Langlands correspondence for GL_n in the case of function fields

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In the Langlands' programme, one encounters two types of fields: *local* fields [1] and *global* fields; this terminology comes from the fact that elements of global fields should be thought as meromorphic functions on curves and elements of local fields as *germs* of meromorphic functions on curves — this is true in the case of function fields. The link between them is the following: the completion of a global field for an (equivalence class) of absolute value (*place*) is a local field [2]; we also have some local-global results like Hasse–Minkowski principle that two quadratic forms over a global field are equivalent if and only if they are equivalent on any completion. Since the beginning of this reading group, we have encountered one type of global field, which are number fields. In that case, the extensions were either archimedean fields (e. g. $\mathbb{R}$ or $\mathbb{C}$) or non-archimedean fields (e. g. finite extensions of $\mathbb{Q}_p$ where p is a prime number); we (topologically) packed all this information in the ring $\mathbb{A}_k$ of adèles of k.

The problem with this type of global field is that the objects we get (Galois representations, automorphic representations, etc.) are much more difficult to study because they have archimedean and non-archimedean parts and since the topologies of archimedean and non-archimedean fields are *really* different (the non-archimedean world is more algebraic — e.g. the (closed) unit ball of the field is a subring — and at first glance not so nice for analytic arguments (totally disconnected), whereas archimedean fields that are much nicer in that respect. Therefore, the methods to work at an archimedean place and at a non-archimedean place are usually quite different and it can/could be tricky to see how to combine them.

In that talk, we study the other type of global fields: functions fields. Function fields have only non-archimedean completions (which are fields of formal Laurent series with coefficients in a finite field) and in fact, in that case, the Langlands correspondence for GL_n is proven — the case $n = 2$ was proven by Drinfeld in the late 80's and (Laurent) Lafforgue proved it for all n in the early 2000's.

1. Function fields.

We start with some generalities on function fields. Let $q := p^r$ where p is a prime number and $r \geq 1$.

Definition **(1.0.1.)** — *A* function field k *is the field of rational functions of a smooth projective irreducible curve* C *over a finite field* $\mathbb{F}_q$*, which means that* $k = \mathcal{O}_{C,\eta}$ *where* η *is the generic point of* C.

Example — The field of rational functions on the curve $\mathbb{P}^1_{\mathbb{F}_q}$ is equal to $\mathbb{F}_q(t)$.

In the sequel of the document, by a *point* of a curve, we will always mean a *closed* point of this curve.

[1] A *local field* is a locally compact topological field whose topology is not discrete. These fields can in fact be equipped with an absolute value and we can even list them: the non-archimedean ones are finite extensions of $\mathbb{Q}_p$ where p is a prime number (*inequal/mixed characteristic case*) or fields of the form $\mathbb{F}_q((t))$ (*equal characteristic case*) and the archimedean ones are either $\mathbb{R}$ or $\mathbb{C}$.

[2] In fact, any local field is obtained this way.

As a consequence, the residue field of a *point* (in this sense) is always a finite extension of $\mathbb{F}_q$ so is itself a finite field.

Let us now compare functions fields and number fields in the case where $C = \mathbb{P}^1_{\mathbb{F}_p}$ (the corresponding function field being $\mathbb{F}_p(t)$) and where the number field is trivial. We summarize the comparison in the following table:

	Function field side	Number field side
Valuated field	$(\mathbb{F}_p(t), v_t)$	$(\mathbb{Q}, v_p)$
Subring of elements with non-negative valuation	$\mathbb{F}_p[t] \subseteq \mathbb{F}_p(t)$	$\mathbb{Z} \subseteq \mathbb{Q}$
Completion	$\mathbb{F}_p(t) \hookrightarrow \mathbb{F}_p((t))$	$\mathbb{Q} \hookrightarrow \mathbb{Q}_p$
Valuation (sub)ring	$\mathbb{F}_p[\![t]\!] \hookrightarrow \mathbb{F}_p((t))$	$\mathbb{Z}_p \hookrightarrow \mathbb{Q}_p$
Uniformizing element	t	p
Residue field	$\mathbb{F}_p$	$\mathbb{F}_p$

On one hand, the completions of $\mathbb{F}_p(t)$ that we can get are parametrized by maximal ideals of $\mathbb{F}_p[t]$ (irreducible monic polynomials); on the other hand, the non-archimedean completions of $\mathbb{Q}$ that we can get are parametrized by maximal ideal of $\mathbb{Z}$ (prime numbers).

There are also differences between $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$: the former is of characteristic zero, the latter is of characteristic p; moreover, addition in the former is much more complicated (carry) than in the latter (term-wise).

For an arbitrary smooth projective irreducible curve C over $\mathbb{F}_q$, the different completions of its function field are parametrized by its points and a completion corresponding to a $\mathbb{F}_{q^n}$-point x is isomorphic to $\mathbb{F}_{q^n}((t_x))$ where $t_x \in \mathcal{O}_{C,x}$ is a uniformizer.

We usually only consider field of rational functions of *curves* because for surfaces, it is already too complicated: they are labelled by a chains $x \in \gamma_x$ where x is a point of the surface and γ is a germ of curve of S at x; the corresponding completion is isomorphic to a field of formal Laurent series in *two* variables and we thus leave the realm of discrete valuation rings, etc.

2. Galois representations for function fields.

In this section, we introduce the Galois representations and all the extra data linked to it (Frobenius classes, eigenvalues of them, etc.) in the case of a function field.

2.1. Galois theory and covers of curves.

Let C be a smooth connected projective curve over $\mathbb{F}_q$ and k be the field of rational functions on C.

2.1.1. — Since $\overline{k}/k$ is an (infinite) Galois extension, the fundamental theorem of infinite Galois theory yields a decreasing bijection between the sets of closed subgroups of $\mathrm{Gal}(\overline{k}/k)$ and field extensions of k contained in $\overline{k}$. Moreover, if $C' \to C$ is a covering of C, then the field of function k' of C' is a finite extension of k. In fact, this defines a functor

$$T : C' \to C \in \mathbf{Cov}(C) \mapsto \mathbb{F}_q(C')/k \in \mathbf{Ext}^{\mathrm{f}}(k)$$

where $\mathbf{Cov}(C)$ stands for the category of covers of C and $\mathbf{Ext}^{\mathrm{f}}(k)$ for that of finite extensions of k. Since a curve is completely determined by its function field, we deduce that T is an equivalence of categories [3].

If we consider non-necessarily finite extensions of k on the right, is it enough to add covers of C that are not necessarily of finite type? If yes, to the (maximal) extension $\overline{k}$ of k corresponds a (maximal) cover of C that we denote by $\overline{C} \to C$; we can think of $\mathrm{Gal}(\overline{k}/k)$ as the group of automorphism of this cover.

[3] Maybe one has to put extra adjectives for the extensions of k that we get...

We now try to associate to a point a Frobenius element.

2.1.2. — Let $x \in C$ be a (closed) point; its residue field $\kappa(x)$ is a finite extension of $\mathbb{F}_q$, we denote its cardinality by q_x. We pick a point $\overline{x} \in \overline{C}$ lying over x. The *decomposition group* $D_{\overline{x}}$ of $\overline{x}$ is the subgroup [4]of $\mathrm{Gal}(\overline{k}/k)$ preserving $\overline{x}$. This group of course depends on the choice of $\overline{x}$ lying over x but if $\overline{y}$ is another point of $\overline{C}$ lying over x, then by transitivity of the action of $\mathrm{Gal}(\overline{k}/k)$ on points $\overline{x}$ lying over x, there exists $\tau \in \mathrm{Gal}(\overline{k}/k)$ such that $\overline{y} = \tau(\overline{x})$. In particular, this implies that

$$D_{\overline{y}} := \{\sigma \in \mathrm{Gal}(\overline{k}/k) \mid \sigma(\overline{y}) = \overline{y}\} = \{\sigma \in \mathrm{Gal}(\overline{k}/k) \mid \sigma(\tau(\overline{x})) = \tau(\overline{x})\} = \tau^{-1} D_{\overline{x}} \tau,$$

so the $\mathrm{Gal}(\overline{k}/k)$-conjugacy class of $D_{\overline{x}}$ does not depend on the choice of $\overline{x}$ lying over x. We denote it by D_x.

Lemma **(2.1.3.)** — *The (conjugacy class of the) subgroup D_x is isomorphic to* $\mathrm{Gal}(\overline{k}_x/k_x)$, *where k_x stands for the completion of k at the place corresponding to the closed point x of C.*

Proof. — This comes from the fact that D_x is the group of those automorphisms of $\overline{k}/k$ that are continuous for the topology defined by the valuation associated to x. Indeed, if $\sigma \in D_{\overline{x}}$, we have $\overline{x} \circ \sigma = \overline{x}$ so σ is clearly continuous for the topology defined by $\overline{x}$. Conversely, if $\sigma \in \mathrm{Gal}(\overline{k}/k)$ is continuous for the topology defined by $\overline{x}$, then for all $t \in \overline{k}$, we have

$$|t|_{\overline{x}} < 1 \Rightarrow |t|_{\overline{x}\circ\sigma} = |\sigma(t)|_{\overline{x}} < 1$$

because continuous homomorphisms preserve null-sequences. This implies that the absolute values associated to $\overline{x}$ and to $\overline{x} \circ \sigma$ are equivalent (i.e., there exists $\alpha > 0$ such that $|\cdot|_{\overline{x}} = |\cdot|^{\alpha}_{\overline{x}\circ\sigma}$) but these two absolute values also coincide on k so they are in fact *equal*, which means that $\sigma \in D_{\overline{x}}$.

In particular, such automorphisms can be extended to the completion with respect to x, which gives a group homomorphism $D_x \to \mathrm{Gal}(\overline{k}_x/k_x)$ which is an isomorphism because any continuous k-automorphism of $\overline{k}$ extends *uniquely* to a k_x-automorphism of $\overline{k}_x$ since $\overline{k}$ (resp. k) is a dense subfield of $\overline{k}_x$ (resp. k_x). □

As in Marwan's talk, for all closed point x of C, we have a group homomorphism

$$d_x : D_x \to \mathrm{Gal}(\overline{\kappa(x)}/\kappa(x)) = \mathrm{Gal}(\overline{\mathbb{F}}_{q_x}/\mathbb{F}_{q_x}).$$

Definition **(2.1.4.)** — *Let $x \in C$ be a closed point.*
The inertia subgroup at x *is the kernel of d_x, it is denoted by I_x.*

2.1.5. — If x is a closed point of C, then $\kappa(x)$ is a finite extension of $\mathbb{F}_q$ of cardinality q_x; in particular, the profinite group $\mathrm{Gal}(\overline{k}_x/k_x)$ is topologically generated by the *geometric Frobenius* given by $\mathrm{Fr}_x : t \mapsto t^{-q_x}$. If G is a group and $\sigma : \mathrm{Gal}(\overline{k}/k) \to G$ is a group homomorphism, we say that σ is *unramified at* x if $\sigma(I_x) = \{0\}$. In that case, as in the case of number fields, Fr_x gives a well-defined conjugacy class in G (and not only a I_x-coset), that we denote by $\sigma(\mathrm{Fr}_x)$. Note that this notion is in fact independent of the choice of $\overline{x}$ over x since the kernel of a group homomorphism is a normal subgroup.

2.2. Continuous representations.

In this subsection, we will have a look at finite-dimensional representations of $G_k := \mathrm{Gal}(\overline{k}/k)$ where k is a function field. This is a profinite group; in particular, it has a topology — the Krull topology: a basis of neighbourhoods of the neutral element is given by normal subgroups of finite index. In fact, we will only consider *continuous* representations of G_k.

There is one little problem: if we consider complex-valued representations, the fact that the Krull topology and the Euclidean topology are very different (the Krull topology is totally disconnected) makes the continuity

[4]If we see $\overline{x}$ as a valuation on $\overline{k}$, then this condition on $\sigma \in \mathrm{Gal}(\overline{k}/k)$ is written $\overline{x} \circ \sigma = \overline{x}$.

condition very restrictive. Indeed, from the theory of Lie groups (using the exponential map), we know that there exists an open neighbourhood V of I_n that does not contain any non-trivial subgroup of $\mathrm{GL}_n(\mathbb{C})$ so by continuity of ρ, the preimage $\rho^{-1}(V)$ must contain a normal subgroup H of G_k which has finite index and that contains the neutral element. Therefore, $\rho(H)$ is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ contained in V so it has to be trivial. This implies that ρ factorizes through the finite group G_k/H.

To get rid of this problem, we replace the field $\mathbb{C}$ by a field which has a topology much more compatible with that of G_k, namely $\mathbb{Q}_\ell$ for $p \nmid \ell$ [5]. Note that this is consistent with the fact that in the number field case, the Galois representations we were considering were (ℓ-adic) étale cohomologies, which are $\mathbb{Q}_\ell$-vector spaces.

Let us now check if this replacement actually fixed the problem we were facing.

2.2.1. — Let $\rho : \mathbb{Z}_p \to \mathbb{Q}_\ell^\times$ where $p \nmid \ell$ be a continuous representation. Let $\mu \in \mathbb{Q}_\ell$ such that $\mu - 1 \in \ell\mathbb{Z}_\ell$ — so $\mu \in \mathbb{Z}_\ell^\times$. Then, the ultrametric inequality yields

$$|\mu^{p^n} - 1|_\ell \leq \max_{1 \leq i \leq p^n - 1} (|\mu - 1|_\ell^{p^n}, |\tbinom{p^n}{i}(-1)^{p^n - i}\mu^i|_\ell) = \max_{1 \leq i \leq p^n - 1} (|\mu - 1|_\ell^{p^n}, |\mu|_\ell^i) = |\mu - 1|_\ell^{p^n} \leq \tfrac{1}{p^n}.$$

where the last inequality holds since $(\mu - 1)^{p^n} \in \ell^{p^n}\mathbb{Z}_\ell$. This implies that there exists a unique [6] continuous group homomorphism $\rho : \mathbb{Z}_p \to \mathbb{Q}_\ell^\times$ such that $\rho(1) = \mu$. In particular, this representation ρ does not factor through a finite quotient of $\mathbb{Z}_p$.

Let us therefore introduce the following definition:

Definition **(2.2.2.)** — *Let ℓ be a prime number not divisible by p. Let $n \geq 0$.*
A n-dimensional ℓ-adic representation *of G_k is a continuous group homomorphism $\rho : G_k \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ satisfying the following two conditions:*

- *its image is contained in $\mathrm{GL}_n(E)$ where E is a finite extension of $\mathbb{Q}_\ell$*
- *it is ramified only at finitely many closed points of C.*

For all $n \geq 0$, we denote by $\mathscr{G}_n$ the set of equivalence classes of irreducible n-dimensional ℓ-adic representations ρ of G_k such that the image of $\det(\rho)$ is a finite group.

2.2.3. — Given an element ρ of $\mathscr{G}_n$, we consider the collection $\{\rho_x := \rho(\mathrm{Fr}_x) \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \mid x \in C$[7]$\}$ and the collection of their eigenvalues $\{(\lambda_1(\rho_x), \dots, \lambda_n(\rho_x)) \in \overline{\mathbb{Q}}_\ell^n \mid x \in C\}$ for ρ unramified. Chebotarev's density theorem implies that if two ℓ-adic representations have the same collection of eigenvalues for all but finitely many (closed) points $x \in C$, then they are isomorphic.

3. Cuspidal automorphic representations for functions fields.

In Henry's talk, we have seen that the other side of the Langlands correspondence was dealing with cuspidal automorphic representations of the adelic group $\mathrm{GL}_n(\mathbb{A}_F)$ where F is a number field. The definition of the rings of adèles also makes sense for a function field — in this case, we will only have non-archimedean factors.

As before, we denote by k the field of functions of a smooth connected projective curve C over $\mathbb{F}_q$. For every closed point x of C, which corresponds to a valuation on k, we denote by k_x the completion of k at x and $\mathcal{O}_x$ the ring of algebraic integers of k_x. We consider a uniformizing element t_x of $\mathcal{O}_x$, which induces isomorphisms

$$k_x \cong \kappa(x)((t_x)) \text{ and } \mathcal{O}_x \cong \kappa(x)[\![t_x]\!],$$

[5] For $p \mid \ell$, ℓ-adic cohomology is a total mess (lack of functoriality), that's why crystalline cohomology was introduced by Berthelot.

[6] The condition $\rho(1) = \mu$ implies that $\rho(p^n) = \mu^{p^n}$ for all $n \geq 0$ so the estimation above implies that $|\rho(p^n) - \rho(0)|_\ell \leq p^{-n}$. Now, the continuity of ρ and the fact that the p-adic valuation is discrete implies that ρ is uniquely determined by the condition $\rho(1) = \mu$.

[7] We only consider closed points.