.5. of [5]; in other words: good luck. First we have to show that our period integrals are independent of the choice of points $P_1, \ldots, P_k, Q$ (we use a wall-crossing formula). Then we have to evaluate these integrals.

References

[1] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes. A pair of Calabi-Yau manifolds as exactly soluble superconformal theory. *Nuclear Physics B359*, pages 21–74, 1991.

[2] D. A. Cox, J. B. Little, and H. K. Schenck. Toric varieties. Number 124 in Graduate Studies in Mathematics. AMS, 2011.

[3] W. Fulton. Introduction to toric varieties. Number 131 in Annals of Mathematics Studies. Princeton University Press, 1993.

[4] M. Gross. Mirror symmetry for $\mathbb{P}^2$ and tropical geometry. Number 224 in Advances in Mathematics, pages 169–245. Elsevier, 2010.

[5] M. Gross. Tropical geometry and mirror symmetry. Number 114 in CBMS Regional Conference Series in Mathematics. AMS, 2011.

[6] A. Strominger, S.-T. Yau, and E. Zaslow. Mirror symmetry is T-duality. *Nuclear Physics B479*, pages 243–259, 1996.