condition very restrictive. Indeed, from the theory of Lie groups (using the exponential map), we know that there exists an open neighbourhood V of I_n that does not contain any non-trivial subgroup of $\mathrm{GL}_n(\mathbb{C})$ so by continuity of ρ, the preimage $\rho^{-1}(V)$ must contain a normal subgroup H of G_k which has finite index and that contains the neutral element. Therefore, $\rho(H)$ is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ contained in V so it has to be trivial. This implies that ρ factorizes through the finite group G_k/H.

To get rid of this problem, we replace the field $\mathbb{C}$ by a field which has a topology much more compatible with that of G_k, namely $\mathbb{Q}_\ell$ for $p \nmid \ell$ [5]. Note that this is consistent with the fact that in the number field case, the Galois representations we were considering were (ℓ-adic) étale cohomologies, which are $\mathbb{Q}_\ell$-vector spaces.

Let us now check if this replacement actually fixed the problem we were facing.

2.2.1. — Let $\rho : \mathbb{Z}_p \to \mathbb{Q}_\ell^\times$ where $p \nmid \ell$ be a continuous representation. Let $\mu \in \mathbb{Q}_\ell$ such that $\mu - 1 \in \ell\mathbb{Z}_\ell$ — so $\mu \in \mathbb{Z}_\ell^\times$. Then, the ultrametric inequality yields

$$|\mu^{p^n} - 1|_\ell \leq \max_{1\leq i\leq p^n-1}(|\mu - 1|_\ell^{p^n}, |\tbinom{p^n}{i}(-1)^{p^n-i}\mu^i|_\ell) = \max_{1\leq i\leq p^n-1}(|\mu - 1|_\ell^{p^n}, |\mu|_\ell^i) = |\mu - 1|_\ell^{p^n} \leq \tfrac{1}{p^n}.$$

where the last inequality holds since $(\mu - 1)^{p^n} \in \ell^{p^n}\mathbb{Z}_\ell$. This implies that there exists a unique [6] continuous group homomorphism $\rho : \mathbb{Z}_p \to \mathbb{Q}_\ell^\times$ such that $\rho(1) = \mu$. In particular, this representation ρ does not factor through a finite quotient of $\mathbb{Z}_p$.

Let us therefore introduce the following definition:

Definition **(2.2.2.)** — *Let ℓ be a prime number not divisible by p. Let $n \geq 0$.*
A n-dimensional ℓ-adic representation *of G_k is a continuous group homomorphism $\rho : G_k \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ satisfying the following two conditions:*

- *its image is contained in $\mathrm{GL}_n(E)$ where E is a finite extension of $\mathbb{Q}_\ell$*
- *it is ramified only at finitely many closed points of C.*

For all $n \geq 0$, we denote by $\mathscr{G}_n$ the set of equivalence classes of irreducible n-dimensional ℓ-adic representations ρ of G_k such that the image of $\det(\rho)$ is a finite group.

2.2.3. — Given an element ρ of $\mathscr{G}_n$, we consider the collection $\{\rho_x := \rho(\mathrm{Fr}_x) \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \mid x \in C^7\}$ and the collection of their eigenvalues $\{(\lambda_1(\rho_x), \dots, \lambda_n(\rho_x)) \in \overline{\mathbb{Q}}_\ell^n \mid x \in C\}$ for ρ unramified. Chebotarev's density theorem implies that if two ℓ-adic representations have the same collection of eigenvalues for all but finitely many (closed) points $x \in C$, then they are isomorphic.

3. Cuspidal automorphic representations for functions fields.

In Henry's talk, we have seen that the other side of the Langlands correspondence was dealing with cuspidal automorphic representations of the adelic group $\mathrm{GL}_n(\mathbb{A}_F)$ where F is a number field. The definition of the rings of adèles also makes sense for a function field — in this case, we will only have non-archimedean factors.

As before, we denote by k the field of functions of a smooth connected projective curve C over $\mathbb{F}_q$. For every closed point x of C, which corresponds to a valuation on k, we denote by k_x the completion of k at x and $\mathscr{O}_x$ the ring of algebraic integers of k_x. We consider a uniformizing element t_x of $\mathscr{O}_x$, which induces isomorphisms

$$k_x \cong \kappa(x)((t_x)) \text{ and } \mathscr{O}_x \cong \kappa(x)[\![t_x]\!],$$

[5] For $p \mid \ell$, ℓ-adic cohomology is a total mess (lack of functoriality), that's why crystalline cohomology was introduced by Berthelot.

[6] The condition $\rho(1) = \mu$ implies that $\rho(p^n) = \mu^{p^n}$ for all $n \geq 0$ so the estimation above implies that $|\rho(p^n) - \rho(0)|_\ell \leq p^{-n}$. Now, the continuity of ρ and the fact that the p-adic valuation is discrete implies that ρ is uniquely determined by the condition $\rho(1) = \mu$.

[7] We only consider closed points.

where $\kappa(x) \cong \mathbb{F}_{q_x}$ is the residue field at x, which is a finite extension of $\mathbb{F}_q$.

Remark. — Note that in that case, the diagonal embedding of k into $\mathbb{A}_k$ is just the expansion of a rational function on C at all closed points of the curve.

3.1. Cuspidal automorphic representations.

We now introduce cuspidal automorphic representations. We denote by $K \subseteq \mathrm{GL}_n(\mathbb{A}_k)$ the maximal compact subgroup, which is equal to $\prod_{x\in C} \mathrm{GL}_n(\mathcal{O}_x)$ where the product is taken over *closed* points of C.

Let $\chi : \mathrm{Z}(\mathrm{GL}_n(\mathbb{A}_k)) = \mathbb{A}_k^\times \to \mathbb{C}^\times$ be a character of finite order. This hypothesis on the finiteness of the order is to match with the fact that $\det(\rho)$ has finite order on the Galois side.

Definition **(3.1.1.)** — *We denote by* $\mathscr{C}_\chi(\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A}_k))$ *the set of functions* $f : \mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A}_k) \to \mathbb{C}$ *satisfying the following conditions:*

1. *f is a locally constant function (*smoothness*)*
2. *the right translates of f under the action of K span a finite-dimensional vector space (K-*finiteness*)* [8]
3. *for all $g \in \mathrm{GL}_n(\mathbb{A}_k)$ and all $z \in \mathrm{Z}(\mathrm{GL}_n(\mathbb{A}))$, we have $f(gz) = \chi(z)f(g)$ (*central character*)*
4. *for all non-zero natural numbers n_1 and n_2 such that $n_1 + n_2 = n$, if we denote by N_{n_1,n_2} the unipotent radical of the standard parabolic subgroup P_{n_1,n_2} of $\mathrm{GL}_n(\mathbb{A}_k)$* [9]*, then we have*

$$\int_{N_{n_1,n_2}(k)\backslash N_{n_1,n_2}(\mathbb{A}_k)} f(ug)\mathrm{d}u = 0$$

   *for all $g \in \mathrm{GL}_n(\mathbb{A}_k)$. (*cuspidality*)*

Remark. — By a theorem of Langlands, irreducible automorphic representations of $\mathrm{GL}_n(\mathbb{A}_k)$ comes in two types: either they are cuspidal or they come from the tensor product of irreducible automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A}_k)$ and $\mathrm{GL}_{n_2}(\mathbb{A}_k)$ with $n_1 + n_2 = n$. In particular, this cuspidality condition rules out the possibility that our representation comes from smaller group of invertible adelic matrices.

Comparing this definition with that given in the case of number fields (or rather for the trivial number field), we see that all the requirements concerning the archimedean factor disappeared. In particular, the boundedness condition (that was here in the case of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ solely in view of the correspondence between irreducible cuspidal representations and cuspidal modular forms, in order to get rid of the *Maass forms* [10] that correspond to no modular forms (in fact, they correspond to automorphic functions (not necessarily holomorphic on the upper half plane)) and have usually slow growth but are not bounded. Moreover, in view of the Langlands correspondence, these Maass forms correspond to no two-dimensional Galois representations of $\mathbb{Q}$.

3.1.2. — The left action of $\mathrm{GL}_n(\mathbb{A}_k)$ on the quotient $\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A}_k)$ induces a right action of $\mathrm{GL}_n(\mathbb{A}_k)$ on the $\mathbb{C}$-vector space [11] $\mathscr{C}_\chi(\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A}_k))$ via the action in the argument. This exhibits $\mathscr{C}_\chi(\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A}_k))$ as a representation of $\mathrm{GL}_n(\mathbb{A}_k)$. We know that this representation decomposes as a direct sum of (Hilbert space) representations. [12]

[8] This seems to be a technical condition imposed here to reduce technicalities, i.e., dealing with *Hilbert space* representations. In general, the subspace of K-finite vector is dense.

[9] Any parabolic subgroup (i. e. a subgroup containing a Borel subgroup) of GL_n is of the form $P_{n_1,\dots,n_r}$ where $n_1, \dots, n_r$ are non-zero natural numbers such that $n_1 + \cdots + n_r = n$ and $P_{n_1,\dots,n_r}$ is the subgroup of upper-triangular invertible matrices that have invertible blocks of respective sizes $n_1, \dots, n_r$ on the diagonal and arbitrary coefficients above these.

[10] These automorphic representations appear as principal series of the Harish-Chandra pair $(\mathfrak{gl}_2, \mathrm{O}_2(\mathbb{R}))$.

[11] Is this a Hilbert space?

[12] This does not follow directly from the reductivity of GL_n since the representation is not finite-dimensional.

Definition **(3.1.3.)** — *The irreducible representations appearing in the decomposition of* $\mathscr{C}_\chi(\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A}_k))$ *are called the* irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_k)$.

Now comes a theorem by Piatetski-Shapiro and Shalika.

Theorem **(3.1.4.)** — *All the irreducible cuspidal automorphic representations of* $\mathrm{GL}_n(\mathbb{A}_k)$ *appear with multiplicity one.*

We denote by $\mathscr{A}_n$ the set of (isomorphism classes of) these representations.

3.1.5. — Let $\pi \in \mathscr{A}_n$. According to the decomposition of $\mathrm{GL}_n(\mathbb{A}_k)$ in local factors, we have a decomposition of π as a restricted tensor product [13]

$$\pi = \bigotimes_{x\in C}{}' \pi_x$$

where for all closed point $x \in C$, π_x is an irreducible representation of $\mathrm{GL}_n(k_x)$. Furthermore, there is a finite subset of closed points S of C such that π_x is *unramified* (or *spherical*) for $x \in C \setminus S$, i.e. it contains a non-zero vector v_x that is stable under the maximal compact subgroup $\mathrm{GL}_n(\mathscr{O}_x)$ of $\mathrm{GL}_n(k_x)$. Such a vector is called a *spherical vector* and is unique up to scaling, we fix it once and for all. [14]

3.2. The spherical Hecke algebra, Hecke operators and Hecke eigenvalues.

As in the case of number fields, the unramified factors of π will benefit from an extra symmetry: that of the *spherical Hecke algebra*. In general these Hecke algebra should be a p-adic analogue of Lie algebras (which of course still exist but are too small according to `ncatlab`).

Definition **(3.2.1.)** — *Let* $x \in C$ *be a closed point.*
The spherical Hecke algebra at x *is the space* $\mathscr{H}_x$ *of compactly supported on* $\mathrm{GL}_n(k_x)$ *which are bi-invariant with respect to the maximal compact subgroup* $\mathrm{GL}_n(\mathscr{O}_x)$.

The structure of algebra comes from the existence of a convolution product that we define now.

Definition **(3.2.2.)** — *Let* f *and* g *be elements of* $\mathscr{H}_x$.
The convolution product $f \star g$ *of* f *and* g *is the element of* $\mathscr{H}_x$ *defined by*

$$f \star g \;:\; h \in \mathrm{GL}_n(k_x) \mapsto \int_{\mathrm{GL}_n(k_x)} f(ht^{-1})g(t)\mathrm{d}t \in \mathbb{C},$$

where $\mathrm{d}t$ *stands for the Haar measure on the locally compact topological group* $\mathrm{GL}_n(k_x)$ *normalized such that the volume of the maximal compact subgroup is equal to 1.*

In fact, one can give a quite explicit description of the algebra $\mathscr{H}_x$.

3.2.3. — For all $i \in [\![1, n]\!]$, let us denote by $H_{i,x}$ the characteristic function of the $\mathrm{GL}_n(\mathscr{O}_x)$-double coset $M_n^i(\mathscr{O}_x)$ defined as

$$M_n^i(\mathscr{O}_x) := \mathrm{GL}_n(\mathscr{O}_x)\cdot \mathrm{diag}(\underbrace{t_x, \dots, t_x}_{i}, \underbrace{1, \dots, 1}_{n-i})\cdot \mathrm{GL}_n(\mathscr{O}_x).$$

This does not depend on the choice of a uniformizer t_x of $\mathscr{O}_x$ because two uniformizers differ by a unit. A special case (proven by Tamagawa and Satake) of the Satake isomorphism gives that $\mathscr{H}_x$ is a $\mathbb{C}$-algebra isomorphic to $\mathbb{C}[H_{1,x}, \dots, H_{n-1,x}, H_{n,x}^{\pm 1}]$, this (non-trivial) result essentially relies on the p-adic Cartan decomposition: any element of $\mathrm{GL}_n(\mathscr{O}_x)\backslash\mathrm{GL}_n(k_x)/\mathrm{GL}_n(\mathscr{O}_x)$ has a unique representative of the form $\mathrm{diag}(t_x^{\lambda_1}, \dots, t_x^{\lambda_n})$ where $\lambda_1 \geq \cdots \geq \lambda_n$ are integers — if λ_n is a natural number, so are all the other λ_i and $H_{n,x}^{-1}$ is not needed[15].

[13] The space $\bigotimes'_{x\in C} \pi_x$ is the subspace of $\bigotimes_{x\in C} \pi_x$ spanned by the tensors of the form $\otimes_{x\in C} w_x$ where $w_x = v_x$ for all but finitely many $x \in X \setminus S$.

[14] In general (i.e., with no K-finiteness hypothesis), there exists *at most* one (possibly none) spherical vector.

[15] Is this the reason why the negative power can only be carried by $H_{n,x}$?

3.2.4. — Let us now define an action of the local factor π_x on $\mathscr{H}_x$ via the formula

$$f_x \cdot v := \int_{\mathrm{GL}_n(k_x)} f_x(g)(g \cdot v)\mathrm{d}g,$$

which makes sense since π_x is a representation of $\mathrm{GL}_n(k_x)$. The vector $f_x \cdot v$ is still $\mathrm{GL}_n(\mathcal{O}_x)$-invariant since f_x is. Moreover, if π_x is irreducible, then the space of $\mathrm{GL}_n(\mathcal{O}_x)$-invariant subspace of vector of π_x is one-dimensional, we consider a generator v_x of it. In particular, since $f_x \cdot v_x$ is a $\mathrm{GL}_n(\mathcal{O}_x)$-invariant vector of v_x, there exists $\varphi(f_x) \in \mathbb{C}$ such that

$$f_x \cdot v_x = \varphi(f_x)v_x.$$

This allows us to define a $\mathbb{C}$-linear form

$$\varphi : f_x \in \mathscr{H}_x \mapsto \varphi(f_x) \in \mathbb{C}.$$

In fact, φ is a morphism of $\mathbb{C}$-algebras. Since $\mathscr{H}_x \cong \mathbb{C}[H_{1,x}, \dots, H_{n-1,x}, H_{n,x}^{\pm 1}]$, the morphism φ is completely determined by the n-tuple $(\varphi(H_{1,x}), \dots, \varphi(H_{n-1,x}), \varphi(H_{n,x})) \in \mathbb{C}^{n-1} \times \mathbb{C}^\times$. These are the eigenvalues of *Hecke operators*, which are defined as the action of $H_{i,x}$ on v_x. We group these eigenvalues as an *unordered* n-tuple of complex numbers $z_1(\pi_x), \dots, z_n(\pi_x)$ defined by the equality [16]

$$\forall i \in [\![1, n]\!], H_{i,x} \cdot v_x = |\kappa(x)|^{\frac{i(n-i)}{2}} \sigma_i(z_1(\pi_x), \dots, z_n(\pi_x))v_x$$

where σ_i is the i-th elementary symmetric polynomial. This induces an isomorphism of $\mathbb{C}$-algebras

$$\mathscr{H}_x \cong \mathbb{C}[z_1(\pi_x)^{\pm 1}, \dots, z_n(\pi_x)^{\pm 1}]^{\mathfrak{S}_n}.$$

In particular, through the identification of the right-hand side with the algebra of characters of finite-dimensional representations of $\mathrm{GL}_n(\mathbb{C})$ and up to the normalization factor, $H_{i,x}$ corresponds to the character of the i-th fundamental representation and $(z_1, \dots, z_n)$ may be thought of as a semi-simple conjugacy class in $\mathrm{GL}_n(\mathbb{C})$.

We can now define the objects that will appear on the automorphic side of Langlands' correspondence.

Definition **(3.2.5.)** — *Let π be an irreducible cuspidal automorphic representation and S be a finite set of closed points of C outside of which the local factors of π are unramified.*
The Hecke eigenvalues *of π are the eigenvalues $z_1(\pi_x), \dots, z_n(\pi_x)$ for all $x \notin S$.*

The following strong multiplicity theorem by Piatetski-Shapiro prove that these Hecke eigenvalues contain enough information.

Theorem **(3.2.6.)** — *Two irreducible cuspidal automorphic representations having the same Hecke eigenvalues are isomorphic.*

4. The Langlands correspondence for functions fields.

In the section, we can finally formulate the Langlands conjecture for the group GL_n in the case of function fields. It was proven by Drinfeld in the late eighties for $n = 2$ and by (Laurent) Lafforgue in the early two thousands for arbitrary n building on Drinfeld's notion of shtuka.

[16]The factor $q_x^{\frac{i(n-i)}{2}}$ is here for aesthetic reasons in the formulation of Langlands' correspondence.

Theorem **(4.0.1.)** — *There exists a bijection between the sets $\mathscr{A}_n$ and $\mathscr{G}_n$ which satisfies the following matching condition: if a Galois ℓ-adic representation $\rho \in \mathscr{G}_n$ corresponds to an irreducible cuspidal automorphic representation $\pi \in \mathscr{A}_n$, then ρ and π have the same non-ramification locus and for each point in this locus, after the choice of a set-theoretic isomorphism* [17] *$\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$, we have*

$$\{\lambda_1(\rho_x), \dots, \lambda_n(\rho_x)\} = \{z_1(\pi_x), \dots, z_n(\pi_x)\}.$$

Remark. — This theorem could seem a bit weird because of the choice of a set-theoretic isomorphism between $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ but in fact, it is not needed since Drinfeld and later Lafforgue proved that the Hecke eigenvalues are algebraic numbers and the field of algebraic numbers is a subfield of $\overline{\mathbb{Q}}_\ell$ so we can do the comparison directly.

This statement is quite remarkable, especially since it is *false* in the case of a number field: some automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ do not correspond to any Galois representation — for example, for $n = 2$, the Maass forms do not correspond to any Galois representation. Beyond the exclusive presence of non-archimedean places, the case of function fields is also much better because geometrical methods (related to the curve whose field of rational functions is our function field) can be used. This leads to the next part of the seminar: the *geometric Langlands correspondence*, which aims at a purely geometric formulation of Langlands correspondence, for algebraic curves over an arbitrary field — not necessarily finite. In the geometric Langlands correspondance, we also want to replace these *sets* $\mathscr{G}_n$ and $\mathscr{A}_n$ by *categories*.

[17] This isomorphism exists since these two fields have the continuum cardality.